

## FRACTIONAL INTEGRATION AND DIFFERENTIATION OF THE $(p, q)$ -EXTENDED BESSEL FUNCTION

JUNESANG CHOI AND RAKESH K. PARMAR

ABSTRACT. We aim to present some formulas for Saigo hypergeometric fractional integral and differential operators involving  $(p, q)$ -extended Bessel function  $J_{\nu, p, q}(z)$ , which are expressed in terms of Hadamard product of the  $(p, q)$ -extended Gauss hypergeometric function and the Fox-Wright function  ${}_p\Psi_q(z)$ . A number of interesting special cases of our main results are also considered. Further, it is emphasized that the results presented here, which are seemingly complicated series, can reveal their involved properties via those of the two known functions in their respective Hadamard product.

### 1. Introduction and preliminaries

Fractional calculus, which has a long history, is an important branch of mathematical analysis (calculus) where differentiations and integrations can be of arbitrary non-integer order. During the past four decades or so, fractional calculus has been widely and extensively investigated and has gained importance and popularity due mainly to its demonstrated applications in numerous and diverse fields of science and engineering such as turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems, and astrophysics (see, for details, [6, 9, 10, 13–15, 21, 25]).

We recall Saigo fractional integral and differential operators involving the hypergeometric function  ${}_2F_1$  (see [20]):

$$(1) \quad \left(I_{0+}^{\alpha, \beta, \eta} f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}) f(t) dt,$$

$$(2) \quad \left(I_-^{\alpha, \beta, \eta} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}) f(t) dt,$$

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Received March 6, 2017; Revised August 7, 2017; Accepted November 24, 2017.

2010 *Mathematics Subject Classification.* Primary 26A33, 33B20, 33C20; Secondary 26A09, 33B15, 33C05.

*Key words and phrases.*  $(p, q)$ -extended Bessel function,  $(p, q)$ -extended hypergeometric function, Fox-Wright function, fractional calculus operators, Hadamard product.

and

$$(3) \quad \begin{aligned} (D_{0+}^{\alpha, \beta, \eta} f)(x) &= (I_{0+}^{-\alpha, -\beta, \alpha + \eta} f)(x) \\ &= \left(\frac{d}{dx}\right)^n (I_{0+}^{-\alpha + n, -\beta - n, \alpha + \eta - n} f)(x) \quad (n = [\Re(\alpha)] + 1), \end{aligned}$$

$$(4) \quad \begin{aligned} (D_-^{\alpha, \beta, \eta} f)(x) &= (I_-^{-\alpha, -\beta, \alpha + \eta} f)(x) \\ &= (-1)^n \left(\frac{d}{dx}\right)^n (I_-^{-\alpha + n, -\beta - n, \alpha + \eta} f)(x) \quad (n = [\Re(\alpha)] + 1). \end{aligned}$$

Here and in what follows,  $[x]$  denotes the greatest integer less than or equal to a real number  $x$ . Setting  $\beta = -\alpha$  in (1), (2), (3), and (4) yields the familiar Riemann-Liouville fractional integrals and derivatives of order  $\alpha \in \mathbb{C}$  with  $\Re(\alpha) > 0$  and  $x \in \mathbb{R}^+$  (see, e.g., [7–9, 21]):

$$(5) \quad (I_{0+}^{\alpha, -\alpha, \eta} f)(x) = (I_{0+}^{\alpha} f)(x) \equiv \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt,$$

$$(6) \quad (I_-^{\alpha, -\alpha, \eta} f)(x) = (I_-^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} f(t) dt,$$

and

$$(7) \quad \begin{aligned} (D_{0+}^{\alpha, -\alpha, \eta} f)(x) &= (D_{0+}^{\alpha} f)(x) = \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f(t) dt \\ &= \left(\frac{d}{dx}\right)^n (I_{0+}^{n-\alpha} f)(x) \quad (n = [\Re(\alpha)] + 1), \end{aligned}$$

$$(8) \quad \begin{aligned} (D_-^{\alpha, -\alpha, \eta} f)(x) &= (D_-^{\alpha} f)(x) \\ &= (-1)^n \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_x^{\infty} (t-y)^{n-\alpha-1} f(t) dt \\ &= (-1)^n \left(\frac{d}{dx}\right)^n (I_-^{n-\alpha} f)(x) \quad (n = [\Re(\alpha)] + 1). \end{aligned}$$

Here and in the following, let  $\mathbb{C}$ ,  $\mathbb{R}^+$ , and  $\mathbb{N}$  be the sets of complex numbers, positive real numbers, and positive integers, respectively, and let  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ .

Setting  $\beta = 0$  in (1), (2), (3), and (4) yields the so-called Erdélyi-Kober fractional integrals and derivatives of order  $\alpha \in \mathbb{C}$  with  $\Re(\alpha) > 0$  and  $x \in \mathbb{R}^+$  (see, e.g., [9, 21]):

$$(9) \quad (I_{0+}^{\alpha, 0, \eta} f)(x) = (I_{\eta, \alpha}^+ f)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{\eta} f(t) dt,$$

$$(10) \quad (I_-^{\alpha, 0, \eta} f)(x) = (K_{\eta, \alpha}^- f)(x) \equiv \frac{x^{\eta}}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} t^{-\alpha-\eta} f(t) dt,$$

and

$$\begin{aligned}
 (D_{0+}^{\alpha, 0, \eta} f)(x) &= (D_{\eta, \alpha}^+ f)(x) \\
 (11) \quad &= \left(\frac{d}{dx}\right)^n (I_{0+}^{-\alpha+n, -\alpha, \alpha+\eta-n} f)(x) \quad (n = [\Re(\alpha)] + 1),
 \end{aligned}$$

$$\begin{aligned}
 (D_-^{\alpha, 0, \eta} f)(x) &= (D_{\eta, \alpha}^- f)(x) \\
 (12) \quad &= (-1)^n \left(\frac{d}{dx}\right)^n (I_-^{-\alpha+n, -\alpha, \alpha+\eta} f)(x) \quad (n = [\Re(\alpha)] + 1),
 \end{aligned}$$

$$\begin{aligned}
 (D_{\eta, \alpha}^+ f)(x) &= x^{-\eta} \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_0^x t^{\alpha+\eta} (x-t)^{n-\alpha-1} f(t) dt \\
 (13) \quad & \quad (n = [\Re(\alpha)] + 1),
 \end{aligned}$$

$$\begin{aligned}
 (D_{\eta, \alpha}^- f)(x) &= x^{\eta+\alpha} \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_x^\infty t^{-\eta} (t-x)^{n-\alpha-1} f(t) dt \\
 (14) \quad & \quad (n = [\Re(\alpha)] + 1).
 \end{aligned}$$

Recently, many authors have investigated the  $(p, q)$ -variant (when  $p = q$ , the  $p$ -variant) associated with a set of related higher transcendental hypergeometric type special functions (see, for details, [1–4, 11, 17–19, 22]). In particular, Maširević et al. [12] introduced and studied the  $(p, q)$ -extended Bessel function  $J_{\nu, p, q}(z)$  of the first kind of order  $\nu$  with  $\min\{p, q\} \geq 0$  and  $\Re(\nu) > -\frac{1}{2}$  when  $p = q = 0$  in the form:

$$\begin{aligned}
 J_{\nu, p, q}(z) &= \frac{\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \sum_{n=0}^{\infty} (-1)^n B\left(n + \frac{1}{2}, \nu + \frac{1}{2}; p, q\right) \frac{z^{2n}}{(2n)!} \\
 (15) \quad &= \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu + 1)} \sum_{n=0}^{\infty} \frac{B\left(n + \frac{1}{2}, \nu + \frac{1}{2}; p, q\right)}{\left(\frac{1}{2}\right)_n B\left(\frac{1}{2}, \nu + \frac{1}{2}\right) n!} \left(-\frac{z^2}{4}\right)^n,
 \end{aligned}$$

where  $B(x, y; p, q)$  is the  $(p, q)$ -extended Beta function introduced by Choi et al. [5]

$$(16) \quad B(x, y; p, q) = \int_0^1 t^{x-1} (1-t)^{y-1} e^{-\frac{t}{p} - \frac{q}{1-t}} dt$$

$$(\min\{\Re(x), \Re(y)\} > 0; \min\{\Re(p), \Re(q)\} \geq 0).$$

It is noted that  $B(x, y; 0, 0) = B(x, y)$  is the familiar Beta function (see, e.g., [23, Section 1.1]). They [5] established various properties of  $B(x, y; p, q)$  in (16) such as integral representations, Mellin transforms (see, e.g., [18]), complete monotonicity, Turán type inequality (see, e.g., [16]) and associated non-homogeneous differential-difference equations. Clearly, the case  $p = 0 = q$  in

