

A NOTE OF LITTLEWOOD-PALEY FUNCTIONS ON TRIEBEL-LIZORKIN SPACES

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ABSTRACT. In this note we prove that several classes of Littlewood-Paley square operators defined by the kernels without any regularity are bounded on Triebel-Lizorkin spaces $F_{\alpha}^{p,q}(\mathbb{R}^n)$ and Besov spaces $B_{\alpha}^{p,q}(\mathbb{R}^n)$ for $0 < \alpha < 1$ and $1 < p, q < \infty$.

1. Introduction

It is well known that the theory of the Littlewood-Paley functions has been an important part of harmonic analysis. One can consult [13–15] for its history and significance. The L^p mapping properties for these operators have also been studied extensively by many authors (see [3–8, 11, 12, 19] for example). In this note we shall prove the boundedness of the Littlewood-Paley square functions on Triebel-Lizorkin spaces and Besov spaces. Let $\psi \in L^1(\mathbb{R}^n)$ and satisfy

$$\int_{\mathbb{R}^n} \psi(x) dx = 0.$$

We consider a square function of Littlewood-Paley type

$$g_{\psi}(f)(x) = \left(\int_0^{\infty} |\psi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where $\psi_t(x) = t^{-n}\psi(t^{-1}x)$.

A well-known result for g_{ψ} proved by Benedek, Calderón and Panzone [2] is the following:

Theorem A. *Suppose that ψ satisfies*

$$|\psi(x)| \leq C(1 + |x|)^{-n-\epsilon} \text{ for some } \epsilon > 0,$$

Received March 10, 2017; Accepted November 3, 2017.

2010 *Mathematics Subject Classification.* 42B20, 42B25.

Key words and phrases. Littlewood-Paley function, Triebel-Lizorkin spaces, Besov spaces, extrapolation.

Supported by the NNSF of China (Grant No. 11701333), Scientific Research Foundation of Shandong University of Science and Technology for Recruited Talents (Grant No. 2015RCJJ053) and Support Program for Outstanding Young Scientific and Technological Top-notch Talents of College of Mathematics and Systems Science (Grant No. Sxy2016K01).

$$\int_{\mathbb{R}^n} |\psi(x-y) - \psi(x)| dx \leq C|y|^\epsilon \quad \text{for some } \epsilon > 0.$$

Then g_ψ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$.

Later on, Fan and Sato [8] relaxed the conditions imposed on ψ in Theorem A and proved the following result.

Theorem B. *Suppose that the function ψ satisfies the following conditions:*

- (i) $\int_{|x| \geq 1} |\psi(x)| |x|^\epsilon dx < \infty$ for some $\epsilon > 0$;
- (ii) $(\int_{|x| < 1} |\psi(x)|^u dx)^{1/u} < \infty$ for some $u > 1$;
- (iii) $|\psi(x)| \leq h(x)\Omega(x')$ for all $x \in \mathbb{R}^n \setminus \{0\}$, where $x' = x/|x|$, for some non-negative function h on $(0, \infty)$ and Ω on S^{n-1} (the unit sphere in \mathbb{R}^n) such that

- (a) $h(r)$ is non-increasing on $(0, \infty)$ and $h(|x|) \in L^1(\mathbb{R}^n)$,
- (b) $\Omega \in L^s(S^{n-1})$ for some $1 < s \leq \infty$.

Then g_ψ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$.

Recently, Sato [12] used a minimum condition on ψ to obtain the following result.

Theorem C. *Suppose that $|\psi(x)| \leq h(x)\Omega(x')$ for all $x \in \mathbb{R}^n \setminus \{0\}$, where h is a non-negative, non-increasing function on $(0, \infty)$ with supported in $(0, 1]$ and Ω is a non-negative function on S^{n-1} . We assume that $h(|x|) \in L^1(\mathbb{R}^n)$, $\Omega \in L^1(S^{n-1})$ and $\psi \in L^s(\mathbb{R}^n)$ for some $1 < s \leq \infty$. Put $m_\psi(x) = h(|x|)\Omega(x')$. Then*

$$\|g_\psi(f)\|_{L^p(\mathbb{R}^n)} \leq C_p (s/(s-1))^{1/2} (\|\psi\|_{L^q(\mathbb{R}^n)} + \|m_\psi\|_{L^1(\mathbb{R}^n)}) \|f\|_{L^p(\mathbb{R}^n)}$$

for all $1 < p < \infty$, where the constant $C_p > 0$ is independent of s, ψ, h, Ω .

By extrapolation, Theorem C yields a more general result.

Theorem D. *Suppose that $|\psi(x)| \leq h(|x|)\Omega(x')$ for all $x \in \mathbb{R}^n \setminus \{0\}$, where h is a non-negative, non-increasing function on $(0, \infty)$ with supported in $(0, 1]$ and Ω is a non-negative function on S^{n-1} . We further assume that $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$ and $h(|x|) \in L^s(\mathbb{R}^n)$ for some $1 < s \leq \infty$. Then g_ψ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$.*

Here $L(\log^+ L)^\beta(S^{n-1})$ for $\beta > 0$ denotes the class of all functions Ω on S^{n-1} satisfying

$$\|\Omega\|_{L(\log^+ L)^\alpha(S^{n-1})} := \int_{S^{n-1}} |\Omega(\theta)| (\log(2 + |\Omega(\theta)|))^\alpha d\sigma(\theta) < \infty,$$

where $d\sigma$ denotes the Lebesgue surface measure on S^{n-1} . Clearly, for any $\alpha > \beta > 0$ and $1 < q \leq \infty$,

$$L^q(S^{n-1}) \subsetneq L(\log^+ L)^\alpha(S^{n-1}) \subsetneq L(\log^+ L)^\beta(S^{n-1}) \subsetneq L^1(S^{n-1}).$$

