Bull. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/BKMS.b170203
pISSN: 1015-8634 / eISSN: 2234-3016

A HOMOLOGICAL CHARACTERIZATION OF KRULL DOMAINS

Fang Gui Wang and De Chuan Zhou

Abstract. Let R be a commutative ring. In this paper, the w-projective Basis Lemma for w-projective modules is given. Then it is shown that for a domain, nonzero w-projective ideals and nonzero w-invertible ideals coincide. As an application, it is proved that R is a Krull domain if and only if every submodule of finitely generated projective modules is w-projective.

1. Introduction

In this paper, we assume that R is a commutative ring with identity 1. Krull domains are very important in the multiplicative ideal theory. Their original definition is as follows: A domain R is called a Krull domain if R satisfies the following properties: (1) R_P is a discrete valuation ring whenever P is a height one prime of R; (2) $R = \bigcap R_P$, where the intersection is taken over the height one primes of R; (3) for any nonzero element $r \in R$, the set of minimal prime ideals over rR is finite. This definition may be not good enough to be understood. So there are many scholars to characterize Krull domains. For example, B. G. Kang proved that R is a Krull domain if and only if every nonzero ideal of R is t-invertible [4, Theorem 3.6]. Then Anderson-Cook showed that t-invertible ideals are equivalent to w-invertible ones [1, Theorem 2.18]. Therefore R is a Krull domain if and only if every nonzero ideal of R is w-invertible. Note that R is a Dedekind domain if and only if every nonzero ideal of R is invertible. Thus in a loose sense, we can regard Krull domains as w-Dedekind domains. It is well known that there is a homological characterization of Dedekind domains: R is a Dedekind domain if and only if every nonzero ideal of R is projective. Then a natural question arises whether Krull domains have a analogous characterization. In [8], F. G. Wang and H. Kim generalized projective modules to w-projective modules by the w-operation. An R-module

Received March 7, 2017; Revised April 12, 2017; Accepted June 8, 2017.
2010 *Mathematics Subject Classification.* 13A15, 13C10, 13D99.
Key words and phrases. w-projective module, w-invertible, Krull domain, w-module.
Supported by the National Natural Science Foundation of China(Grant No.11671283 and No. 11661014).

M is said to be w-projective if $\mathrm{Ext}_R^1(L(M), N)$ is GV-torsion for any torsion-free w-module N, where $L(M) = (M/\mathrm{tor_{GV}}(M))_w$. The w-projective modules have many similar properties as projective modules. For example, if M is a w-projective module, then $M_\mathfrak{m}$ is free over $R_\mathfrak{m}$ for any maximal w-ideal $\mathfrak{m}$ of R. While, there are still problems about w-projective modules should be studied. We can ask whether w-projective modules have the corresponding w-Projective Basis Lemma. For a domain R, nonzero projective ideals and nonzero invertible ideals coincide, then we can also ask whether nonzero w-projective ideals are exactly w-invertible ideals. In fact, this paper shows that the answers of these two questions are positive (Theorem 2.2 and Theorem 2.7). Then on the basis of these results, it is proved that R is a Krull domain if and only if every submodule of finitely generated projective modules is w-projective (Theorem 3.3). Now we introduce some definitions and notations from [9]. Let J be a finitely generated ideal of R. If the natural homomorphism $\varphi : R \to J^* = \mathrm{Hom}_R(J, R)$ is an isomorphism, then J is called a GV-*ideal*, denoted by $J \in$GV(R). Let M be an R-module. Define

$$\mathrm{tor_{GV}}(M) = \{x \in M \mid Jx = 0 \text{ for some } J \in \mathrm{GV}(R)\}.$$