(15) reduces immediately to the classical Bessel function  $J_\nu(z)$  of the first kind (see, e.g., [22, 26])

$$(17) \quad J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{\nu+2n}}{n! \Gamma(\nu+n+1)}.$$

Also, for our present investigation, we need the concept of Hadamard product (or convolution) of two analytic functions. It can help us in decomposing newly established series into two known functions. Let

$$f(z) := \sum_{n=0}^{\infty} a_n z^n \quad (|z| < R_f) \quad \text{and} \quad g(z) := \sum_{n=0}^{\infty} b_n z^n \quad (|z| < R_g)$$

be two given power series whose radii of convergence are given by  $R_f$  and  $R_g$ , respectively. Then their Hadamard product is a power series defined by

$$(18) \quad (f * g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n = (g * f)(z) \quad (|z| < R),$$

whose radius of convergence  $R$  is

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} (|a_n b_n|)^{\frac{1}{n}} \leq \left( \limsup_{n \rightarrow \infty} (|a_n|)^{\frac{1}{n}} \right) \left( \limsup_{n \rightarrow \infty} (|b_n|)^{\frac{1}{n}} \right) = \frac{1}{R_f \cdot R_g}$$

and so  $R \geq R_f \cdot R_g$ . In particular, if one of the power series defines an entire function, then the Hadamard product series defines an entire function, too.

In this paper, we aim to investigate compositions of the generalized fractional integration and differentiation operators (1), (2), (3) and (4) involving  $(p, q)$ -extended Bessel function  $J_{\nu, p, q}(z)$ . Also, those results for the classical Riemann-Liouville and Erdélyi-Kober fractional integral and differential operators involving the  $(p, q)$ -extended Bessel function  $J_{\nu, p, q}(z)$ , which correspond to the main identities, are deduced. Further, we show that those compositions are expressed in terms of the Hadamard product (18) of the  $(p, q)$ -extended Gauss hypergeometric function (see [5, p. 354, Eq. (7.1)])

$$(19) \quad F_{p, q}(a, b; c; z) = \sum_{n=0}^{\infty} (a)_n \frac{B(b+n, c-b; p, q)}{B(b, c-b)} \frac{z^n}{n!} \quad (|z| < 1; \Re(c) > \Re(b) > 0)$$

and Fox-Wright function  ${}_p\Psi_q(z)$  ( $p, q \in \mathbb{N}_0$ ) (see, for details, [9, 15]; see also [21, 24]):

$$(20) \quad {}_p\Psi_q \left[ \begin{matrix} (\alpha_1, A_1), \dots, (\alpha_p, A_p); \\ (\beta_1, B_1), \dots, (\beta_q, B_q); \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + A_1 n) \cdots \Gamma(\alpha_p + A_p n)}{\Gamma(\beta_1 + B_1 n) \cdots \Gamma(\beta_q + B_q n)} \frac{z^n}{n!}$$

$$\left( A_j \in \mathbb{R}^+ \quad (j = 1, \dots, p); \quad B_j \in \mathbb{R}^+ \quad (j = 1, \dots, q); \quad 1 + \sum_{j=1}^q B_j - \sum_{j=1}^p A_j \geq 0 \right),$$

where the equality in the convergence condition holds for

$$|z| < \nabla := \left( \prod_{j=1}^p A_j^{-A_j} \right) \left( \prod_{j=1}^q B_j^{B_j} \right).$$

Here and in the following,  $(\lambda)_\nu$  denotes the Pochhammer symbol defined (for  $\lambda, \nu \in \mathbb{C}$ ), in terms of the Gamma function  $\Gamma$ , by

$$(21) \quad (\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda + 1) \cdots (\lambda + \nu - 1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases}$$

## 2. Fractional integrations of the $J_{\nu, p, q}(z)$

Here, composition formulas of generalized fractional integrals (1) and (2) involving the  $(p, q)$ -extended Bessel function  $J_{\nu, p, q}(z)$  of the first kind of order  $\nu$  are presented. To do this, we begin by stating some image formulas regarding (1) and (2) which may be known formulas and are given in the following lemma.

