Bull. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/BKMS.b170221
pISSN: 1015-8634 / eISSN: 2234-3016

NEW ANALYTIC APPROXIMATE SOLUTIONS TO THE GENERALIZED REGULARIZED LONG WAVE EQUATIONS

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Abstract. In this paper, the new optimal perturbation iteration method has been applied to solve the generalized regularized long wave equation. Comparing the new analytic approximate solutions with the known exact solutions reveals that the proposed technique is extremely accurate and effective in solving nonlinear wave equations. We also show that,unlike many other methods in literature, this method converges rapidly to exact solutions at lower order of approximations.

1. Introduction

Nonlinear partial differential equations (NPDEs) represent many scientific phenomena in applied mathematics and physics and it is still very difficult to solve most of them either numerically or analytically. By 2000s, many authors have used a variety of methods to analyze the approximate solutions of NPDEs, such as homotopy analysis method(HAM) [29], homotopy perturbation Sumudu transform method [36], modified simple equation method [22], homotopy perturbation method (HPM) [16, 27], Adomian decomposition method (ADM) [19], variational iteration method (VIM) [35] and the sine-cosine method [5]. Because of the inadequacy of these methods, the newly developed techniques such as perturbation iteration method [1, 2], optimal homotopy asymptotic method [7, 18, 24], optimal perturbation iteration method (OPIM) [8, 13, 33, 34] have recently drawn more attention to determine more accurate and effective solutions for nonlinear models.

One of the famous NPDEs is the generalized regularized long wave (GRLW) equation which can be given as

$$u_t + u_x + \alpha(u^p)_x - \beta u_{xxt} = 0,\ (x,t) \in (a,b) \times (0,T), \tag{1.1}$$

where α, β are positive constants and p is a positive integer. GRLW equation was first used as a model for small-amplitude long-waves on the surface of water in a channel by Peregrine [28]. Additionally, Eq. (1.1) presents mathematical

Received March 13, 2017; Accepted November 3, 2017.
2010 *Mathematics Subject Classification.* 70K60, 65N99, 33E15, 35C08.
Key words and phrases. optimal perturbation iteration method, partial differential equations, regularized long wave equations, solitons.

models for predicting a variety of physical phenomena such as longitudinal dispersive waves in elastic rods, pressure waves in liquid-gas bubble mixtures and rotating flow down a tube. In (1.1), the nonlinear term $\alpha(u^p)_x$ causes steepening of the wave form. The last term βu_{xxt} is called as dispersion effect term and this term makes the wave form spread. The solitons emerge by virtue of the balance between dispersion and nonlinearity [26]. These solitons exist in many types of systems from sky to laboratory [6]. GRLW equation can also be used instead of KdV equation in many physical systems [10].

GRLW equation reduced to the regularized long wave equation (RLW) and the modified regularized long wave (MRLW) equation for $p = 1, 2$ respectively. The RLW and MRLW equations describe the development of an undular bore. The behavior of an undular bore is characterised with constants α, β in Eq. (1.1). These equations have been used for modeling in many fields of science and engineering such as magneto-hydrodynamics waves in plasma, rotating flow down a tube, ion-acoustic waves in plasma, longitudinal dispersive waves in elastic rods, pressure waves in liquidgas bubble mixture, lossless propagation of shallow water waves and thermally excited phonon packets in low temperature nonlinear crystals [12, 23, 25, 31].

There are few analytical solutions available for special kinds of GRLW in the literature. Therefore, the numerical solutions of these equations have been subject of many studies. Several numerical techniques including the Petrov-Galerkin method [14], finite difference method [20], finite element methods [11,15,17], the radial basis function collocation method [32], expansion methods [3, 4] and the cubic B-spline finite element method [30] have been established for the approximate solution of these equations.

In this study, we put forward a new approach to optimal perturbation iteration method to make it practicable for NPDEs. Generalized regularized long wave (GRLW) equation is specifically addressed to demonstrate the effectiveness of OPIM in solving such NPDEs. Examples show that the new approximate solutions obtained via proposed method is more accurate and impressive than many other techniques in literature.

2. Analysis of OPIM on GRLW equation

Optimal perturbation iteration method (OPIM) is a combination of perturbation iteration [1, 2] and optimal homotopy asymptotic methods [7, 18, 24]. It has been lately developed and it has been efficiently implemented to strongly nonlinear differential equations [8, 33, 34]. In this section, we give a formulation of OPIM for the general NPDEs. At the same time, we will have created the algorithm for GRLW equations.