Thus $\mathrm{tor_{GV}}(M)$ is a submodule of M. And M is said to be GV-*torsion* (resp., GV-*torsion-free*) if $\mathrm{tor_{GV}}(M) = M$ (resp., $\mathrm{tor_{GV}}(M) = 0$). Clearly R is a GV-torsion-free R-module ([9, Corollary 1.5]). A GV-torsion-free module M is called a *w-module* if $\mathrm{Ext}_R^1(R/J, M) = 0$ for any $J \in \mathrm{GV}(R)$. The w-envelope of a GV-torsion-free module M is the set given by

$$M_w = \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in \mathrm{GV}(R)\},$$

where $E(M)$ is the injective hull of M. It is easy to see that M is a w-module if and only if $M_w = M$. A nonzero ideal I of R is said to be *a prime w-ideal* if I is both a prime ideal and a w-ideal; and *a maximal w-ideal* if I is maximal in the set of all proper w-ideals of R. Each maximal w-ideal is prime. We denote the set of maximal w-ideals by w-Max(R). For a domain R with the quotient field K, a fractional ideal I of R is called *w-invertible* if $(II^{-1})_w = R$, where $I^{-1} = \{x \in K | xI \subseteq R\}$.

2. w-projective module

In [8], the notion of w-projective modules was introduced by F. G. Wang and H. Kim. An R-module M is said to be w-projective if $\mathrm{Ext}_R^1(L(M), N)$ is GV-torsion for any torsion-free w-module N, where $L(M) = (M/\mathrm{tor_{GV}}(M))_w$.

Now, we recall several concepts from [6]. Let M and N be R-modules and let $f \in \mathrm{Hom}_R(M, N)$. If for each maximal w-ideal $\mathfrak{m}$ of R, $f_\mathfrak{m} : M_\mathfrak{m} \to N_\mathfrak{m}$ is a monomorphism (resp., an epimorphism, an isomorphism), we call f a *w-monomorphism* (resp., a *w-epimorphism*, a *w-isomorphism*). Meanwhile, a sequence $A \to B \to C$ of R-modules and R-homomorphisms is said to be *w-exact* if the sequence $A_\mathfrak{m} \to B_\mathfrak{m} \to C_\mathfrak{m}$ is exact for any maximal w-ideal $\mathfrak{m}$ of R. A module M is said to be *of finite type* if there exist a finitely generated free module F and a w-epimorphism $f : F \to M$ ([6, Definition 1.3]). Obviously, a finitely generated R-module is of finite type.

Clearly projective modules and GV-torsion modules always are w-projective. It is easy to verify that, for R-modules M and N, and for $f \in \mathrm{Hom}_R(M,N)$, if f is a w-isomorphism, then M is w-projective if and only if N is w-projective.

It is known that an R-module P is called *a projective module* if for any exact sequence $B \xrightarrow{g} C \to 0$ of R-modules and $f \in \mathrm{Hom}_R(P,C)$, there exists $h \in \mathrm{Hom}_R(P,B)$ such that $gh = f$. For w-modules, we can give a characterization of w-projective modules similar to the definition of projective modules above.

Let A, B and B_1 be R-modules and let $\alpha \in \mathrm{Hom}_R(B, B_1)$. For any $f \in \mathrm{Hom}_R(A,B)$, set $\alpha_*(f) = \alpha f$. Then α_* is a homomorphism from $\mathrm{Hom}_R(A,B)$ to $\mathrm{Hom}_R(A,B_1)$.

Proposition 2.1. *The following statements are equivalent for a w-module M.*

(1) *M is w-projective.*

(2) *For any exact sequence $B \xrightarrow{g} C \to 0$ of GV-torsion-free modules and $f \in \mathrm{Hom}_R(M,C)$, there exists some $J \in \mathrm{GV}(R)$ such that for any $b \in J$, $gh = bf$ for some $h \in \mathrm{Hom}_R(P,B)$.*

(3) *For any exact sequence $F \xrightarrow{g} M \to 0$ where F is free, there exists some $J \in \mathrm{GV}(R)$ such that for any $b \in J$, $gh = b\mathbf{1}_M$ for some $h \in \mathrm{Hom}_R(M,F)$.*

(4) *For any exact sequence $0 \to A \to B \to C \to 0$ of GV-torsion-free modules,*

$$0 \to \mathrm{Hom}_R(M,A) \to \mathrm{Hom}_R(M,B) \to \mathrm{Hom}_R(M,C) \to 0$$

is w-exact.