**Lemma 2.1.** *Let  $\alpha, \beta, \eta \in \mathbb{C}$  and  $x \in \mathbb{R}^+$ .*

(a) *If  $\Re(\alpha) > 0$  and  $\Re(\sigma) > \max\{0, \Re(\beta - \eta)\}$ , then*

$$(22) \quad (I_{0+}^{\alpha, \beta, \eta} t^{\sigma-1})(x) = \frac{\Gamma(\sigma)\Gamma(\sigma + \eta - \beta)}{\Gamma(\sigma - \beta)\Gamma(\sigma + \alpha + \eta)} x^{\sigma - \beta - 1}.$$

*In particular, we have*

$$(23) \quad (I_{0+}^{\alpha} t^{\sigma-1})(x) = \frac{\Gamma(\sigma)}{\Gamma(\sigma + \alpha)} x^{\sigma + \alpha - 1} \quad (\min\{\Re(\alpha), \Re(\sigma)\} > 0)$$

*and*

$$(24) \quad (I_{\eta, \alpha}^+ t^{\sigma-1})(x) = \frac{\Gamma(\sigma + \eta)}{\Gamma(\sigma + \alpha + \eta)} x^{\sigma-1} \quad (\Re(\alpha) > 0, \Re(\sigma) > -\Re(\eta)).$$

(b) *If  $\Re(\alpha) > 0$  and  $\Re(\sigma) < 1 + \min\{\Re(\beta), \Re(\eta)\}$ , then*

$$(25) \quad (I_{-}^{\alpha, \beta, \eta} t^{\sigma-1})(x) = \frac{\Gamma(\beta - \sigma + 1)\Gamma(\eta - \sigma + 1)}{\Gamma(1 - \sigma)\Gamma(\alpha + \beta + \eta - \sigma + 1)} x^{\sigma - \beta - 1}.$$

*In particular, for  $x > 0$ , we have*

$$(26) \quad (I_{-}^{\alpha} t^{\sigma-1})(x) = \frac{\Gamma(1 - \alpha - \sigma)}{\Gamma(1 - \sigma)} x^{\sigma + \alpha - 1} \quad (0 < \Re(\alpha) < 1 - \Re(\sigma))$$

*and*

$$(27) \quad (K_{\eta, \alpha}^- t^{\sigma-1})(x) = \frac{\Gamma(\eta - \sigma + 1)}{\Gamma(\alpha + \eta - \sigma + 1)} x^{\sigma-1} \quad (\Re(\sigma) < 1 + \Re(\sigma)).$$

**Theorem 2.2.** *Let  $\alpha, \beta, \eta, \sigma, \nu, \omega \in \mathbb{C}$  be such that  $\min\{\Re(p), \Re(q)\} > 0$ ,  $\Re(\nu) > -1$ ,  $\Re(\alpha) > 0$  and  $\Re(\sigma + \nu) > \max\{0, \Re(\beta - \eta)\}$ . Then*

$$(28) \quad \left( I_{0+}^{\alpha, \beta, \eta} \{t^{\sigma-1} J_{\nu, p, q}(\omega t)\} \right) (x)$$

$$\begin{aligned}
&= \sqrt{\pi} x^{\sigma+\nu-\beta-1} \frac{\left(\frac{\omega}{2}\right)^\nu}{\Gamma(\nu+1)} F_{p,q} \left[ \begin{matrix} 1, \frac{1}{2}; \\ \nu+1; \end{matrix} -\frac{\omega^2 x^2}{4} \right] \\
&\quad * {}_2\Psi_3 \left[ \begin{matrix} (\sigma+\nu, 2), (\sigma+\nu+\eta-\beta, 2); \\ (\frac{1}{2}, 1), (\sigma+\nu-\beta, 2), (\sigma+\nu+\alpha+\eta, 2); \end{matrix} -\frac{\omega^2 x^2}{4} \right],
\end{aligned}$$

where  $*$  denotes the Hadamard product in (18) and whose left-sided hypergeometric fractional integral is assumed to be convergent.