In this paper we focus on the boundedness of the Littlewood-Paley function g_ψ on Triebel-Lizorkin spaces. It is well known that the Triebel-Lizorkin spaces

and Besov spaces contain many important function spaces, such as Lebesgue spaces, Hardy spaces, Sobolev spaces and Lipschitz spaces. Let us recall some definitions. For $\alpha \in \mathbb{R}$ and $0 < p, q \leq \infty$ ($p \neq \infty$), we define the homogeneous Triebel-Lizorkin spaces $\dot{F}_\alpha^{p,q}(\mathbb{R}^n)$ and homogeneous Besov spaces $\dot{B}_\alpha^{p,q}(\mathbb{R}^n)$ by (1.1)

$$\dot{F}_\alpha^{p,q}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} = \left\| \left(\sum_{i \in \mathbb{Z}} 2^{-i\alpha q} |\Psi_i * f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty \right\};$$

(1.2)

$$\dot{B}_\alpha^{p,q}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^n)} = \left(\sum_{i \in \mathbb{Z}} 2^{-i\alpha q} \|\Psi_i * f\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} < \infty \right\},$$

where $\mathcal{S}'(\mathbb{R}^n)$ is the tempered distribution class on \mathbb{R}^n , $\widehat{\Psi}_i(\xi) = \phi(2^i \xi)$ for $i \in \mathbb{Z}$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ satisfies the conditions:

- (i) $0 \leq \phi(x) \leq 1$;
- (ii) $\text{supp}(\phi) \subset \{x \in \mathbb{R}^n : 1/2 \leq |x| \leq 2\}$;
- (iii) $\phi(x) > c > 0$ if $3/5 \leq |x| \leq 5/3$.

The inhomogeneous versions of Triebel-Lizorkin spaces and Besov spaces, which are denoted by $F_\alpha^{p,q}(\mathbb{R}^n)$ and $B_\alpha^{p,q}(\mathbb{R}^n)$, respectively, are obtained by adding the term $\|\Theta * f\|_{L^p(\mathbb{R}^n)}$ to the right hand side of (1.1) or (1.2) with $\sum_{i \in \mathbb{Z}}$ replaced by $\sum_{i \geq 1}$, where $\Theta \in \mathcal{S}(\mathbb{R}^n)$ (the Schwartz class), $\text{supp}(\hat{\Theta}) \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2\}$, $\hat{\Theta}(x) > c > 0$ if $|x| \leq 5/3$. The following properties are well known (see [9, 16] for example):

$$(1.3) \quad \dot{F}_0^{p,2}(\mathbb{R}^n) = L^p(\mathbb{R}^n) \text{ for } 1 < p < \infty;$$

$$(1.4) \quad \dot{F}_\alpha^{p,p}(\mathbb{R}^n) = \dot{B}_\alpha^{p,p}(\mathbb{R}^n) \text{ for } \alpha \in \mathbb{R} \text{ and } 1 < p < \infty;$$

$$(1.5) \quad F_\alpha^{p,q}(\mathbb{R}^n) \sim \dot{F}_\alpha^{p,q}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \text{ and} \\ \|f\|_{F_\alpha^{p,q}(\mathbb{R}^n)} \sim \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)} \quad (\alpha > 0);$$

$$(1.6) \quad B_\alpha^{p,q}(\mathbb{R}^n) \sim \dot{B}_\alpha^{p,q}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \text{ and} \\ \|f\|_{B_\alpha^{p,q}(\mathbb{R}^n)} \sim \|f\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)} \quad (\alpha > 0).$$

In 2009, Zhang and Chen [20] proved the following result.

Theorem E. *Let $\Omega \in H^1(S^{n-1})$ satisfying $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$ and $\psi(x) = h(|x|)\Omega(x')$. Suppose that there exist $\epsilon, \gamma, C > 0$ such that $|h(t)| \leq Ct^{-n+\epsilon}(1+t)^{-2\epsilon}$ and*

$$\int_{\mathbb{R}} |(t+r)^{n-1}h(t+r) - t^{n-1}h(t)| dt \leq C|r|^\gamma.$$

Then g_ψ is bounded on $F_\alpha^{p,q}(\mathbb{R}^n)$ for all $0 < \alpha < 1$ and $1 < p, q < \infty$.

Here $H^1(S^{n-1})$ denotes the Hardy space on S^{n-1} , which is the set of all $L^1(S^{n-1})$ functions Ω satisfying

$$\|\Omega\|_{H^1(S^{n-1})} := \int_{S^{n-1}} \sup_{0 \leq r < 1} \left| \int_{S^{n-1}} \Omega(\theta) \frac{1-r^2}{|rw-\theta|^n} d\sigma(\theta) \right| d\sigma(w) < \infty.$$

It was known that

$$H^1(S^{n-1}) \not\subseteq L(\log^+ L)^\beta(S^{n-1}) \not\subseteq H^1(S^{n-1}) \text{ for any } 0 < \beta < 1.$$

It follows from Theorems B-D and (1.3) that the operator g_ψ is bounded on $\dot{F}_0^{p,2}(\mathbb{R}^n)$ for all $1 < p < \infty$ under the same assumptions on ψ as in one of Theorems B-D. A natural question is the following:

Question F. *Is the operator g_ψ bounded on $\dot{F}_\alpha^{p,q}(\mathbb{R}^n)$ for some $\alpha \neq 0$ and $q \neq 2$ under the same assumptions on ψ as in one of Theorems B-D?*

Question F is the main motivation for this work. This problem will be addressed by our main result.

Theorem 1.1. (i) *Suppose that ψ satisfies the condition of Theorem B. Then*

$$\|g_\psi(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} \leq C \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)}$$

for all $0 < \alpha < 1$ and $1 < p, q < \infty$, where the constant $C > 0$ depends on s, ψ, h, Ω .