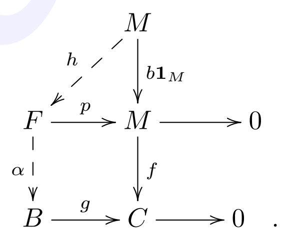
(5) *For any w-module N, $\mathrm{Ext}^1_R(M,N)$ is GV-torsion.*

Proof. (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) See [7, Theorem 6.7.9].

(2) $\Leftrightarrow$ (4) Note that the sequence $\mathrm{Hom}_R(M,B) \to \mathrm{Hom}_R(M,C) \to 0$ is w-exact if and only if $\mathrm{Hom}_R(M,C)/\mathrm{Im}(g_*)$ is GV-torsion; if and only if for any $f \in \mathrm{Hom}_R(M,C)$, there exists some $J \in \mathrm{GV}(R)$ such that for any $b \in J$, $gh = bf$ for some $h \in \mathrm{Hom}_R(M,B)$.

(2) $\Rightarrow$ (3) It is obvious.

(3) $\Rightarrow$ (2) Let $g : B \to C$ be an epimorphism and let $f : M \to C$ be a homomorphism. Then there exists an epimorphism $p : F \to M$ where F is free. Consider the following diagram:



Since F is a projective module, there exists a homomorphism $\alpha : F \to B$ such that $g\alpha = fp$. By assumption, there exists some $J \in \mathrm{GV}(R)$ such that for any $b \in J$, $ph = b\mathbf{1}_M$ for some $h \in \mathrm{Hom}_R(M, F)$. Set $\beta = \alpha h : M \to B$. Thus $g\beta = bf$. □

For an R-module M, let $M^* = \mathrm{Hom}_R(M, R)$.

Theorem 2.2 (w-Projective Basis Lemma)**.** *Let M be a w-module. Then M is a w-projective module if and only if there exist elements $\{x_i \mid i \in \Gamma\} \subseteq M$ and some $J = (d_1, d_2, \ldots, d_n) \in \mathrm{GV}(R)$ such that for each $k \in \{1, 2, \ldots, n\}$, there exist R-homomorphisms $\{f_{ki} \in M^* \mid i \in \Gamma\}$ such that*

(1) *if $x \in M$, then almost all $f_{ki}(x) = 0$;*

(2) *if $x \in M$, then $d_k x = \sum\limits_i f_{ki}(x)x_i$.*

Proof. Suppose M is a w-projective module and $0 \to A \to F \xrightarrow{g} M \to 0$ is an exact sequence, where F is free with a basis $\{e_i | i \in \Gamma\}$. Set $x_i = g(e_i)$. By Proposition 2.1, $g_* : \mathrm{Hom}_R(M, F) \to \mathrm{Hom}_R(M, M)$ is a w-epimorphism. Then there exists some $J = (d_1, \ldots, d_n) \in \mathrm{GV}(R)$ such that $J\mathbf{1}_M \subseteq \mathrm{Im}(g_*)$. Hence for $1 \leqslant k \leqslant n$, $d_k\mathbf{1}_M = g_*(\alpha_k) = g\alpha_k$ for some $\alpha_k \in \mathrm{Hom}_R(M, F)$. For any $x \in M$, set $\alpha_k(x) = \sum\limits_i r_{ki}e_i$, $r_{ki} \in R$. Then almost all $r_{ki} = 0$. Set $f_{ki}(x) = r_{ki}$. Then $\{f_{ki} \in M^*\}$ and $d_k x = d_k\mathbf{1}_M(x) = g(\alpha_k(x)) = \sum\limits_i f_{ki}(x)x_i$.