*Proof.* Applying (15) to (1) and changing the order of integration and summation, which is valid under the given conditions here, and using (22), we find

$$\begin{aligned}
(29) \quad &\left( I_{0+}^{\alpha,\beta,\eta} \left\{ t^{\sigma-1} J_{\nu,p,q}(\omega t) \right\} \right) (x) \\
&= \frac{\left(\frac{\omega}{2}\right)^\nu}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{B\left(k+\frac{1}{2}, \nu+\frac{1}{2}; p, q\right)}{\left(\frac{1}{2}\right)_k B\left(\frac{1}{2}, \nu+\frac{1}{2}\right) k!} \left(-\frac{\omega^2}{4}\right)^k \left( I_{0+}^{\alpha,\beta,\eta} t^{\sigma+\nu+2k-1} \right) (x) \\
&= x^{\sigma+\nu-\beta-1} \frac{\left(\frac{\omega}{2}\right)^\nu}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{(1)_k B\left(k+\frac{1}{2}, \nu+\frac{1}{2}; p, q\right)}{B\left(\frac{1}{2}, \nu+\frac{1}{2}\right) k!} \\
&\quad \times \frac{\Gamma(\sigma+\nu+2k)\Gamma(\sigma+\nu+\eta-\beta+2k)}{\left(\frac{1}{2}\right)_k \Gamma(\sigma+\nu-\beta+2k)\Gamma(\sigma+\nu+\alpha+\eta+2k) k!} \left(-\frac{\omega^2 x^2}{4}\right)^k.
\end{aligned}$$

Expressing the last summation in (29) in terms of the Hadamard product (18) with the functions (19) and (20), and considering (21) for  $\left(\frac{1}{2}\right)_k$ , we obtain the right-hand side of (28).  $\square$

**Theorem 2.3.** Let  $\alpha, \beta, \eta, \sigma, \nu, \omega \in \mathbb{C}$  be such that  $\min\{\Re(p), \Re(q)\} > 0$ ,  $\Re(\nu) > -1$ ,  $\Re(\alpha) > 0$  and  $\Re(\sigma-\nu) < 1 + \min\{\Re(\beta), \Re(\eta)\}$ . Then

$$\begin{aligned}
(30) \quad &\left( I_-^{\alpha,\beta,\eta} \left\{ t^{\sigma-1} J_{\nu,p,q} \left( \frac{\omega}{t} \right) \right\} \right) (x) \\
&= \sqrt{\pi} x^{\sigma-\nu-\beta-1} \frac{\left(\frac{\omega}{2}\right)^\nu}{\Gamma(\nu+1)} F_{p,q} \left[ \begin{matrix} 1, \frac{1}{2}; \\ \nu+1; \end{matrix} -\frac{\omega^2}{4x^2} \right] \\
&\quad * {}_2\Psi_3 \left[ \begin{matrix} (1+\beta-\sigma+\nu, 2), (1+\eta-\sigma+\nu, 2); \\ (\frac{1}{2}, 1), (1-\sigma+\nu, 2), (1+\alpha+\beta+\eta-\sigma+\nu, 2); \end{matrix} -\frac{\omega^2}{4x^2} \right],
\end{aligned}$$

where  $*$  denotes the Hadamard product in (18) and whose left-sided hypergeometric fractional integral is assumed to be convergent.

*Proof.* Applying (15) to (2) and changing the order of integration and summation, which is valid under the given conditions here, and using (25), we find

$$\begin{aligned}
&\left( I_-^{\alpha,\beta,\eta} \left\{ t^{\sigma-1} J_{\nu,p,q} \left( \frac{\omega}{t} \right) \right\} \right) (x) \\
&= \frac{\left(\frac{\omega}{2}\right)^\nu}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{B\left(k+\frac{1}{2}, \nu+\frac{1}{2}; p, q\right)}{\left(\frac{1}{2}\right)_k B\left(\frac{1}{2}, \nu+\frac{1}{2}\right) k!} \left(-\frac{\omega^2}{4}\right)^k \left( I_-^{\alpha,\beta,\eta} t^{\sigma-\nu-2k-1} \right) (x)
\end{aligned}$$