(a) Consider the following nonlinear partial differential equation in closed form:

$$F\left(u_{xxt}, u_x, u_t, u, \varepsilon\right) = 0, \tag{2.1}$$

where $u = u(x,t)$ and ε is the perturbation parameter which is artificially inserted into (2.1). In this case, $\varepsilon = 1$ can be submitted into Eq. (1.1) as:

$$F = u_t + \varepsilon\left(u_x + \alpha(u^p)_x - \beta u_{xxt}\right) = 0. \tag{2.2}$$

(b) Approximate solution with one correction term in the perturbation expansion is taken as

$$u_{n+1} = u_n + \varepsilon (u_c)_n, \tag{2.3}$$

where $n \in \mathrm{N}\cup\{0\}$. Upon substitution of (2.3) into (2.2), expanding in a Taylor series with first derivatives only gives the following algorithm:

$$F + F_u\,(u_c)_n\,\varepsilon + F_{u_x}\,((u_c)_n)_x\,\varepsilon + F_{u_t}\,((u_c)_n)_t\,\varepsilon + F_{u_{xxt}}\,((u_c)_n)_{xxt}\,\varepsilon + F_\varepsilon\varepsilon = 0, \tag{2.4}$$

where

$$F_u = \frac{\partial F}{\partial u},\ F_{u_x} = \frac{\partial F}{\partial u_x},\ F_{u_t} = \frac{\partial F}{\partial u_t},\ F_{u_{xxt}} = \frac{\partial F}{\partial u_{xxt}},\ F_\varepsilon = \frac{\partial F}{\partial \varepsilon}.$$

Calculating all derivatives and functions at $\varepsilon = 0$ yields

$$((u_c)_n)_t = \beta(u_n)_{xxt} - (u_n)_x - \alpha((u)_n{}^p)_x - (u_n)_t. \tag{2.5}$$

(2.5) is called as perturbation iteration algorithm (PIA) for RGLW equation (1.1). In order to initiate the iteration procedure, a first trial function u_0 is selected appropriately according to the prescribed conditions. Then the first correction term $(u_c)_0$ can be computed from the algorithm (2.5) by using u_0 and given condition(s).

c) Use the following equation

$$u_{n+1} = u_n + P_n(u_c)_n \tag{2.6}$$

to enhance the accuracy of the results and effectiveness of the method. Here $P_0, P_1, P_2, \ldots$ are convergence control parameters which ensure us to adjust the convergence.

Progressing for $n = 0, 1, \ldots$, more approximate solutions are determined as:

$$\begin{aligned} &u_1 = u(x,t;P_0) = u_0 + P_0(u_c)_0 \\ &u_2(x,t;P_0,P_1) = u_1 + P_1\,(u_c)_1 \\ &\quad\vdots \\ &u_m(x,t;P_0,\ldots,P_{m-1}) = u_{m-1} + P_{m-1}(u_c)_{m-1}. \end{aligned} \tag{2.7}$$

d) Putting the approximate solution u_m into Eq. (2.1), the general problem results in the following residual:

$$Re(x,t;P_0,\ldots,P_{m-1}) = F\left((u_m)_{xxt}, (u_m)_x, (u_m)_t, (u_m)\right) \tag{2.8}$$

Evidently, when $Re(x,t;P_0,\ldots,P_{m-1}) = 0$ then the approximation $u_m(x,t;P_0,\ldots,P_{m-1})$ will be the exact solution.However, it doesn't usually happen in nonlinear equations,but the functional can be minimized as:

$$J(P_0,\ldots,P_{m-1}) = \int_0^T \int_a^b Re^2(x,t;P_0,\ldots,P_{m-1})dxdt, \tag{2.9}$$

where a, b and T are selected from the domain of the problem. Optimum values of $P_0, P_1, \ldots$ can be acquired from the conditions

$$\frac{\partial J}{\partial P_0} = \frac{\partial J}{\partial P_1} = \cdots = \frac{\partial J}{\partial P_{m-1}} = 0. \tag{2.10}$$

The constants $P_0, P_1, \ldots$ can also be stated from

$$\begin{aligned} Re(x_0, t_0; P_i) &= Re(x_1, t_1; P_i) = \cdots \\ &= Re(x_{m-1}, t_{m-1}; P_i) = 0,\ i = 0, 1, \ldots, m-1, \end{aligned} \tag{2.11}$$

where $x_i, t_i \in (a, b) \times (0, T)$. For more information about finding these constants, we refer to [18, 24].

Inserting the constants into the last one of Eqs. (2.7), the approximate solution of order m is obtained. Having identified the optimal parameters in this fashion, we call the new iterative technique (2.4) together with (2.9) or (2.11) the optimal perturbation iteration method (OPIM).

3. Applications

In this section, we give some numerical experiments to test the accuracy of the proposed method. For illustration purposes, we will consider the special cases of the GRLW equation. We will first obtain the PIM results and then the OPIM solutions depending on them.