(ii) *Suppose that ψ satisfies the condition of Theorem C. Then*

$$\|g_\psi(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} \leq C(s/(s-1))^{1/2} (\|\psi\|_{L^s(\mathbb{R}^n)} + \|m_\psi\|_{L^1(\mathbb{R}^n)}) \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)}$$

for all $0 < \alpha < 1$ and $1 < p, q < \infty$, where the constant $C > 0$ is independent of s, ψ, h, Ω .

Actually, Theorem 1.1 will be derived from the following more abstract one.

Theorem 1.2. *Let $A > 0$ and $v \geq 1$. Suppose that ψ satisfies the following conditions:*

(i) *there exist $\epsilon, \delta > 0$ and $C > 0$ independent of A, v such that*

$$(1.7) \quad \int_{2^{kv}}^{2^{(k+1)v}} |\hat{\psi}(t\xi)|^2 \frac{dt}{t} \leq CA^2 v \min(1, |2^{kv}\xi|^{\epsilon/v}, |2^{kv}\xi|^{-\delta/v})$$

for all $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n$;

(ii) *there exists a constant $C > 0$ independent of A, v such that*

$$(1.8) \quad \left\| \left(\sum_{l \in \mathbb{Z}} \left\| \sup_{t > 0} |\psi_t| * g_{l,\zeta} \right\|_{L^r(\mathfrak{R}_n)}^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \leq CA \left\| \left(\sum_{l \in \mathbb{Z}} \|g_{l,\zeta}\|_{L^r(\mathfrak{R}_n)}^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}$$

for all $1 < p, q, r < \infty$, where $\mathfrak{R}_n = \{\zeta \in \mathbb{R}^n; 1/2 < |\zeta| \leq 1\}$.

Then for all $0 < \alpha < 1$ and $1 < p, q < \infty$, there exists a constant $C > 0$ independent of A, v such that

$$\|g_\psi(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} \leq CA v^{1/2} \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)}.$$

Applying Theorem 1.1, Theorems B-C and (1.5), we can get the following result immediately.

Theorem 1.3. (i) *Suppose that ψ satisfies the condition of Theorem B. Then*

$$\|g_\psi(f)\|_{F_\alpha^{p,q}(\mathbb{R}^n)} \leq C \|f\|_{F_\alpha^{p,q}(\mathbb{R}^n)}$$

for all $0 < \alpha < 1$ and $1 < p, q < \infty$, where the constant C depends on s, ψ, h, Ω .

(ii) *Suppose that ψ satisfies the condition of Theorem C. Then*

$$\|g_\psi(f)\|_{F_\alpha^{p,q}(\mathbb{R}^n)} \leq C(s/(s-1))^{1/2} (\|\psi\|_{L^s(\mathbb{R}^n)} + \|m_\psi\|_{L^1(\mathbb{R}^n)}) \|f\|_{F_\alpha^{p,q}(\mathbb{R}^n)}$$

for all $0 < \alpha < 1$ and $1 < p, q < \infty$, where the constant C is independent of s, ψ, h, Ω .

By (ii) of Theorems 1.1 and 1.3 and applying extrapolation argument, we can prove the following result.

Corollary 1.1. *Let $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$ and $\Omega \geq 0$. Suppose that $|\psi(x)| \leq h(|x|)\Omega(x')$ for all $x \in \mathbb{R}^n \setminus \{0\}$, where h is a non-negative, non-increasing function on $(0, \infty)$ with supported in $(0, 1]$ and $h(|x|) \in L^2(\mathbb{R}^n)$. Then g_ψ is bounded on $\dot{F}_\alpha^{p,q}(\mathbb{R}^n)$ and $F_\alpha^{p,q}(\mathbb{R}^n)$ for all $0 < \alpha < 1$ and $1 < p, q < \infty$. Moreover,*

$$\|g_\psi f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} \leq C \|H\|_{L^2(\mathbb{R}^n)} (1 + \|\Omega\|_{L(\log^+ L)^{1/2}(S^{n-1})}) \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)}$$

for all $0 < \alpha < 1$ and $1 < p, q < \infty$, where $H(x) = h(|x|)$ and the constant C is independent of ψ, h, Ω .

In particular, when $\psi(x) = \Omega(x')|x|^{\rho-n}\chi_{\{|x|<1\}}$ with $\rho > 0$, $\Omega \in L^1(S^{n-1})$ and $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$, the Littlewood-Paley operator g_ψ reduces to the classical parametric Marcinkiewicz integral operator μ_Ω^ρ . As an application of Corollary 1.1, we have:

Corollary 1.2. *Let $\rho > 0$, $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$ and satisfy $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$. Then μ_Ω^ρ is bounded on $\dot{F}_\alpha^{p,q}(\mathbb{R}^n)$ and $F_\alpha^{p,q}(\mathbb{R}^n)$ for all $0 < \alpha < 1$ and $1 < p, q < \infty$.*

Observing that

$$(1.9) \quad |\Delta_\zeta(g_\psi(f))(x)| \leq g_\psi(\Delta_\zeta(f))(x)$$

for all $x, \zeta \in \mathbb{R}^n$, where $\Delta_\zeta(f)$ denotes the difference of f , i.e., $\Delta_\zeta(f)(x) = f(x + \zeta) - f(x)$ for all $x, \zeta \in \mathbb{R}^n$. By (1.9) and (ii) of Lemma 2.1, the L^p boundedness of g_ψ automatically implies its boundedness on $\dot{B}_\alpha^{p,q}(\mathbb{R}^n)$. This together with Theorems B-D and (1.6) yields the following result immediately.

Theorem 1.4. *Under the same conditions of Theorem 1.1 and Corollaries 1.1-1.2, the operator g_ψ is bounded on $\dot{B}_\alpha^{p,q}(\mathbb{R}^n)$ and $B_\alpha^{p,q}(\mathbb{R}^n)$ for all $0 < \alpha < 1$ and $1 < p, q < \infty$.*

The rest of this paper is organized as follows. After presenting some auxiliary lemmas following from [18], we shall prove Theorems 1.1-1.2 and Corollary 1.1 in Section 3. We would like to remark that the main method employed in this paper is a combination of ideas and arguments from [1], [11], [12], [18], among others. Compare our main results with Theorem E, our main results and proofs are greatly different from Theorem E and its proof. In [20] the proof of Theorem E relies heavily on BCP's method developed in [10] and the rotation method developed in [17], but the above methods do not work for our main results.