Conversely, assume the existence of $\{x_i \mid i \in \Gamma\} \subseteq M$, $J = (d_1, d_2, \ldots, d_n) \in \mathrm{GV}(R)$ and $\{f_{ki} \in M^*\}$ for each $k \in \{1, 2, \ldots, n\}$. Let F be free with a basis $\{e_i \mid i \in \Gamma\}$ and let $\beta : F \to M$ be a homomorphism with $\beta(e_i) = x_i$. For $1 \leqslant k \leqslant n$, define $h_k : M \to F$ by $h_k(x) = \sum\limits_i f_{ki}(x)e_i$ for any $x \in M$. Then $h_k \in \mathrm{Hom}_R(M, F)$ and $\beta h_k(x) = \beta(\sum\limits_i f_{ki}(x)e_i) = \sum\limits_i f_{ki}(x)x_i = d_k x$. Thus $\beta h_k = d_k\mathbf{1}_M$. Whence M is a w-projective module by Proposition 2.1. □

Let M be an R-module. Let τ denote the trace map of M, that is, $\tau : M \otimes_R M^* \to R$ defined by

$$\tau(x \otimes f) = f(x), x \in M, f \in M^*.$$

Recall that M is said to be *invertible* (resp., *w-invertible*) if τ is an isomorphism (a w-isomorphism). It is well known that M is invertible if and only if M is projective of constant rank one. An R-module M is said to have *constant rank* n (resp., *w-rank n*) if for any maximal ideal (resp., any maximal w-ideal) $\mathfrak{m}$ of R, $M_\mathfrak{m}$ is free of rank n over $R_\mathfrak{m}$. If M has w-rank n, then we say w-rank$(M) = n$. In [8], F. G. Wang and H. Kim proved that M is w-invertible if and only if M is w-projective of finite type and has w-rank one [8, Theorem 4.13]. Actually, for a GV-torsion-free module M, if M is a w-projective module and has w-rank one, then M is of finite type. In order to prove this result, we begin with the following two theorems.

Theorem 2.3. *Let M be GV-torsion-free. If w-rank$(M) = 1$ and $(\mathrm{Im}\tau)_w = R$, then M is of finite type.*

Proof. If $(\mathrm{Im}\tau)_w = R$, then there exists some $J = (d_1, \dots, d_n) \in \mathrm{GV}(R)$ such that $J \subseteq \mathrm{Im}\tau$. For any d_k, set $d_k = \tau(\sum_{i=1}^{m_k} x_{ki} \otimes f_{ki})$, where $x_{ki} \in M$ and $f_{ki} \in M^*$. Then $d_k = \sum_{i=1}^{m_k} f_{ki}(x_{ki})$. For any $\mathfrak{m} \in w\text{-Max}(R)$, we have $J \nsubseteq \mathfrak{m}$. Then $d_k \notin \mathfrak{m}$ for some k. Hence $f_{ki}(x_{ki}) \notin \mathfrak{m}$ for some i. Therefore $\frac{f_{ki}(x_{ki})}{1}$ is a unit of $R_\mathfrak{m}$. For this k and i, define $g_{ki} : M_\mathfrak{m} \to R_\mathfrak{m}$ by $g_{ki}(\frac{x}{s}) = \frac{f_{ki}(x)}{s}$, where $x \in M$, $s \in R\backslash\mathfrak{m}$. Obviously, g_{ki} is an $R_\mathfrak{m}$-epimorphism. Then g_{ki} is an $R_\mathfrak{m}$-isomorphism since w-rank$(M) = 1$. So $\frac{x_{ki}}{1}R = M_\mathfrak{m}$. Let N be the submodule generated by these x_{ki}. Then N is a finitely generated submodule of M and $N_\mathfrak{m} = M_\mathfrak{m}$ for any $\mathfrak{m} \in w\text{-Max}(R)$. Hence M is of finite type. □

Theorem 2.4. *Let M be a w-projective w-module. Then the following statements hold.*

(1) $M = ((\mathrm{Im}\tau)M)_w$.

(2) *If w-rank$(M) = 1$, then $(\mathrm{Im}\tau)_w = R$.*

Proof. (1) By Theorem 2.2, there exist elements $\{x_i \,|\, i \in \Gamma\} \subseteq M$ and some $J = (d_1, d_2, \dots, d_n) \in \mathrm{GV}(R)$ such that for each $k \in \{1, 2, \dots, n\}$, there exist R-homomorphisms $\{f_{ki} \in M^* \,|\, i \in \Gamma\}$ such that almost all $f_{ki}(x) = 0$ and $d_k x = \sum\limits_i f_{ki}(x)x_i$ for any $x \in M$. Then $Jx \subseteq (\mathrm{Im}\tau)M$. Hence $M = ((\mathrm{Im}\tau)M)_w$.