Example 3.1. As the first case, consider the GRLW equation (1.1) with $p = 2, \alpha = \beta = 1$. In this case, the solitary wave solution can be computed for the equation:

$$u_t + u_x + \left(u^2\right)_x - u_{xxt} = 0; \quad x \geq 0,\ 0 \leq t \leq 1 \tag{3.1}$$

with the initial condition

$$u(x, 0) = \frac{3}{2} \operatorname{sech}^2\left(\frac{x+1}{2\sqrt{2}}\right). \tag{3.2}$$

The exact solution of this problem can be given as [9, 21]

$$u(x, t) = \frac{3}{2} \operatorname{sech}^2\left(\frac{x+1-2t}{2\sqrt{2}}\right). \tag{3.3}$$

Initially, trial function u_0 can be taken as

$$u_0 = \frac{3}{2} \operatorname{sech}^2\left(\frac{x+1}{2\sqrt{2}}\right). \tag{3.4}$$

Using the algorithm (2.5) and Eq. (3.4), first order problem arises as:

$$\begin{aligned} &((u_c)_0)_t = \frac{3\operatorname{sech}^2\left(\frac{1+x}{2\sqrt{2}}\right)\tanh\left(\frac{1+x}{2\sqrt{2}}\right)}{2\sqrt{2}} + \frac{9\operatorname{sech}^4\left(\frac{1+x}{2\sqrt{2}}\right)\tanh\left(\frac{1+x}{2\sqrt{2}}\right)}{2\sqrt{2}}; \\ &(u_c)_0(x, 0) = 0. \end{aligned} \tag{3.5}$$

Solving (3.5) gives the first correction term as:

$$(u_c)_0 = t\left[\frac{3\text{sech}^2\left(\frac{1+x}{2\sqrt{2}}\right)\tanh\left(\frac{1+x}{2\sqrt{2}}\right)}{2\sqrt{2}} + \frac{9\text{sech}^4\left(\frac{1+x}{2\sqrt{2}}\right)\tanh\left(\frac{1+x}{2\sqrt{2}}\right)}{2\sqrt{2}}\right]. \tag{3.6}$$

PIM Solutions:

After obtaining $(u_c)_0$, the first order approximate solution becomes:

$$\begin{aligned}(u_1)_{PIM} &= u_0 + (u_c)_0 \\ &= \tfrac{3}{2}\text{sech}^2\left(\tfrac{x+1}{2\sqrt{2}}\right) \\ &\quad + t\left[\frac{3\text{sech}^2\left(\frac{1+x}{2\sqrt{2}}\right)\tanh\left(\frac{1+x}{2\sqrt{2}}\right)}{2\sqrt{2}} + \frac{9\text{sech}^4\left(\frac{1+x}{2\sqrt{2}}\right)\tanh\left(\frac{1+x}{2\sqrt{2}}\right)}{2\sqrt{2}}\right].\end{aligned} \tag{3.7}$$

Using the algorithm (2.5) and u_1, second order approximate solution can be calculated as:

$$\begin{aligned}(u_2)_{PIM} &= u_1 + (u_c)_1 \\ &= \tfrac{3}{2}\text{sech}^2\left(\tfrac{1+x}{2\sqrt{2}}\right) - \tfrac{3}{16}t^2\text{sech}^4\left(\tfrac{1+x}{2\sqrt{2}}\right) - \tfrac{9}{8}t^2\text{sech}^6\left(\tfrac{1+x}{2\sqrt{2}}\right) \\ &\quad - \tfrac{27}{16}t^2\text{sech}^8\left(\tfrac{1+x}{2\sqrt{2}}\right) - \frac{63t\tanh\left(\frac{x+1}{2\sqrt{2}}\right)\text{sech}^6\left(\frac{x+1}{2\sqrt{2}}\right)}{8\sqrt{2}} \\ &\quad - \frac{3t\tanh\left(\frac{x+1}{2\sqrt{2}}\right)\text{sech}^4\left(\frac{x+1}{2\sqrt{2}}\right)}{2\sqrt{2}} - \frac{3t^3\tanh\left(\frac{x+1}{2\sqrt{2}}\right)\text{sech}^6\left(\frac{x+1}{2\sqrt{2}}\right)}{8\sqrt{2}} \\ &\quad - \frac{9t^3\tanh\left(\frac{x+1}{2\sqrt{2}}\right)\text{sech}^8\left(\frac{x+1}{2\sqrt{2}}\right)}{4\sqrt{2}} - \frac{27t^3\tanh\left(\frac{x+1}{2\sqrt{2}}\right)\text{sech}^{10}\left(\frac{x+1}{2\sqrt{2}}\right)}{8\sqrt{2}} \\ &\quad + \tfrac{3}{8}t^2\tanh^2\left(\tfrac{x+1}{2\sqrt{2}}\right)\text{sech}^2\left(\tfrac{x+1}{2\sqrt{2}}\right) \\ &\quad + \tfrac{81}{8}t^2\tanh^2\left(\tfrac{x+1}{2\sqrt{2}}\right)\text{sech}^6\left(\tfrac{x+1}{2\sqrt{2}}\right) + \cdots .\end{aligned} \tag{3.8}$$

OPIM Solutions:

With the help of Eqs. (2.5), (2.7), (3.6), first and second order approximate solutions can be obtained as:

$$\begin{aligned}&(u_1)_{OPIM} \\ &= u_0 + P_0(u_c)_0 \\ &= \tfrac{3}{2}\text{sech}^2\left(\tfrac{x+1}{2\sqrt{2}}\right) \\ &\quad + P_0 t\left[\frac{3\text{sech}^2\left(\frac{1+x}{2\sqrt{2}}\right)\tanh\left(\frac{1+x}{2\sqrt{2}}\right)}{2\sqrt{2}} + \frac{9\text{sech}^4\left(\frac{1+x}{2\sqrt{2}}\right)\tanh\left(\frac{1+x}{2\sqrt{2}}\right)}{2\sqrt{2}}\right],\end{aligned} \tag{3.9}$$

(3.10)
$$
\begin{aligned}
(u_2)_{OPIM} &\\
= u_1 + P_1 &\left[\begin{array}{l}
-\frac{1}{16}27P_0t^2\mathrm{sech}^8\left(\frac{x+1}{2\sqrt{2}}\right)-\frac{9}{8}P_0t^2\mathrm{sech}^6\left(\frac{x+1}{2\sqrt{2}}\right)-\frac{3}{16}P_0t^2\mathrm{sech}^4\left(\frac{x+1}{2\sqrt{2}}\right)\\
+\frac{9t\tanh\left(\frac{x+1}{2\sqrt{2}}\right)\mathrm{sech}^4\left(\frac{x+1}{2\sqrt{2}}\right)}{2\sqrt{2}}+\frac{3t\tanh\left(\frac{x+1}{2\sqrt{2}}\right)\mathrm{sech}^2\left(\frac{x+1}{2\sqrt{2}}\right)}{2\sqrt{2}}\\
-3\sqrt{2}P_0t\tanh\left(\frac{x+1}{2\sqrt{2}}\right)\mathrm{sech}^4\left(\frac{x+1}{2\sqrt{2}}\right)-\frac{63P_0t\tanh\left(\frac{x+1}{2\sqrt{2}}\right)\mathrm{sech}^6\left(\frac{x+1}{2\sqrt{2}}\right)}{8\sqrt{2}}\\
-\frac{27P_0^2t^3\tanh\left(\frac{x+1}{2\sqrt{2}}\right)\mathrm{sech}^{10}\left(\frac{x+1}{2\sqrt{2}}\right)}{8\sqrt{2}}+\frac{3}{8}P_0t^2\tanh^2\left(\frac{x+1}{2\sqrt{2}}\right)\mathrm{sech}^2\left(\frac{x+1}{2\sqrt{2}}\right)\\
+\frac{3P_0^2t^3\tanh^3\left(\frac{x+1}{2\sqrt{2}}\right)\mathrm{sech}^4\left(\frac{x+1}{2\sqrt{2}}\right)}{4\sqrt{2}}+\frac{9P_0t\tanh^3\left(\frac{x+1}{2\sqrt{2}}\right)\mathrm{sech}^4\left(\frac{x+1}{2\sqrt{2}}\right)}{\sqrt{2}}+\cdots
\end{array}\right].
\end{aligned}
$$

In the light of the information in Section 2, the unknown constants can be optimally obtained. P_0 can be found by constructing the residual

(3.11)
$$Re(x,t;P_0) = (u_1)_t + (u_1)_x + \left((u_1)^2\right)_x - (u_1)_{xxt}$$

for the first order OPIM solution. Using collocation method with $x_0 = 25$, $t_0 = 0.25$, Eq. (2.11) becomes

(3.12)
$$
\begin{aligned}
&Re(25, 0.25; P_0)\\
&= -4.39773833\times10^{-8} + 1.4214507\times10^{-8}P_0 - 1.70943984\times10^{-16}P_0^2 = 0.
\end{aligned}
$$

Solving (3.12) gives $P_0 = 3.09384$ and $P_0 = 8.3153\times10^7$. Substituting $P_0 = 3.09384$ into (3.9) yields

(3.13)
$$
\begin{aligned}
(u_1)_{OPIM} = \mathrm{sech}^2\left(\frac{x+1}{2\sqrt{2}}\right)&\Bigg[1.5 + 3.28151t\tanh\left(\frac{x+1}{2\sqrt{2}}\right)\\
&+78.7563t\sinh^4\left(\frac{x+1}{2\sqrt{2}}\right)\mathrm{csch}^3\left(\frac{x+1}{\sqrt{2}}\right)\Bigg].
\end{aligned}
$$

In a similar way, one can get the second order OPIM solution as:

(3.14)
$$
\begin{aligned}
&(u_2)_{OPIM}\\
&= -0.00180923\mathrm{sech}^{11}\left(\frac{x+1}{2\sqrt{2}}\right)\times\\
&\left[\begin{array}{l}
12580.3t^3\sinh\left(\frac{x+1}{2\sqrt{2}}\right)-1918.02t^3\sinh\left(\frac{3(x+1)}{2\sqrt{2}}\right)-344.261t^3\sinh\left(\frac{5(x+1)}{2\sqrt{2}}\right)\\
-9.83602t^3\sinh\left(\frac{7(x+1)}{2\sqrt{2}}\right)+1044.t^2\cosh\left(\frac{5(x+1)}{2\sqrt{2}}\right)+102.t^2\cosh\left(\frac{7(x+1)}{2\sqrt{2}}\right)\\
+\left(-5064.t^2-408.064\right)\cosh\left(\frac{x+1}{2\sqrt{2}}\right)+\left(-180.t^2-272.043\right)\cosh\left(\frac{3(x+1)}{2\sqrt{2}}\right)\\
-3442.57t\sinh\left(\frac{3(x+1)}{2\sqrt{2}}\right)-1266.36t\sinh\left(\frac{5(x+1)}{2\sqrt{2}}\right)-71.6573t\sinh\left(\frac{7(x+1)}{2\sqrt{2}}\right)\\
-116.59\cosh\left(\frac{5(x+1)}{2\sqrt{2}}\right)-29.1474\cosh\left(\frac{7(x+1)}{2\sqrt{2}}\right)-3.2386\cosh\left(\frac{9(x+1)}{2\sqrt{2}}\right)\\
-7.34419t\sinh\left(\frac{9(x+1)}{2\sqrt{2}}\right)-2240.53t\sinh\left(\frac{x+1}{2\sqrt{2}}\right)+2.t^2\cosh\left(\frac{9(x+1)}{2\sqrt{2}}\right)
\end{array}\right].
\end{aligned}
$$

Table 1. Absolute errors of the first and second order PIM and OPIM approximate solutions at $x = 25$ for Example 1.

t	PIM-1st	PIM-2nd	OPIM-1st	OPIM-2nd
0.1	5.05×10^{-9}	2.6957×10^{-9}	4.1581×10^{-9}	4.2717×10^{-10}
0.2	1.1535×10^{-8}	6.5156×10^{-9}	6.881×10^{-9}	6.3401×10^{-9}
0.3	1.9673×10^{-8}	1.1677×10^{-8}	7.9506×10^{-9}	5.9871×10^{-10}
0.4	2.9716×10^{-8}	1.8433×10^{-8}	7.1159×10^{-9}	2.9616×10^{-9}
0.5	4.1953×10^{-8}	2.7072×10^{-8}	4.0875×10^{-9}	3.0257×10^{-10}
0.6	5.6717×10^{-8}	3.7926×10^{-8}	1.4678×10^{-9}	1.2308×10^{-9}
0.7	7.4391×10^{-8}	5.138×10^{-8}	9.934×10^{-9}	2.5269×10^{-10}
0.8	9.5418×10^{-8}	6.7876×10^{-8}	2.1753×10^{-8}	4.2352×10^{-10}
0.9	1.203×10^{-7}	8.7924×10^{-8}	3.7434×10^{-8}	6.4065×10^{-9}
1.	1.4964×10^{-7}	1.121×10^{-7}	5.7565×10^{-8}	9.0995×10^{-9}

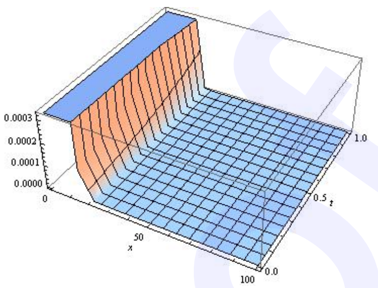


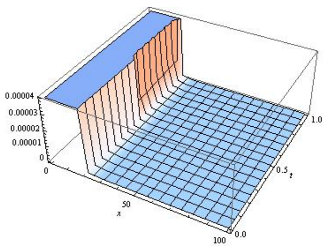
Figure 3.1. Exact solution for Example 1.

By following the similar procedure, higher order approximate solutions can be reached. Due to huge amount of calculations, one has to use a symbolic computer program such as Mathematica, Maple, Matlab etc. A comparison of the PIM solutions with the OPIM solutions and absolute errors are given in Table 1 and Table 2. It is clear from these tables that both approximate solutions are found to be in good agreement with analytical solution. In addition to that, OPIM gives slightly better results than PIM. PIM, OPIM and exact solutions are also sketched in Figures (3.1), (3.2) and (3.3).

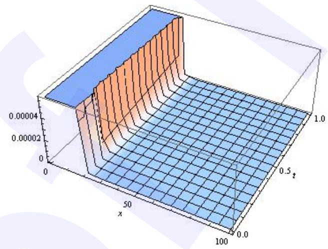
This problem has been also considered by several authors with different techniques such as ADM and VIM [21, 35]. The new approximate solutions indicate that OPIM provides more accurate solution than those in [9, 21, 35], especially for big amplitudes and small values of time. One can also increase the accuracy of OPIM results by taking more terms from Eqs. (2.7). However, a clear conclusion can be drawn from the obtained results that the OPIM provides highly accurate approximate solutions for GRLW equations even in the first and second iterations.