Throughout the paper, we denote p' by the conjugate index of p , which satisfies $1/p + 1/p' = 1$. The letter C or c , sometimes with certain parameters, will stand for positive constants not necessarily the same one at each occurrence, but are independent of the essential variables. In what follows, we set $\mathfrak{R}_n = \{\zeta \in \mathbb{R}^n; 1/2 < |\zeta| \leq 1\}$.

2. Preliminary lemmas

To prove our main results, we need some useful characterizations of Triebel-Lizorkin spaces and Besov spaces, which are followed from [18].

Lemma 2.1. (i) *Let $0 < \alpha < 1$, $1 < p < \infty$, $1 < q \leq \infty$ and $1 \leq r < \min(p, q)$. Then*

$$\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} \sim \left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left(\int_{\mathfrak{R}_n} |\Delta_{2^{-l}\zeta}(f)|^r d\zeta \right)^{q/r} \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}.$$

(ii) *Let $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $1 \leq r \leq p$. Then*

$$\|f\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^n)} \sim \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left\| \left(\int_{\mathfrak{R}_n} |\Delta_{2^{-l}\zeta}(f)|^r d\zeta \right)^{1/r} \right\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q}.$$

The result for the following vector-valued inequalities of the Hardy-Littlewood maximal functions followed from [18] will also be needed.

Lemma 2.2. *Let $y' \in S^{n-1}$ and $M_{y'}$ be the directional Hardy-Littlewood maximal function along θ defined by*

$$M_{y'}(f)(x) = \sup_{r>0} \frac{1}{2r} \int_{|t|<r} |f(x - ty')| dt.$$

Then

$$\left\| \left(\sum_{j \in \mathbb{Z}} \|M_{y'}(f_{j,\zeta})\|_{L^r(\mathfrak{R}_n)}^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| \left(\sum_{j \in \mathbb{Z}} \|f_{j,\zeta}\|_{L^r(\mathfrak{R}_n)}^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}$$

for all $1 < p, q, r < \infty$, where $C > 0$ is independent of y' .

3. Proofs of main results

This section is devoted to proving our main results. Let us begin with the proof of Theorem 1.2.

Proof of Theorem 1.2. By (1.9) and (i) of Lemma 2.1, we have

$$\begin{aligned}
& \|g_\psi(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} \\
& \leq C \left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left(\int_{\mathfrak{R}_n} |\Delta_{2^{-l}\zeta}(g_\psi(f))| d\zeta \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\
(3.1) \quad & \leq C \left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left(\int_{\mathfrak{R}_n} |g_\psi(\Delta_{2^{-l}\zeta}(f))| d\zeta \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\
& \leq C \left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left(\int_{\mathfrak{R}_n} \left(\int_0^\infty |\psi_t * \Delta_{2^{-l}\zeta}(f)|^2 \frac{dt}{t} \right)^{1/2} d\zeta \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}
\end{aligned}$$

for all $0 < \alpha < 1$ and $1 < p, q < \infty$. Therefore, to prove Theorem 1.2, it suffices to show that there exists a constant $C > 0$ independent of A and v such that

$$\begin{aligned}
(3.2) \quad & \left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left(\int_{\mathfrak{R}_n} \left(\int_0^\infty |\psi_t * \Delta_{2^{-l}\zeta} f|^2 \frac{dt}{t} \right)^{1/2} d\zeta \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\
& \leq CA v^{1/2} \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)}
\end{aligned}$$

for all $0 < \alpha < 1$ and $1 < p, q < \infty$.

Let $\eta_0 \in C^\infty(\mathbb{R})$ be an even function satisfying $0 \leq \eta_0(t) \leq 1$, $\eta_0(0) = 1$ and $\eta_0(t) = 0$ for $|t| \geq 1$. Set $\eta(\xi) = 1$ for $|\xi| \leq 1$, $\eta(\xi) = \eta_0\left(\frac{|\xi|-1}{2^v-1}\right)$, where $a > 1$. Then, η satisfies $\chi_{\{|\xi| \leq 1\}}(\xi) \leq \eta(\xi) \leq \chi_{\{|\xi| \leq 2^v\}}(\xi)$ and $|\partial^\alpha \eta(\xi)| \leq c_\alpha (2^v - 1)^{-|\alpha|}$ for $\xi \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$, where c_α is independent of v . We define the sequence of functions $\{\phi_k\}_{k \in \mathbb{Z}}$ on \mathbb{R}^n by

$$\phi_k(\xi) = \eta(2^{-(k+1)v}\xi) - \eta(2^{-kv}\xi), \quad \xi \in \mathbb{R}^n.$$

Observing that $\text{supp}(\phi_k) \subset \{2^{kv} \leq |\xi| \leq 2^{(k+2)v}\}$, $\text{supp}(\phi_k) \cap \text{supp}(\phi_j) = \emptyset$ for $|j - k| \geq 2$ and $\sum_{k \in \mathbb{Z}} \phi_k(\xi) = 1$ for $\xi \in \mathbb{R}^n \setminus \{0\}$. Define the multiplier operator Γ_k on \mathbb{R}^n by

$$\widehat{\Gamma_k f}(\xi) = \phi_k(|\xi|) \hat{f}(\xi).$$

It follows from [18, Lemma 2.5] that for $1 < p, q, r < \infty$, there exists a constant $C > 0$ independent of v such that

$$\begin{aligned}
(3.3) \quad & \left\| \left(\sum_{j \in \mathbb{Z}} \left\| \left(\sum_{k \in \mathbb{Z}} |\Gamma_k f_{j,\zeta}|^2 \right)^{1/2} \right\|_{L^r(\mathfrak{R}_n)}^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\
& \leq C \left(\frac{2^v}{2^v - 1} \right)^{n+2} \left\| \left(\sum_{j \in \mathbb{Z}} \|f_{j,\zeta}\|_{L^r(\mathfrak{R}_n)}^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