(2) If $(\mathrm{Im}\tau)_w \neq R$, then $\mathrm{Im}\tau \subseteq \mathfrak{m}$ for some $\mathfrak{m} \in w\text{-Max}(R)$. By (1), $M_\mathfrak{m} = (((\mathrm{Im}\tau)M)_w)_\mathfrak{m} = (\mathrm{Im}\tau)_\mathfrak{m} M_\mathfrak{m} \subseteq \mathfrak{m}R_\mathfrak{m}M_\mathfrak{m} \subseteq M_\mathfrak{m}$. Hence $\mathfrak{m}R_\mathfrak{m}M_\mathfrak{m} = M_\mathfrak{m}$. By Nakayama's lemma, $M_\mathfrak{m} = 0$. But this is a contradiction to w-rank$(M) = 1$. So $(\mathrm{Im}\tau)_w = R$. □

By combining Theorem 2.3 and Theorem 2.4, we can get the following corollary.

Corollary 2.5. *If a GV-torsion-free module M is w-projective and has w-rank one, then M is of finite type.*

Proposition 2.6. *Let R be a domain and let I be a nonzero w-projective ideal of R. Then w-rank$(I) = 1$. Therefore I is of finite type.*

Proof. We can assume that I is a w-ideal. By Proposition 2.1, $\mathrm{Ext}_R^1(I, N)$ is GV-torsion for any w-module N. Then $I_\mathfrak{m}$ is free over $R_\mathfrak{m}$ for any $\mathfrak{m} \in w\text{-Max}(R)$ by [7, Theorem 6.7.11]. Hence $I_\mathfrak{m} = R_\mathfrak{m}$, which implies that w-rank$(I) = 1$. Therefore I is of finite type by Corollary 2.5. □

By Proposition 2.6 and [8, Theorem 4.13], we can get that for a domain R, w-projective ideals and w-invertible ideals coincide. This result is a good correspondence with the fact that projective ideals are exactly invertible ideals. Actually, we can also get this result by applying the w-Projective Basis Lemma.

Theorem 2.7. *Let R be a domain and let I be a nonzero fractional ideal of R. Then I is w-projective if and only if I is w-invertible.*

Proof. Assume that I is a w-projective w-ideal. By Theorem 2.2, there exist elements $\{x_i \,|\, i \in \Gamma\} \subseteq I$ and some $J = (d_1, d_2, \ldots, d_n) \in \mathrm{GV}(R)$ such that for each $k \in \{1, 2, \ldots, n\}$, there exist R-homomorphisms $\{f_{ki} \in I^* \,|\, i \in \Gamma\}$ such that almost all $f_{ki}(x) = 0$ and $d_k x = \sum\limits_i f_{ki}(x)x_i$ for any $x \in I$. Note that $x_{ki} = \frac{f_{ki}(a)}{a} \in I^{-1}$ is independent of the choice of a nonzero element $a \in I$ for any $k = 1, \ldots, n$, almost all $f_{ki}(a) = 0$. Assume that $d_k a = \sum_{i=1}^{m_k} f_{ki}(a)x_i = a\sum_{i=1}^{m_k} x_{ki}x_i$. Then $d_k = \sum_{i=1}^{m_k} x_{ki}x_i \in II^{-1}$. So $J \subseteq II^{-1}$. Therefore $(II^{-1})_w = R$.

Conversely, assume that $(II^{-1})_w = R$. Without loss of generality, we can also assume that I is a w-ideal of R. Then there exists some $J = (b_1, \ldots, b_n) \in \mathrm{GV}(R)$ such that b_k can be expressed as $b_k = \sum_{i=1}^{m_k} a_{ki}x_{ki}$ for any $k = 1, \ldots, n$, where $a_{ki} \in I, x_{ki} \in I^{-1}$. Define $\varphi : I^{-1} \to \mathrm{Hom}_R(I, R) = I^*$ by $\varphi(x)(y) = yx$, where $x \in I^{-1}$, $y \in I$. Obviously, φ is an isomorphism. Set $f_{ki} = \varphi(x_{ki})$. Then $f_{ki} \in I^*$ and $f_{ki}(a) = \varphi(x_{ki})(a) = ax_{ki}$ for any $a \in I$. So $b_k a = \sum_{i=1}^{m_k} aa_{ki}x_{ki} = \sum_{i=1}^{m_k} f_{ki}(a)a_{ki}$. By Theorem 2.2, I is w-projective. □

