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ON GENERALIZED KRULL POWER SERIES RINGS

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Abstract. Let R be an integral domain. We prove that the power series ring $R[\![X]\!]$ is a Krull domain if and only if $R[\![X]\!]$ is a generalized Krull domain and t-dim $R \leq 1$, which improves a well-known result of Paran and Temkin. As a consequence we show that one of the following statements holds: (1) the concepts "Krull domain" and "generalized Krull domain" are the same in power series rings, (2) there exists a non-t-SFT domain R with t-dim $R > 1$ such that t-dim $R[\![X]\!] = 1$.

1. Introduction

For an integral domain R, an overring of R is a ring lying between R and its quotient field. Let R be an integral domain. Consider the following properties on a family $\{R_\alpha\}_{\alpha\in\Lambda}$ of valuation overrings of R.

(a) $R = \cap_{\alpha\in\Lambda} R_\alpha$.
(b) The family $\{R_\alpha\}_{\alpha\in\Lambda}$ has finite character, i.e., if $0 \neq a \in R$, then a is a unit in all but finitely many R_α.
(c) Each R_α is essential for R, i.e., R_α is the localization of R at $M_\alpha \cap R$, where M_α is the maximal ideal of R_α.
(d) Each R_α is one-dimensional.
(d') Each R_α is one-dimensional and discrete.

If there is a family $\{R_\alpha\}_{\alpha\in\Lambda}$ of valuation overrings of R satisfying (a), (b), (c), and (d), then R is called a generalized Krull domain. If (d) is replaced by (d') in the preceding sentence, then we obtain the definition of a Krull domain R. It is clear that a Krull domain is a generalized Krull domain. A one-dimensional nondiscrete valuation domain is a simple example of a generalized Krull domain that is not a Krull domain.

Let R be an integral domain and let $R[\![X]\!]$ be the power series ring over R. Gilmer showed that if R is a Krull domain, then so is the power series ring $R[\![X]\!]$ [5]. In fact, the converse also holds (see [3] or [6]). Hence, R is a Krull domain if and only if $R[\![X]\!]$ is a Krull domain. The notion of generalized Krull domain was introduced by Weissauer [18] and its importance is in field

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arithmetic and Galois theory (see, for example, [4, 15–17]). There was a long standing open problem in commutative algebra and field arithmetic, namely, is the power series ring $R[\![X]\!]$ a generalized Krull domain if R is [4, Problem 15.5.9(a)]? An answer to this question was given by Paran and Temkin in [16], where they showed that $R[\![X]\!]$ is not necessarily a generalized Krull domain if R is. In fact, they proved that $R[\![X]\!]$ is a Krull domain if and only if both R and $R[\![X]\!]$ are generalized Krull domains [16]. Hence, if R is a generalized Krull domain that is not a Krull domain, then $R[\![X]\!]$ is never a generalized Krull domain.

In this paper, we show that $R[\![X]\!]$ is a Krull domain if and only if $R[\![X]\!]$ is a generalized Krull domain and $t\text{-}\dim R \leq 1$ (the definition of $t\text{-}\dim R$, the t-dimension of R, is given in Section 2). Since every generalized Krull domain has t-dimension at most one, our result improves Paran and Temkin's result. As a consequence we show that one of the following statements holds.

(1) The concepts "Krull domain" and "generalized Krull domain" are the same in power series rings.
(2) There exists a non-t-SFT domain R with $t\text{-}\dim R > 1$ such that $t\text{-}\dim R[\![X]\!] = 1$.

Note that for any ring R, we always have $\dim R[\![X]\!] \geq \dim R$, where $\dim R$ denotes the Krull dimension of R. If (2) is true, then we have $t\text{-}\dim R > t\text{-}\dim R[\![X]\!]$, which shows a strange behavior of the t-dimension of the power series ring $R[\![X]\!]$.

Arnold proved that if R is a non-SFT ring, then the Krull dimension of $R[\![X]\!]$ is infinite [1] (see also [2,9,10,12–14]). In [11], Kang and Park conjectured that a similar result holds for non-t-SFT domains, namely, if R is a non-t-SFT domain, then $t\text{-}\dim R[\![X]\!] = \infty$. If Kang-Park's conjecture is true, then (2) never holds and hence the concepts "Krull domain" and "generalized Krull domain" are the same in power series rings.

If R is a t-SFT domain, then $t\text{-}\dim R[\![X]\!] \geq t\text{-}\dim R$ [11]. However, it is still unknown whether this inequality holds in general. In other words, so far there are no examples of an integral domain R such that $t\text{-}\dim R > t\text{-}\dim R[\![X]\!]$. It follows from our result that if the concepts "Krull domain" and "generalized Krull domain" are different in power series rings, then one can obtain such an example.

2. Results

We first give definition of an integral domain of Krull type and a Prüfer v-multiplication domain (PvMD), and collect some well-known results related to these kinds of integral domains that will be used later in the proof of our main result.