Table 2. Absolute errors of the first and second order PIM and OPIM approximate solutions at $x = 100$ for Example 1.

t	PIM-1st	PIM-2nd	OPIM-1st	OPIM-2nd
0.1	4.6919×10^{-32}	2.5045×10^{-32}	3.8632×10^{-32}	3.9688×10^{-33}
0.2	1.0717×10^{-31}	6.0536×10^{-32}	6.393×10^{-32}	5.8905×10^{-33}
0.3	1.8278×10^{-31}	1.0849×10^{-31}	7.3868×10^{-32}	5.5626×10^{-33}
0.4	2.7609×10^{-31}	1.7126×10^{-31}	6.6113×10^{-32}	2.7515×10^{-32}
0.5	3.8978×10^{-31}	2.5152×10^{-31}	3.7976×10^{-32}	2.8112×10^{-32}
0.6	5.2695×10^{-31}	3.5236×10^{-31}	1.3637×10^{-32}	1.1435×10^{-32}
0.7	6.9116×10^{-31}	4.7737×10^{-31}	9.2295×10^{-32}	2.3477×10^{-32}
0.8	8.8652×10^{-31}	6.3063×10^{-31}	2.021×10^{-31}	3.9349×10^{-31}
0.9	1.1177×10^{-30}	8.1689×10^{-31}	3.478×10^{-31}	5.9522×10^{-32}
1.	1.3903×10^{-30}	1.0416×10^{-30}	5.3483×10^{-31}	8.4543×10^{-32}

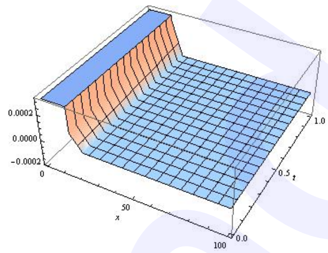


(A) First order PIM solution

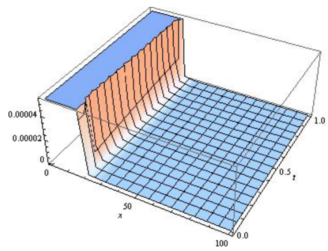


(B) First order OPIM solution

Figure 3.2. First order approximate solutions for Example 1; $0 \leq x \leq 100$, $0 \leq t \leq 1$.



(A) Second order PIM solution



(B) Second order OPIM solution.

Figure 3.3. Second order approximate solutions for Example 1; $0 \leq x \leq 100$, $0 \leq t \leq 1$.

Example 3.2. Consider the GRLW equation (1.1) with $p = 8, \alpha = \beta = 1$ as follows:

$$u_t + u_x + \left(u^8\right)_x - u_{xxt} = 0 \tag{3.15}$$

with the initial condition

$$u(x,0) = \sqrt[7]{18}\,\mathrm{sech}^{\frac{2}{7}}\left(\frac{7(x+1)}{\sqrt{5}}\right). \tag{3.16}$$

The exact solution of this problem is given by [9, 21]

$$u(x,t) = \sqrt[7]{18}\,\mathrm{sech}^{\frac{2}{7}}\left(\frac{7(x+1-5t)}{\sqrt{5}}\right). \tag{3.17}$$

PIM Solutions:

In order to start the iterations,(3.16) can be taken as a trial function u_0. By repeating similar processes as in the previous example, one can calculate the following approximate solutions:

$$(u_1)_{PIM} = u_0 + t\left[\begin{array}{c} \dfrac{2\sqrt[7]{2}3^{2/7}\sinh\left(\frac{7(x+1)}{\sqrt{5}}\right)\mathrm{sech}^{\frac{9}{7}}\left(\frac{7(x+1)}{\sqrt{5}}\right)}{\sqrt{5}} \\ +\dfrac{288\sqrt[7]{2}3^{2/7}\sinh\left(\frac{7(x+1)}{\sqrt{5}}\right)\mathrm{sech}^{\frac{23}{7}}\left(\frac{7(x+1)}{\sqrt{5}}\right)}{\sqrt{5}} \end{array}\right], \tag{3.18}$$

(3.19)

$$\begin{aligned}(u_2)_{PIM} = u_1 &- \tfrac{7}{5}\sqrt[7]{2}3^{2/7}t^2\mathrm{sech}^{\frac{2}{7}}\left(\tfrac{7(x+1)}{\sqrt{5}}\right) - \tfrac{1008}{5}\sqrt[7]{2}3^{2/7}t^2\mathrm{sech}^{\frac{16}{7}}\left(\tfrac{7(x+1)}{\sqrt{5}}\right) \\ &+ \left[\begin{array}{l} -12694\cosh\left(\frac{28(x+1)}{\sqrt{5}}\right) + 4313\cosh\left(\frac{42(x+1)}{\sqrt{5}}\right) + 4\cosh\left(\frac{56(x+1)}{\sqrt{5}}\right) \\ -28850\cosh\left(\frac{28(x+1)}{\sqrt{5}}\right) + 8331\cosh\left(\frac{42(x+1)}{\sqrt{5}}\right) + 4\cosh\left(\frac{56(x+1)}{\sqrt{5}}\right) + \cdots \\ -4316\sqrt[7]{2}3^{2/7}\sqrt{5}\cosh\left(\frac{28(x+1)}{\sqrt{5}}\right)\mathrm{sech}^{\frac{2}{7}}\left(\frac{7(x+1)}{\sqrt{5}}\right) \end{array}\right]\end{aligned}$$

and so on.