$$\begin{aligned}
 &= x^{\sigma-\nu-\beta-1} \frac{\left(\frac{\omega}{2}\right)^\nu}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{(1)_k \mathbf{B}\left(k+\frac{1}{2}, \nu+\frac{1}{2}; p, q\right)}{\mathbf{B}\left(\frac{1}{2}, \nu+\frac{1}{2}\right) k!} \\
 &\quad \times \frac{\Gamma(\beta-\sigma+\nu+1+2k)\Gamma(\eta-\sigma+\nu+1+2k)}{\left(\frac{1}{2}\right)_k \Gamma(1-\sigma+\nu+2k)\Gamma(\alpha+\beta+\eta-\sigma+\nu+1+2k) k!} \left(-\frac{\omega^2}{4x^2}\right)^k.
 \end{aligned}$$

Applying a similar argument as in the proof of (28) to the last summation leads to the right-hand side of (30).  $\square$

The results in Theorems 2.2 and 2.3 can be specialized to give a number of identities involving known fractional integral operators. Here, we present four identities as in the following corollaries, without their proofs.

**Corollary 2.4.** *Let  $\alpha, \sigma, \nu, \omega \in \mathbb{C}$  be such that  $\min\{\Re(p), \Re(q)\} > 0$ ,  $\Re(\nu) > -1$ ,  $\Re(\alpha) > 0$ ,  $\Re(\sigma + \nu) > 0$ . Then*

$$\begin{aligned}
 (31) \quad & \left(I_{0+}^\alpha \left\{t^{\sigma-1} J_{\nu, p, q}(\omega t)\right\}\right)(x) \\
 &= \sqrt{\pi} x^{\sigma+\nu+\alpha-1} \frac{\left(\frac{\omega}{2}\right)^\nu}{\Gamma(\nu+1)} \\
 &\quad \times F_{p, q} \left[ \begin{matrix} 1, \frac{1}{2}; \\ \nu+1; \end{matrix} -\frac{\omega^2 x^2}{4} \right] * {}_1\Psi_2 \left[ \begin{matrix} (\sigma+\nu, 2); \\ \left(\frac{1}{2}, 1\right), (\sigma+\nu+\alpha, 2); \end{matrix} -\frac{\omega^2 x^2}{4} \right],
 \end{aligned}$$

where  $*$  denotes the Hadamard product in (18) and whose left-sided hypergeometric fractional integral is assumed to be convergent.

**Corollary 2.5.** *Let  $\alpha, \eta, \sigma, \nu, \omega \in \mathbb{C}$  be such that  $\min\{\Re(p), \Re(q)\} > 0$ ,  $\Re(\nu) > -1$ ,  $\Re(\alpha) > 0$  and  $\Re(\sigma + \nu) > -\Re(\eta)$ . Then*

$$\begin{aligned}
 (32) \quad & \left(I_{\eta, \alpha}^+ \left\{t^{\sigma-1} J_{\nu, p, q}(\omega t)\right\}\right)(x) \\
 &= \sqrt{\pi} x^{\sigma+\nu-1} \frac{\left(\frac{\omega}{2}\right)^\nu}{\Gamma(\nu+1)} \\
 &\quad \times F_{p, q} \left[ \begin{matrix} 1, \frac{1}{2}; \\ \nu+1; \end{matrix} -\frac{\omega^2 x^2}{4} \right] * {}_1\Psi_2 \left[ \begin{matrix} (\sigma+\nu+\eta, 2); \\ \left(\frac{1}{2}, 1\right), (\sigma+\nu+\alpha+\eta, 2); \end{matrix} -\frac{\omega^2 x^2}{4} \right],
 \end{aligned}$$

where  $*$  denotes the Hadamard product in (18) and whose left-sided hypergeometric fractional integral is assumed to be convergent.