By Minkowski's inequality, it follows that for all $0 < \alpha < 1$ and $1 < p, q < \infty$,

$$\begin{aligned}
(3.4) \quad & \left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left(\int_{\mathfrak{R}_n} \left(\int_0^\infty |\psi_t * \Delta_{2^{-l}\zeta}(f)|^2 \frac{dt}{t} \right)^{1/2} d\zeta \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\
&= \left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left(\int_{\mathfrak{R}_n} \left(\sum_{k \in \mathbb{Z}} \int_{2^{kv}}^{2^{(k+1)v}} \left| \psi_t * \sum_{j \in \mathbb{Z}} \Gamma_{j-k} \Delta_{2^{-l}\zeta}(f) \right|^2 \frac{dt}{t} \right)^{1/2} d\zeta \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\
&\leq \sum_{j \in \mathbb{Z}} \left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left(\int_{\mathfrak{R}_n} \left(\sum_{k \in \mathbb{Z}} \int_{2^{kv}}^{2^{(k+1)v}} |\psi_t * \Gamma_{j-k} \Delta_{2^{-l}\zeta}(f)|^2 \frac{dt}{t} \right)^{1/2} d\zeta \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

Define the mixed norm $\|\cdot\|_{E_{p,q}^\alpha}$ for measurable functions on $\mathbb{R}^n \times \mathfrak{R}_n \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{R}_+$ by

$$\|g\|_{E_{p,q}^\alpha} := \left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left(\int_{\mathfrak{R}_n} \left(\sum_{k \in \mathbb{Z}} \int_0^\infty |g(t, x, \zeta, l, k)|^2 \frac{dt}{t} \right)^{1/2} d\zeta \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}.$$

For any $j \in \mathbb{Z}$, let

$$A_j(f)(t, x, \zeta, l, k) := \psi_t * \Gamma_{j-k} \Delta_{2^{-l}\zeta}(f)(x) \chi_{[2^{kv}, 2^{(k+1)v}]}(t).$$

Thus, we have

$$(3.5) \quad \left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left(\int_{\mathfrak{R}_n} \left(\int_0^\infty |\psi_t * \Delta_{2^{-l}\zeta}(f)|^2 \frac{dt}{t} \right)^{1/2} d\zeta \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \leq \sum_{j \in \mathbb{Z}} \|A_j(f)\|_{E_{p,q}^\alpha}$$

for all $0 < \alpha < 1$ and $1 < p, q < \infty$.

By our assumption (1.7), (ii) of Lemma 2.1, Hölder's inequality, Minkowski's inequality, Fubini's theorem and Plancherel's theorem, it yields that

$$\begin{aligned}
(3.6) \quad & \|A_j(f)\|_{E_{2,2}^\alpha}^2 \\
&= \left\| \left(\sum_{l \in \mathbb{Z}} 2^{2l\alpha} \left(\int_{\mathfrak{R}_n} \left(\sum_{k \in \mathbb{Z}} \int_{2^{kv}}^{2^{(k+1)v}} |\psi_t * \Gamma_{j-k} \Delta_{2^{-l}\zeta}(f)|^2 \frac{dt}{t} \right)^{1/2} d\zeta \right)^2 \right)^{1/2} \right\|_{L^2(\mathbb{R}^n)}^2 \\
&= \int_{\mathbb{R}^n} \sum_{l \in \mathbb{Z}} 2^{2l\alpha} \left(\int_{\mathfrak{R}_n} \left(\sum_{k \in \mathbb{Z}} \int_{2^{kv}}^{2^{(k+1)v}} |\psi_t * \Gamma_{j-k} \Delta_{2^{-l}\zeta}(f)(x)|^2 \frac{dt}{t} \right)^{1/2} d\zeta \right)^2 dx \\
&\leq C \sum_{l \in \mathbb{Z}} 2^{2l\alpha} \int_{\mathfrak{R}_n} \sum_{k \in \mathbb{Z}} \int_{2^{kv}}^{2^{(k+1)v}} \int_{\mathbb{R}^n} |\psi_t * \Gamma_{j-k} \Delta_{2^{-l}\zeta}(f)(x)|^2 dx \frac{dt}{t} d\zeta \\
&\leq C \sum_{l \in \mathbb{Z}} 2^{2l\alpha} \int_{\mathfrak{R}_n} \sum_{k \in \mathbb{Z}} \int_{E_{j-k}} \int_{2^{kv}}^{2^{(k+1)v}} |\widehat{\psi}(tx)|^2 \frac{dt}{t} |\widehat{\Delta_{2^{-l}\zeta}(f)}(x)|^2 dx d\zeta \\
&\leq CA^2 v 2^{-c|j|} \sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left\| \left(\int_{\mathfrak{R}_n} |\Delta_{2^{-l}\zeta}(f)|^2 d\zeta \right)^{1/2} \right\|_{L^2(\mathbb{R}^n)}^2 \\
&\leq CA^2 v 2^{-c|j|} \|f\|_{\dot{B}_{\alpha}^{2,2}(\mathbb{R}^n)}^2,
\end{aligned}$$

where $E_{j-k} = \{x \in \mathbb{R}^n : 2^{(j-k)v} \leq |x| \leq 2^{(j-k+2)v}\}$ and $c = \min(\epsilon, \delta)$. Here the constant $C > 0$ is independent of A and v . (3.6) together with (1.4) yields

$$(3.7) \quad \|A_j(f)\|_{E_{2,2}^\alpha} \leq CAv^{1/2}2^{-c|j|/2}\|f\|_{\dot{F}_c^{2,2}(\mathbb{R}^n)}.$$