OPIM Solutions:

By the help of Eqs. (2.5), (2.7), (3.16), we get

$$(u_1)_{OPIM} = u_0 + tP_0\left[\begin{array}{c} \dfrac{2\sqrt[7]{2}3^{2/7}\sinh\left(\frac{7(x+1)}{\sqrt{5}}\right)\mathrm{sech}^{\frac{9}{7}}\left(\frac{7(x+1)}{\sqrt{5}}\right)}{\sqrt{5}} \\ +\dfrac{288\sqrt[7]{2}3^{2/7}\sinh\left(\frac{7(x+1)}{\sqrt{5}}\right)\mathrm{sech}^{\frac{23}{7}}\left(\frac{7(x+1)}{\sqrt{5}}\right)}{\sqrt{5}} \end{array}\right], \tag{3.20}$$

Table 3. Absolute errors of the first and second order PIM and OPIM approximate solutions at $x = 30$ for Example 2.

t	PIM-1st	PIM-2nd	OPIM-1st	OPIM-2nd
0.1	7.9394×10^{-13}	6.6752×10^{-13}	4.1004×10^{-13}	4.6839×10^{-15}
0.2	2.12×10^{-12}	1.8538×10^{-12}	2.8795×10^{-13}	3.2893×10^{-15}
0.3	4.2783×10^{-12}	3.8589×10^{-12}	6.6637×10^{-13}	7.6119×10^{-15}
0.4	7.7382×10^{-12}	7.1522×10^{-12}	2.9222×10^{-12}	3.338×10^{-14}
0.5	1.3233×10^{-11}	1.2467×10^{-11}	7.2138×10^{-12}	8.2402×10^{-14}
0.6	2.1912×10^{-11}	2.0953×10^{-11}	1.4688×10^{-11}	1.6778×10^{-13}
0.7	3.557×10^{-11}	3.4404×10^{-11}	2.7142×10^{-11}	3.1005×10^{-13}
0.8	5.7015×10^{-11}	5.5629×10^{-11}	4.7383×10^{-11}	5.4126×10^{-13}
0.9	9.0638×10^{-11}	8.9019×10^{-11}	7.9802×10^{-11}	9.1158×10^{-13}
1.	1.433×10^{-10}	1.4144×10^{-10}	1.3126×10^{-10}	1.4994×10^{-12}

(3.21)

$$(u_2)_{OPIM} = u_1 + P_2 \left[\begin{array}{l} \frac{2\sqrt[7]{2}3^{2/7} t \sinh\left(\frac{7(x+1)}{\sqrt{5}}\right) \operatorname{sech}^{\frac{9}{7}}\left(\frac{7(x+1)}{\sqrt{5}}\right)}{\sqrt{5}} - \frac{7}{5}\sqrt[7]{2}3^{2/7} P_0 t^2 \operatorname{sech}^{\frac{2}{7}}\left(\frac{7(x+1)}{\sqrt{5}}\right) \\ - \frac{58\sqrt[7]{2}3^{2/7} P_0 t \sinh\left(\frac{7(x+1)}{\sqrt{5}}\right) \operatorname{sech}^{\frac{9}{7}}\left(\frac{7(x+1)}{\sqrt{5}}\right)}{\sqrt{5}} \\ + \frac{3312}{5}\sqrt[7]{2}3^{2/7} P_0 t^2 \sinh^2\left(\frac{7(x+1)}{\sqrt{5}}\right) \operatorname{sech}^{\frac{30}{7}}\left(\frac{7(x+1)}{\sqrt{5}}\right) + \cdots \\ + \frac{9}{5}\sqrt[7]{2}3^{2/7} P_0 t^2 \sinh^2\left(\frac{7(x+1)}{\sqrt{5}}\right) \operatorname{sech}^{\frac{16}{7}}\left(\frac{7(x+1)}{\sqrt{5}}\right) \\ + \frac{288\sqrt[7]{2}3^{2/7} P_0 t \sinh^3\left(\frac{7(x+1)}{\sqrt{5}}\right) \operatorname{sech}^{\frac{23}{7}}\left(\frac{7(x+1)}{\sqrt{5}}\right)}{5\sqrt{5}} \end{array} \right].$$

For the parameters P_0, P_1, the method given in Section 2 can be used. By using collocation method, we reach the values $P_0 = 9.04508$ and $P_0 = 1.71562, P_1 = -0.61424$ for the $(u_1)_{OPIM}$ and $(u_2)_{OPIM}$, respectively. Substituting these values into Eqs. (3.20) and Eq. (3.21) yields the first and second order OPIM approximate solutions for Example 2. The absolute errors with the exact solution is given in tables (3) and (4) for some constant amplitude. Figures (3.4), (3.5) and (3.6) display the exact solution and the approximate solutions obtained via PIM and OPIM.