3. A homological characterization of Krull domains

Let M be an R-module, $m \in M$, and $r \in R$. We say m is *divisible* by r if $rm^{'} = m$ for some $m^{'} \in M$; we say the module M is *divisible* if each $m \in M$ is divisible by every non-zero-divisor $r \in R$ (i.e., there is no nonzero $s \in S$ with $sr = 0$). If R is a domain, then M is divisible if and only if $\mathrm{Ext}_R^1(R/Ra, M) = 0$ for any $0 \neq a \in R$ [3, Lemma 7.2].

Lemma 3.1. *Let R be a domain. If a GV-torsion-free module C is divisible, then C_w is divisible.*

Proof. Let $L = C_w/C$. Then L is GV-torsion. For any $0 \neq a \in R$, the sequence $0 \to R \to R \to R/Ra \to 0$ is exact. Then the induced sequence

$$\mathrm{Hom}_R(R, L) \longrightarrow \mathrm{Hom}_R(R, L) \longrightarrow \mathrm{Ext}_R^1(R/Ra, L) \to 0$$

is exact. So $\mathrm{Ext}_R^1(R/Ra, L)$ is GV-torsion. Since C is divisible, $\mathrm{Ext}_R^1(R/Ra, C)$ $= 0$ by [3, Lemma 7.2]. Note that

$$0 = \mathrm{Ext}_R^1(R/Ra, C) \to \mathrm{Ext}_R^1(R/Ra, C_w) \to \mathrm{Ext}_R^1(R/Ra, L)$$

is an exact sequence. Then $\mathrm{Ext}_R^1(R/Ra, C_w)$ is GV-torsion. On the other hand, $\mathrm{Ext}_R^1(R/Ra, C_w) = C_w/aC_w$ is GV-torsion-free. So $\mathrm{Ext}_R^1(R/Ra, C_w) = 0$. Hence C_w is divisible again by [3, Lemma 7.2]. □

Lemma 3.2. *Let R be a domain and let C be a divisible R-module.*

(1) *If S is a multiplicatively closed set of R, then C_S is a divisible R_S-module and the natural homomorphism $\vartheta : C \to C_S$ is an epimorphism.*

(2) *If R is a Krull domain and $\mathfrak{m}$ is a maximal w-ideal of R, then $C_{\mathfrak{m}}$ is an injective $R_{\mathfrak{m}}$-module, hence an injective R-module.*

Proof. (1) Let $\frac{x}{s} \in C_S$ and let $\frac{r}{s_1} \in R_S$, where $x \in C, s, s_1 \in S$ and $0 \neq r \in R$. Since C is divisible, $ry = x$ for some $y \in C$. Hence $\frac{ys_1}{s} \in C_S$ and $\frac{ys_1}{s}\frac{r}{s_1} = \frac{x}{s}$. Therefore C_S is a divisible R_S-module. It is easy to verify that ϑ is an epimorphism.

(2) If R is a Krull domain, then $R_{\mathfrak{m}}$ is a Dedekind domain for any $\mathfrak{m} \in w$-Max(R). Note that $R_{\mathfrak{m}}$ is a Dedekind domain if and only if every divisible $R_{\mathfrak{m}}$-module is injective [7, Theorem 5.2.15]. By (1), $C_{\mathfrak{m}}$ is a divisible $R_{\mathfrak{m}}$-module, hence an injective R_m-module. □

Theorem 3.3. *The following statements are equivalent for a domain R:*
(1) *R is a Krull domain.*
(2) *Every divisible w-module is injective.*
(3) *Every submodule of finitely generated projective modules is w-projective.*

Proof. (1)⇒(2) Let $\mathfrak{m}$ be a prime w-ideal of R. Then $\mathfrak{m}$ is a maximal w-ideal by [7, Corollary 7.9.4].