On the other hand, by our assumption (1.8) we have

$$(3.8) \quad \begin{aligned} & \left\| \left(\sum_{l \in \mathbb{Z}} \left\| \sup_{k \in \mathbb{Z}} \sup_{t \in [1, 2^v]} |\psi_{2^{kv}t} * g_{l,\zeta,k}| \right\|_{L^r(\mathfrak{R}_n)}^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ & \leq \left\| \left(\sum_{l \in \mathbb{Z}} \left\| \sup_{t > 0} |\psi_t| * \left(\sup_{k \in \mathbb{Z}} |g_{l,\zeta,k}| \right) \right\|_{L^r(\mathfrak{R}_n)}^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ & \leq CA \left\| \left(\sum_{l \in \mathbb{Z}} \left\| \sup_{k \in \mathbb{Z}} |g_{l,\zeta,k}| \right\|_{L^r(\mathfrak{R}_n)}^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \end{aligned}$$

for any $1 < p, q, r < \infty$. Fix $1 < p, q, r < \infty$, by the duality, Fubini's theorem, Hölder's inequality and our assumption (1.8) again, there exists a sequence of functions $\{f_{l,\zeta}\}_{l,\zeta}$ such that $\|\{f_{l,\zeta}\}\|_{L^{p'}(\ell^{q'}(L^{r'}(\mathfrak{R}_n), \mathbb{R}^n))} = 1$ and

$$\begin{aligned} & \left\| \left(\sum_{l \in \mathbb{Z}} \left\| \sum_{k \in \mathbb{Z}} \int_1^{2^v} |\psi_{2^{kv}t} * g_{l,\zeta,k}| dt \right\|_{L^r(\mathfrak{R}_n)}^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ & = \left\| \left(\sum_{l \in \mathbb{Z}} \left\| \sum_{k \in \mathbb{Z}} \int_1^{2^v} |\psi_{2^{kv}t} * g_{l,\zeta,k}| dt \right\|_{L^r(\mathfrak{R}_n)}^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ & = \sum_{l \in \mathbb{Z}} \int_{\mathbb{R}^n} \int_{\mathfrak{R}_n} \sum_{k \in \mathbb{Z}} \int_1^{2^v} |\psi_{2^{kv}t} * g_{l,\zeta,k}| dt |f_{l,\zeta}(x)| d\zeta dx \\ & = \sum_{l \in \mathbb{Z}} \int_{\mathbb{R}^n} \int_{\mathfrak{R}_n} \sum_{k \in \mathbb{Z}} |g_{l,\zeta,k}| \int_1^{2^v} |\psi_{2^{kv}t}| * |f_{l,\zeta}(-x)| dt d\zeta dx \\ & \leq Cv \sum_{l \in \mathbb{Z}} \int_{\mathbb{R}^n} \int_{\mathfrak{R}_n} \sum_{k \in \mathbb{Z}} |g_{l,\zeta,k}| \sup_{t > 0} |\psi_t| * (|f_{l,\zeta}|)(-x) d\zeta dx \\ & \leq Cv \left\| \left(\sum_{l \in \mathbb{Z}} \left\| \sum_{k \in \mathbb{Z}} |g_{l,\zeta,k}| \right\|_{L^r(\mathfrak{R}_n)}^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ & \quad \left\| \left(\sum_{l \in \mathbb{Z}} \left\| \sup_{t > 0} |\psi_t| * (|f_{l,\zeta}|) \right\|_{L^{r'}(\mathfrak{R}_n)}^{q'} \right)^{1/q'} \right\|_{L^{p'}(\mathbb{R}^n)} \\ & \leq CAv \left\| \left(\sum_{l \in \mathbb{Z}} \left\| \sum_{k \in \mathbb{Z}} |g_{l,\zeta,k}| \right\|_{L^r(\mathfrak{R}_n)}^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}, \end{aligned}$$

where $\tilde{f}_{l,\zeta}(x) = f_{l,\zeta}(-x)$. This together with (3.8) yields

$$(3.9) \quad \begin{aligned} & \left\| \left(\sum_{l \in \mathbb{Z}} \left\| \left(\sum_{k \in \mathbb{Z}} \left(\int_1^{2^v} |\psi_{2^{kv}t} * g_{l,\zeta,k}|^2 dt \right)^{1/2} \right\|_{L^r(\mathfrak{R}_n)}^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ & \leq CA v^{1/2} \left\| \left(\sum_{l \in \mathbb{Z}} \left\| \left(\sum_{k \in \mathbb{Z}} |g_{l,\zeta,k}|^2 \right)^{1/2} \right\|_{L^r(\mathfrak{R}_n)}^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \end{aligned}$$

for any $1 < p, q, r < \infty$, where C is independent of A and v .

Fix $1 < p, q < \infty$, we can choose $1 < r < \min(p, q)$. By Lemma 2.1, (3.3), (3.9) and Hölder's inequality, we obtain