Definition 1. Let R be an integral domain. If there is a family $\{R_\alpha\}_{\alpha\in\Lambda}$ of valuation overrings of R satisfying (a), (b), and (c) in the introduction, then R is said to be of Krull type.

By definition, any generalized Krull domain is of Krull type. In other words, the concept of integral domain of Krull type is weaker than that of generalized Krull domain. However, it is still stronger than that of PvMD as shown below.

Let R be an integral domain with quotient field K. A fractional ideal I of R is an R-submodule of K such that $aI \subseteq R$ for some $0 \neq a \in R$. For each nonzero fractional ideal I of R, let $I^{-1} = \{x \in K \mid xI \subseteq R\}$, $I_v = (I^{-1})^{-1}$, and

$$I_t = \cup\{J_v \mid J \text{ is a nonzero finitely generated fractional subideal of } I\}.$$

An ideal I of R is called a v-ideal (respectively t-ideal) if $I_v = I$ (respectively $I_t = I$). I is a maximal t-ideal if it is maximal among proper t-ideals.

Definition 2. An integral domain R is called a PvMD if every nonzero finitely generated ideal I of R is t-invertible, i.e., $(II^{-1})_t = R$.

The following theorem of Griffin shows that every integral domain of Krull type is a PvMD.

Theorem 3. ([7, Theorem 7]). The following are equivalent for an integral domain R.

(1) R is an integral domain of Krull type.
(2) R is a PvMD and each nonzero element of R belongs to only finitely many maximal t-ideals of R.

For an integral domain R, denote by t-Max(R) the set of maximal t-ideals of R. It is well-known that each maximal t-ideal is a prime ideal, each proper t-ideal is contained in a maximal t-ideal, and a prime ideal minimal over a t-ideal is a prime t-ideal. In particular, any height one prime ideal is a t-ideal (height of a prime ideal P is the supremum of the lengths of chains of prime ideals between (0) and P).

The following is well-known (see [7] or [8, Theorems 3.2 and 3.3]).

Theorem 4. *Let R be an integral domain. Then R is a PvMD if and only if R_P is a valuation domain for each $P \in t\text{-Max}(R)$. Hence, if R is a PvMD, then $R = \cap_{P \in t\text{-Max}(R)} R_P$ is an intersection of essential valuation domains.*

For a prime t-ideal P of R, the t-height of P, denoted by t-ht$P$, is the supremum of the lengths of chains of prime t-ideals between (0) and P (we include (0) as a prime t-ideal). The t-dimension of R is defined by $t\text{-}\dim R = \sup\{t\text{-ht}P \mid P \text{ is a prime } t\text{-ideal of } R\}$.

Let R be an integral domain. If there is a family $\{R_\alpha\}_{\alpha\in\Lambda}$ of valuation overrings of R satisfying (a) and (b), then R is said to be of finite character. A family $\{R_\alpha\}_{\alpha\in\Lambda}$ satisfying (a) and (b) is called a defining family for R.

Theorem 5. *Let R be an integral domain of Krull type. Then R is a generalized Krull domain if and only if $t\text{-}\dim R \leq 1$.*

Proof. The result trivially holds if R is a field. Hence, suppose that R is not a field so that $t\text{-}\dim R > 0$. By Theorems 3 and 4, $R = \cap_{P\in t\text{-Max}(R)} R_P$ is an intersection of valuation domains and each nonzero element of R belongs to only finitely many maximal t-ideals P of R. Hence, $\{R_P\}_{P\in t\text{-Max}(R)}$ is a defining family for R. If R is a generalized Krull domain, then by [6, Corollary 43.9], $\{R_{P_\lambda}\}_{\lambda\in\Lambda}$ is the unique defining family for R, where $\{P_\lambda\}_{\lambda\in\Lambda}$ is the set of height one prime ideals of R. It follows that $t\text{-Max}(R) = \{P_\lambda\}_{\lambda\in\Lambda}$ and hence $t\text{-}\dim R = 1$. Conversely, if $t\text{-}\dim R = 1$, then R_P is one-dimensional for each maximal t-ideal P of R, which implies that R is a generalized Krull domain. □

We now state and prove the main result of this paper.

Theorem 6. *The following are equivalent for an integral domain R.*

(1) *R is a Krull domain.*
(2) *$R[\![X]\!]$ is a Krull domain.*
(3) *$R[\![X]\!]$ is a generalized Krull domain and $t\text{-}\dim R \leq 1$.*