Let C be a divisible w-module of R. Then there is an exact sequence $0 \to A \to C \to C_{\mathfrak{m}} \to 0$ by Lemma 3.2(1).

We can prove that

(a) $\mathrm{Hom}_R(R/\mathfrak{m}, A) = 0$.

If $f \in \mathrm{Hom}_R(R/\mathfrak{m}, A)$, then $sf(\overline{1}) = 0$ for some $s \in R\backslash\mathfrak{m}$. For any $a \in \mathfrak{m}$, we have $af(\overline{1}) = f(\overline{a}) = 0$. Then $(\mathfrak{m} + sR)f(\overline{1}) = 0$. Since $(\mathfrak{m} + sR)_w = R$ and A is a w-module, $f(\overline{1}) = 0$. Hence $f = 0$.

(b) $\mathrm{Ext}^1_R(R/\mathfrak{m}, A)$ is GV-torsion-free.

Consider the following exact sequence

$0 = \mathrm{Hom}_R(R/\mathfrak{m}, A) \to \mathrm{Hom}_R(R, A) \to \mathrm{Hom}_R(\mathfrak{m}, A) \to \mathrm{Ext}^1_R(R/\mathfrak{m}, A) \to 0.$

By [7, Theorem 6.1.18], $\mathrm{Hom}_R(R, A)$ and $\mathrm{Hom}_R(\mathfrak{m}, A)$ both are w-modules. Then $\mathrm{Ext}^1_R(R/\mathfrak{m}, A)$ is GV-torsion-free by [7, Theorem 6.1.17].

(c) $\mathrm{Ext}^1_R(R/\mathfrak{m}, C)$ is GV-torsion.

For any $P \in w$-Max(R), let $S = R\backslash P$. Note that C_S is a w-module. Then $\mathrm{Ext}^1_R(R/\mathfrak{m}, C)_S \cong \mathrm{Ext}^1_{R_S}((R/\mathfrak{m})_S, C_S)$ by [2, Lemma 2.4]. By Lemma 3.2(2), $\mathrm{Ext}^1_{R_S}((R/\mathfrak{m})_S, C_S) = 0$, and then $\mathrm{Ext}^1_R(R/\mathfrak{m}, C)$ is GV-torsion.

Consider the following exact sequence

$$(1) \qquad \begin{aligned} 0 &\to \mathrm{Hom}_R(R/\mathfrak{m}, C) \to \mathrm{Hom}_R(R/\mathfrak{m}, C_{\mathfrak{m}}) \\ &\to \mathrm{Ext}^1_R(R/\mathfrak{m}, A) \to \mathrm{Ext}^1_R(R/\mathfrak{m}, C) \to 0. \end{aligned}$$

If $P = \mathfrak{m}$, then $\mathrm{Hom}_R(R/\mathfrak{m}, C)_S \cong \mathrm{Hom}_{R_S}((R/\mathfrak{m})_S, C_S) \cong \mathrm{Hom}_{R_S}((R/\mathfrak{m})_S, (C_{\mathfrak{m}})_S) \cong \mathrm{Hom}_R(R/\mathfrak{m}, C_{\mathfrak{m}})_S$ by [8, Proposition 1.9]. So $\mathrm{Ext}^1_R(R/\mathfrak{m}, A)_S \cong \mathrm{Ext}^1_R(R/\mathfrak{m}, C)_S$. If $P \neq \mathfrak{m}$, then $(R/\mathfrak{m})_S = 0$. Again by [8, Proposition 1.9], $\mathrm{Hom}_R(R/\mathfrak{m}, C_{\mathfrak{m}})_S \cong \mathrm{Hom}_{R_S}((R/\mathfrak{m})_S, (C_{\mathfrak{m}})_S) = 0$. Hence $\mathrm{Ext}^1_R(R/\mathfrak{m}, A)_S \cong \mathrm{Ext}^1_R(R/\mathfrak{m}, C)_S$. Since $\mathrm{Ext}^1_R(R/\mathfrak{m}, C)_S \cong \mathrm{Ext}^1_{R_S}((R/\mathfrak{m})_S, C_S)$ by [2, Lemma 2.4] and C_S is an injective R_S-module by Lemma 3.2(2), $\mathrm{Ext}^1_R(R/\mathfrak{m}, C)_S = 0$. Hence $\mathrm{Ext}^1_R(R/\mathfrak{m}, A)$ is GV-torion. Thus $\mathrm{Ext}^1_R(R/\mathfrak{m}, A) = 0$ by (b).