$$(3.10) \quad \begin{aligned} & \|A_j(f)\|_{E_{p,q}^\alpha} \\ & = \left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left(\int_{\mathfrak{R}_n} \left(\sum_{k \in \mathbb{Z}} \int_{2^{kv}}^{2^{(k+1)v}} |\psi_t * \Gamma_{j-k} \Delta_{2^{-l}\zeta}(f)|^2 \frac{dt}{t} \right)^{1/2} d\zeta \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ & = \left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left(\int_{\mathfrak{R}_n} \left(\sum_{k \in \mathbb{Z}} \int_1^{2^v} |\psi_{2^{kv}t} * \Gamma_{j-k} \Delta_{2^{-l}\zeta}(f)|^2 \frac{dt}{t} \right)^{1/2} d\zeta \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ & \leq \left\| \left(\sum_{l \in \mathbb{Z}} 2^{2q\alpha} \left\| \left(\sum_{k \in \mathbb{Z}} \int_1^{2^v} |\psi_{2^{kv}t} * \Gamma_{j-k} \Delta_{2^{-l}\zeta}(f)|^2 dt \right)^{1/2} \right\|_{L^r(\mathfrak{R}_n)}^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ & \leq CA v^{1/2} \left\| \left(\sum_{l \in \mathbb{Z}} \left\| \left(\sum_{k \in \mathbb{Z}} |\Gamma_{j-k}(2^{l\alpha} \Delta_{2^{-l}\zeta}(f))|^2 \right)^{1/2} \right\|_{L^r(\mathfrak{R}_n)}^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ & \leq CA v^{1/2} \left(\frac{2^v}{2^v - 1} \right)^{n+2} \left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \|\Delta_{2^{-l}\zeta}(f)\|_{L^r(\mathfrak{R}_n)}^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ & \leq CA v^{1/2} \|f\|_{\dot{F}_{\alpha,q}^{p,q}(\mathbb{R}^n)}, \end{aligned}$$

where the constant $C > 0$ is independent of A, v .

Interpolation between (3.7) and (3.10) implies that for any $1 < p, q < \infty$, there exists $\theta \in (0, 1]$ and a constant $C > 0$ independent of A, v such that

$$(3.11) \quad \|A_j(f)\|_{E_{p,q}^\alpha} \leq CA v^{1/2} 2^{-c\theta|j|/2} \|f\|_{\dot{F}_{\alpha,q}^{p,q}(\mathbb{R}^n)}.$$

Combining (3.11) with (3.5) yields (3.2) and completes the proof of Theorem 1.2. \square

Proof of Theorem 1.1. It was shown in Lemmas 1-3 of [11] that there exists a constant C depending on ψ, h, Ω such that

$$(3.12) \quad \int_{2^k}^{2^{k+1}} |\hat{\psi}(t\xi)|^2 \frac{dt}{t} \leq C \min(1, |2^k \xi|^\epsilon, |2^k \xi|^{-\epsilon})$$

for all $k \in \mathbb{Z}$, $\xi \in \mathbb{R}^n$ and some $\epsilon > 0$ if ψ satisfies the condition of Theorem B. It follows from [12, Lemma 2] that there exists $C > 0$ independent of s, ψ, h, Ω

such that

$$(3.13) \quad \int_{2^{ks'}}^{2^{(k+1)s'}} |\hat{\psi}(t\xi)|^2 \frac{dt}{t} \leq Cs' \|\psi\|_{L^s(\mathbb{R}^n)}^2 \min\{1, |2^{ks'}\xi|^{1/(2s')}, |2^{ks'}\xi|^{-1/(2s')}\}$$

if $\psi \in L^s(\mathbb{R}^n)$ for some $s > 1$.

On the other hand, when ψ satisfies the condition of Theorem B or C, as in Stein [14, pp.63–64], we can show that

$$(3.14) \quad \sup_{t>0} |\psi_t| * f(x) \leq \|h\|_{L^1(\mathbb{R}^n)} v_n^{-1} M_\Omega(f)(x),$$

where v_n is the volume of the unit ball in \mathbb{R}^n and

$$M_\Omega(f)(x) = \sup_{t>0} t^{-n} \int_{|y|<t} |f(x-y)| \Omega\left(\frac{y}{|y|}\right) dy.$$

One can easily check that

$$(3.15) \quad M_\Omega(f)(x) \leq \int_{S^{n-1}} \Omega(y') M_{y'}(f)(x) d\sigma(y'),$$

where $y' = \frac{y}{|y|}$ and $M_{y'}$ is the directional Hardy-Littlewood maximal function along y' . We notice that $\|m_\psi\|_{L^1(\mathbb{R}^n)} = \|h\|_{L^1(\mathbb{R}^n)} \|\Omega\|_{L^1(S^{n-1})} (nv_n)^{-1}$. By (3.14)-(3.15), Lemma 2.2 and Minkowski's inequality, we can obtain

$$(3.16) \quad \left\| \left(\sum_{l \in \mathbb{Z}} \left\| \sup_{t>0} |\psi_t| * f_{l,\zeta} \right\|_{L^r(\mathfrak{R}_n)}^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ \leq C \|m_\psi\|_{L^1(\mathbb{R}^n)} \left\| \left(\sum_{l \in \mathbb{Z}} \|f_{l,\zeta}\|_{L^r(\mathfrak{R}_n)}^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}$$

for any $1 < p, q, r < \infty$, where $C > 0$ is independent of s, ψ, h, Ω .

By (3.12), (3.16) and Theorem 1.2 with $v = A = 1$, there exists a constant $C > 0$ depending on ψ, h, Ω such that

$$\|g_\psi(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} \leq C \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)}$$

for all $0 < \alpha < 1$ and $1 < p, q < \infty$ if ψ satisfies the condition of Theorem B.

It follows also from (3.13), (3.16) and Theorem 1.2 with $v = s'$ and $A = \|\psi\|_{L^s(\mathbb{R}^n)} + \|m_\psi\|_{L^1(\mathbb{R}^n)}$ that there exists a constant $C > 0$ independent of s, ψ, h, Ω such that

$$\|g_\psi(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} \leq C (\|\psi\|_{L^s(\mathbb{R}^n)} + \|m_\psi\|_{L^1(\mathbb{R}^n)}) \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)}$$

for all $0 < \alpha < 1$ and $1 < p, q < \infty$ if ψ satisfies the condition of Theorem C. This finishes the proof of Theorem 1.1. \square

Proof of Corollary 1.1. One can easily check that $H \in L^r(\mathbb{R}^n)$ for any $1 < r \leq 2$. Employing the notation in [12], let $F_1 = \{\theta \in S^{n-1} : |\Omega(\theta)| \leq 2\}$ and $F_k = \{\theta \in S^{n-1} : 2^{k-1} < |\Omega(\theta)| \leq 2^k\}$ for any $k \geq 2$. Let $E_k = \{x \in B(0, 1) :$