It should be noted that as the number of iterations increase, the approximate solution becomes more intricate and the use of the symbolic computer program becomes essential. Mathematica 9.0 is used to cope with the complex computations for illustrations in this paper.

4. Conclusions

In this paper, a new efficient technique, namely optimal perturbation iteration method is introduced for solving nonlinear partial differential equations. Firstly, the classical perturbation iteration technique is modified for handling

Table 4. Absolute errors of the first and second order PIM and OPIM approximate solutions at $x = 80$ for Example 2.

t	PIM-1st	PIM-2nd	OPIM-1st	OPIM-2nd
0.1	3.0029×10^{-32}	2.5248×10^{-32}	1.5509×10^{-32}	1.7716×10^{-34}
0.2	8.0186×10^{-32}	7.0117×10^{-32}	1.0891×10^{-32}	1.2441×10^{-34}
0.3	1.6182×10^{-31}	1.4595×10^{-31}	2.5204×10^{-32}	2.879×10^{-34}
0.4	2.9268×10^{-31}	2.7052×10^{-31}	1.1053×10^{-31}	1.2625×10^{-33}
0.5	5.0054×10^{-31}	4.7157×10^{-31}	2.7285×10^{-31}	3.1167×10^{-33}
0.6	8.2881×10^{-31}	7.9253×10^{-31}	5.5558×10^{-31}	6.3463×10^{-33}
0.7	1.3454×10^{-30}	1.3013×10^{-30}	1.0266×10^{-30}	1.1727×10^{-32}
0.8	2.1565×10^{-30}	2.1041×10^{-30}	1.7922×10^{-30}	2.0472×10^{-32}
0.9	3.4282×10^{-30}	3.367×10^{-30}	3.0184×10^{-30}	3.4479×10^{-32}
1.	5.4203×10^{-30}	5.3497×10^{-30}	4.965×10^{-30}	5.6714×10^{-32}

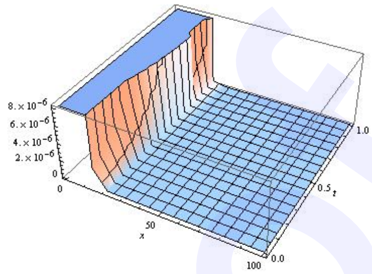
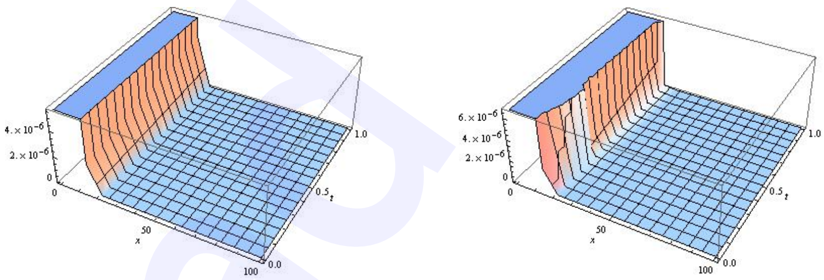


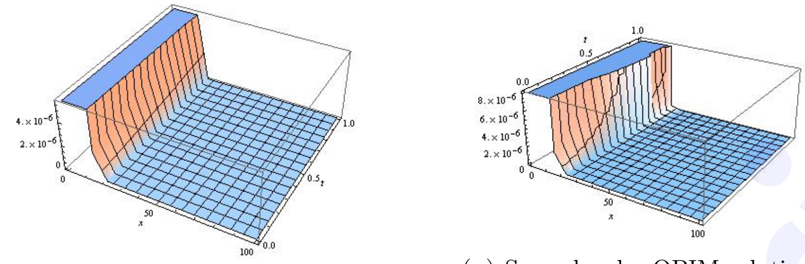
Figure 3.4. Exact solution for Example 2.



(A) First order PIM solution (B) First order OPIM solution

Figure 3.5. First order approximate solutions for Example 2; $0 \leq x \leq 100$, $0 \leq t \leq 1$.

NPDEs. Then, OPIM is generated by inserting new convergence parameters into algorithms of PIM. PIM and OPIM have been successfully implemented



(A) Second order PIM solution

(B) Second order OPIM solution.

Figure 3.6. Second order approximate solutions for Example 2; $0 \leq x \leq 100$, $0 \leq t \leq 1$.

to find the solution of the generalized regularized long wave equation. Illustrations show that OPIM is effective mathematical tool for solving these types of equations. In this method, it is important to get unknown parameters $P_0, P_1, \ldots$, and this makes it time consuming, especially for large n. In our cases, this method converges rapidly at lower order of approximations. On the other hand, since the method is often tedious to use by hand, one has to use a symbolic computer program to obtain approximate solutions. In this study, Mathematica 9.0 has been used to perform the complex calculations in applications. Finally, we can say that the proposed technique is an applicable alternative to existing numerical methods.

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