Proof. The theorem trivially holds if R is a field. Hence, we assume that R is not a field so that $t\text{-}\dim R > 0$. We only need to prove that (3) implies (2). Suppose that $R[\![X]\!]$ is a generalized Krull domain and $t\text{-}\dim R = 1$. Let $\{Q_\alpha\}_{\alpha\in\Lambda}$ be the set of maximal t-ideals of $R[\![X]\!]$. By Theorems 3 and 4,

$$R[\![X]\!] = \bigcap_{\alpha\in\Lambda} R[\![X]\!]_{Q_\alpha}$$

is an intersection of valuation domains and the family $\{R[\![X]\!]_{Q_\alpha}\}_{\alpha\in\Lambda}$ has finite character. By Theorem 5, $\{Q_\alpha\}_{\alpha\in\Lambda}$ is the set of height one prime ideals of $R[\![X]\!]$. Hence, each $R[\![X]\!]_{Q_\alpha}$ is one-dimensional. For each α, if we let

$$V_\alpha = R[\![X]\!]_{Q_\alpha} \cap K,$$

where K is the quotient field of R, then V_α is a valuation overring of R with Krull dimension at most one (see [6, Theorem 19.16]). We have

$$R = R[\![X]\!] \cap K = (\bigcap_{\alpha\in\Lambda} R[\![X]\!]_{Q_\alpha}) \cap K = \bigcap_{\alpha\in\Lambda} (R[\![X]\!]_{Q_\alpha} \cap K) = \bigcap_{\alpha\in\Lambda} V_\alpha.$$

Since the family $\{R[\![X]\!]_{Q_\alpha}\}_{\alpha\in\Lambda}$ has finite character, so is $\{V_\alpha\}_{\alpha\in\Lambda}$. Let P be a maximal t-ideal of R. Since $t\text{-}\dim R = 1$, P is a minimal prime ideal of R. By [6, Theorem 43.7], R_P is a valuation domain and $R_P = V_\alpha$ for some $\alpha \in \Lambda$. Hence, R is a PvMD by Theorem 4. Since $\{V_\alpha\}_{\alpha\in\Lambda}$ has finite character, each nonzero element of R belongs to only finitely many maximal t-ideals P of R. Theorem 3 shows that R is an integral domain of Krull type. By Theorem 5, R is a generalized Krull domain. Since both R and $R[\![X]\!]$ are generalized Krull domain, $R[\![X]\!]$ is a Krull domain by [16, Theorem 1.1]. □

Definition 7. An ideal I of a ring R is called an SFT ideal if there exist a finitely generated ideal J of R with $J \subseteq I$ and a positive integer k such that $a^k \in J$ for each $a \in I$. A ring R is called an SFT ring if every ideal I of R is an SFT ideal.

The definition of SFT ring was given by Arnold in [1], where he showed that if a ring R is a non-SFT ring, then the Krull dimension of the power series ring $R[\![X]\!]$ is infinite. In fact, $\dim R[\![X]\!]$ is uncountably infinite if R is a non-SFT ring [9] (see also [2, 12–14]).

In [11] Kang and Park defined the following concept of t-SFT ideal, which is a t-analogue version of SFT ideal.

Definition 8. A nonzero ideal I of an integral domain R is called a t-SFT ideal if there exist a finitely generated ideal J of R with $J \subseteq I$ and a positive integer k such that $a^k \in J_v$ for each $a \in I_t$. An integral domain R is called a t-SFT domain if every nonzero ideal I of R is a t-SFT ideal.

Remark 9. Note that every SFT domain is a t-SFT domain and the concepts "SFT domain" and "t-SFT domain" are the same in valuation domains, more generally, in Prüfer domains [11].

Theorem 6 implies the following.

Corollary 10. *One of the following statements holds.*

(1) *The concepts "Krull domain" and "generalized Krull domain" are the same in power series rings.*
(2) *There exists a non-t-SFT domain R with t-$\dim R > 1$ such that t-$\dim R[\![X]\!] = 1$.*

Proof. Suppose that (1) does not hold. Then there exists an integral domain R such that $R[\![X]\!]$ is a generalized Krull domain but not a Krull domain. By Theorem 6, t-$\dim R > 1$. Since $R[\![X]\!]$ is a generalized Krull domain that is not a field, t-$\dim R[\![X]\!] = 1$ (Theorem 5). As in the proof of Theorem 6, $R = \cap_\alpha V_\alpha$ is an intersection of valuation overrings of R of Krull dimension at most one. Hence, R is completely integrally closed. If R is a t-SFT domain, then it is a Krull domain by [11, Theorem 2.9] and hence $R[\![X]\!]$ is a Krull domain, a contradiction. Therefore, R is a non-t-SFT domain. □

Remark 11. In [11], Kang and Park conjectured that if R is a non-t-SFT domain, then t-$\dim R[\![X]\!] = \infty$. If Kang-Park's conjecture is true, then (2) never holds and hence the concepts "Krull domain" and "generalized Krull domain" are the same in power series rings.

Remark 12. For an integral domain R, if P is a prime ideal of R, then $P[\![X]\!]$ is a prime ideal of $R[\![X]\!]$. Hence, we always have $\dim R[\![X]\!] \geq \dim R$. For a t-SFT domain R, we also have t-$\dim R[\![X]\!] \geq t$-$\dim R$ [11]. However, it is still unknown whether this inequality holds in general. In other words, so far there are no examples of an integral domain R such that t-$\dim R > t$-$\dim R[\![X]\!]$. According to Corollary 10, if the concepts "Krull domain" and "generalized Krull domain" are different in power series rings, then one can obtain such an example.

We end the paper with the following question.

Question 13. In Corollary 10, which of the statements, (1) or (2), holds?

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