$x \neq 0, x' \in F_k$ for all $k \geq 1$, $B(0, 1) = \{x \in \mathbb{R}^n : |x| \leq 1\}$ and $\Omega_k = \Omega \chi_{F_k}$. We decompose ψ as $\psi = \sum_{k=1}^{\infty} \psi_k$, where

$$\psi_k = \psi \chi_{E_k} - |B(0, 1)|^{-1} \int_{E_k} \psi(x) dx \chi_{B(0, 1)}.$$

Note that

$$\begin{aligned} \int_{\mathbb{R}^n} \psi_k(x) dx &= 0; \\ |\psi(x) \chi_{E_k}| &\leq H(x) \Omega_k(x') \chi_{B(0, 1)}; \\ \left| |B(0, 1)|^{-1} \int_{E_k} \psi(x) dx \chi_{B(0, 1)} \right| &\leq \|H\|_{L^1(\mathbb{R}^n)} \|\Omega\|_{L^1(S^{n-1})} \chi_{B(0, 1)}. \end{aligned}$$

Let

$$h^*(|x|) = (H(x) + \|H\|_{L^1(\mathbb{R}^n)} \chi_{B(0, 1)})(x) \text{ and } \Omega_k^*(x') = \Omega_k(x') + \|\Omega_k\|_{L^1(S^{n-1})}.$$

One can easily check that

$$|\psi_k(x)| \leq h^*(|x|) \Omega_k^*(x'),$$

where h^* is a nonnegative, nonincreasing function on $(0, \infty)$ with supported in $(0, 1]$ and Ω_k^* is a nonnegative function on S^{n-1} . Specially, $h^*(|x|) \in L^1(\mathbb{R}^n)$, $\Omega_k^* \in L^1(S^{n-1})$ and $\psi_k \in L^s(\mathbb{R}^n)$ for any $1 < s \leq 2$. Let $\psi_{k,t} = t^{-n} \psi_k(t^{-1}x)$ and $m_{\psi_k} = h^*(|x|) \Omega_k^*(x')$. By (3.1) and Minkowski's inequality, (3.17)

$$\begin{aligned} &\|g_\psi(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} \\ &\leq C \left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left(\int_{\mathfrak{R}_n} \left(\int_0^\infty |\psi_t * \Delta_{2^{-l}\zeta}(f)|^2 \frac{dt}{t} \right)^{1/2} d\zeta \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ &\leq C \sum_{k=1}^{\infty} \left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left(\int_{\mathfrak{R}_n} \left(\int_0^\infty |\psi_{k,t} * \Delta_{2^{-l}\zeta}(f)|^2 \frac{dt}{t} \right)^{1/2} d\zeta \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \end{aligned}$$

for all $0 < \alpha < 1$ and $1 < p, q < \infty$. By the proofs of Theorems 1.1-1.2, there exists a constant $C > 0$ independent of $s, \psi_k, h^*, \Omega_k^*$ such that

$$\begin{aligned} (3.18) \quad &\left\| \left(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \left(\int_{\mathfrak{R}_n} \left(\int_0^\infty |\psi_t * \Delta_{2^{-l}\zeta}(f)|^2 \frac{dt}{t} \right)^{1/2} d\zeta \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ &\leq C (s/(s-1))^{1/2} (\|\psi_k\|_{L^s(\mathbb{R}^n)} + \|m_{\psi_k}\|_{L^1(\mathbb{R}^n)}) \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} \end{aligned}$$

for all $0 < \alpha < 1$ and $1 < p, q < \infty$. Taking $s = 1 + 1/k$. It follows from (3.17)-(3.18) that

(3.19)

$$\|g_\psi(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} \leq C \sum_{k=1}^{\infty} k^{1/2} (\|\psi_k\|_{L^{1+1/k}(\mathbb{R}^n)} + \|m_{\psi_k}\|_{L^1(\mathbb{R}^n)}) \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)}$$

for all $0 < \alpha < 1$ and $1 < p, q < \infty$, where $C > 0$ is independent of k, ψ_k, h, Ω .

Note that

$$\|m_{\psi_k}\|_{L^1(\mathbb{R}^n)} \leq C \|h^*\|_{L^1(\mathbb{R}^n)} \|\Omega_k^*\|_{L^1(S^{n-1})} \leq C \|H\|_{L^2(\mathbb{R}^n)} \|\Omega_k\|_{L^{1+1/k}(S^{n-1})};$$

$$\|\psi_k\|_{L^{1+1/k}(\mathbb{R}^n)} \leq C \|H\|_{L^2(\mathbb{R}^n)} \|\Omega_k\|_{L^{1+1/k}(S^{n-1})};$$

$$\|\Omega_k\|_{L^{1+1/k}(S^{n-1})} = \begin{cases} C \|\Omega_k\|_{L^1(S^{n-1})}, & \text{if } \|\Omega_k\|_{L^1(S^{n-1})} \geq 2^{-k}; \\ 2^{-k^2/(k+1)}, & \text{if } \|\Omega_k\|_{L^1(S^{n-1})} < 2^{-k}. \end{cases}$$

It follows from (3.19) that

$$\begin{aligned} & \|g_\psi f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} \\ & \leq C \|H\|_{L^2(\mathbb{R}^n)} \sum_{k=1}^{\infty} (k^{1/2} 2^{-k^2/(k+1)} + k^{1/2} \|\Omega_k\|_{L^1(S^{n-1})}) \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} \\ & \leq C \|H\|_{L^2(\mathbb{R}^n)} (1 + \|\Omega\|_{L(\log^+ L)^{1/2}(S^{n-1})}) \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)}. \end{aligned}$$

This yields the conclusion of Corollary 1.1. \square

Acknowledgement. The author would like to express his deep gratitude to the referee for his/her valuable comments and suggestions.

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