Again by the exact sequence (1), $\mathrm{Ext}_R^1(R/\mathfrak{m}, C) = 0$. So C is injective by [7, Theorem 6.8.27].

(2)⇒(3) Let P be a submodule of a finitely generated projective module F. We can assume that P is a w-module. Let N be a w-module and let E be the injective hull of N. Set $C = E/N$. Then C is GV-torsion-free and divisible. By Lemma 3.1, C_w is a divisible w-module. Then C_w is injective by assumption. Set $L = C_w/C$. Then L is GV-torsin. Thus $\mathrm{Hom}_R(F/P, L)$ is GV-torsion. Note that $\mathrm{Hom}_R(F/P, L) \to \mathrm{Ext}_R^1(F/P, C) \to \mathrm{Ext}_R^1(F/P, C_w) = 0$ is an exact sequence. Then $\mathrm{Ext}_R^1(F/P, C)$ is GV-torsion. It is easy to verify that $\mathrm{Ext}_R^1(P, N) \cong \mathrm{Ext}_R^1(F/P, C)$. Then $\mathrm{Ext}_R^1(P, N)$ is GV-torsion. Thus P is w-projective by Proposition 2.1.

(3)⇒(1) For any nonzero ideal I of R, I is w-projective by assumption. Then I is w-invertible by Theorem 2.7. Note that R is a Krull domain if and only if every nonzero ideal of R is w-invertible [7, Theorem 7.9.3]. Then R is a Krull domain. □

Remark. In fact, in [2], S. E. Baghdadi, H. Kim and F. G. Wang proved that R is a Krull domain if and only if every divisible w-module is injective [2, Theorem 2.6]. They quoted [5, Proposition 24] to show the necessity of this result directly. The Proposition 24 in [5] shows that if M is a domain R-module, then M is injective if and only if it is divisorial and divisible. Here we give a direct proof for (1)⇒ (2) of Theorem 3.3.

References

[1] D. D. Anderson and S. J. Cook, *Two star-operations and their induced lattices*, Comm. Algebra **28** (2000), no. 5, 2461-2475.

[2] S. E. Baghdadi, H. Kim, and F. G. Wang, *Divisible is injective over Krull domains*, Comm. Algebra **44** (2016), 4294–4301.

[3] L. Fuchs and L. Salce, *Modules over non-Noetherian domains*, Math. Surveys and Monographs 84, AMS, Providence, RI, 2001.

[4] B. G. Kang, *On the converse of a well-known theorem about Krull domains*, J. Algebra **124** (1989), 284–299.

[5] M. Nishi and M. Shinagawa, *Codivisorial and divisorial modules over completely integrally closed domains* (*I*), Hiroshima Math. J. **5** (1975), no. 2, 269–292.

[6] F. G. Wang, *Finitely presented type modules and w-coherent rings*, J. Sichuan Normal Univ. **33** (2010), 1–9.

[7] F. G. Wang and H. Kim, *Foundations of commutative rings and their modules*, Singapore, Springer, 2016.

[8] ———, *Two generalizations of projective modules and their applications*, J. Pure Appl. Algebra **219** (2015), no. 6, 2099–2123.

[9] H. Y. Yin, F. G. Wang, X. S. Zhu, and Y. H. Chen, *w-modules over commutative rings*, J. Korean Math. Soc. **48** (2011), no. 1, 207–222.

Fang Gui Wang
College of Mathematics and Software Science
Sichuan Normal University
Chengdu 610068, P. R. China
E-mail address: wangfg2004@163.com

De Chuan Zhou
College of Science
Southwest University of Science and Technology
Mianyang 621010, P. R. China
E-mail address: dechuan11119@sina.com