**Corollary 2.6.** *Let  $\alpha, \sigma, \nu, \omega \in \mathbb{C}$  be such that  $\min\{\Re(p), \Re(q)\} > 0$ ,  $\Re(\nu) > -1$ ,  $0 < \Re(\alpha) < 1 - \Re(\sigma - \nu)$ . Then*

$$\begin{aligned}
 (33) \quad & \left(I_-^\alpha \left\{t^{\sigma-1} J_{\nu, p, q}\left(\frac{\omega}{t}\right)\right\}\right)(x) \\
 &= \sqrt{\pi} x^{\sigma-\nu+\alpha-1} \frac{\left(\frac{\omega}{2}\right)^\nu}{\Gamma(\nu+1)} \\
 &\quad \times F_{p, q} \left[ \begin{matrix} 1, \frac{1}{2}; \\ \nu+1; \end{matrix} -\frac{\omega^2}{4x^2} \right] * {}_1\Psi_2 \left[ \begin{matrix} (1-\alpha-\sigma+\nu, 2); \\ \left(\frac{1}{2}, 1\right), (1-\sigma+\nu, 2); \end{matrix} -\frac{\omega^2}{4x^2} \right],
 \end{aligned}$$

where  $*$  denotes the Hadamard product in (18) and whose left-sided hypergeometric fractional integral is assumed to be convergent.

**Corollary 2.7.** *Let  $\alpha, \eta, \sigma, \nu, \omega \in \mathbb{C}$  be such that  $\min\{\Re(p), \Re(q)\} > 0$ ,  $\Re(\nu) > -1$ ,  $\Re(\alpha) > 0$  and  $\Re(\sigma - \nu) < 1 + \Re(\eta)$ . Then*

$$(34) \quad \left( K_{\eta, \alpha}^- \left\{ t^{\sigma-1} J_{\nu, p, q} \left( \frac{\omega}{t} \right) \right\} \right) (x) \\ = \sqrt{\pi} x^{\sigma-\nu-1} \frac{\left(\frac{\omega}{2}\right)^\nu}{\Gamma(\nu+1)} \\ \times F_{p, q} \left[ \begin{matrix} 1, \frac{1}{2}; \\ \nu+1; \end{matrix} -\frac{\omega^2}{4x^2} \right] * {}_1\Psi_2 \left[ \begin{matrix} (1+\eta-\sigma+\nu, 2); \\ (\frac{1}{2}, 1), (1+\alpha+\eta-\sigma+\nu, 2); \end{matrix} -\frac{\omega^2}{4x^2} \right],$$

where  $*$  denotes the Hadamard product in (18) and whose left-sided hypergeometric fractional integral is assumed to be convergent.

### 3. Fractional differentiations of the $J_{\nu, p, q}(z)$

Here, composition formulas of the generalized fractional differentiations (3) and (4) involving  $(p, q)$ -extended Bessel function  $J_{\nu, p, q}(z)$  of the first kind of order  $\nu$  are presented. To do this, we begin by giving the following image formulas asserted in Lemma 3.1, without their proofs.

**Lemma 3.1.** *Let  $\alpha, \beta, \eta \in \mathbb{C}$  and  $x \in \mathbb{R}^+$ . Then*

(a) *If  $\Re(\alpha) > 0$  and  $\Re(\sigma) > -\min\{0, \Re(\alpha + \beta + \eta)\}$ , then*

$$(35) \quad (D_{0+}^{\alpha, \beta, \eta} t^{\sigma-1})(x) = \frac{\Gamma(\sigma)\Gamma(\sigma + \alpha + \beta + \eta)}{\Gamma(\sigma + \beta)\Gamma(\sigma + \eta)} x^{\sigma+\beta-1}.$$

*In particular, we have*

$$(36) \quad (D_{0+}^\alpha t^{\sigma-1})(x) = \frac{\Gamma(\sigma)}{\Gamma(\sigma - \alpha)} x^{\sigma-\alpha-1} \quad (\Re(\alpha) > 0, \Re(\sigma) > 0)$$

*and*

$$(37) \quad (D_{\eta, \alpha}^+ t^{\sigma-1})(x) = \frac{\Gamma(\sigma + \alpha + \eta)}{\Gamma(\sigma + \eta)} x^{\sigma-1} \quad (\Re(\alpha) > 0, \Re(\sigma) > -\Re(\alpha + \eta)).$$

(b) *If  $\Re(\alpha) > 0$ ,  $\Re(\sigma) < 1 + \min\{\Re(-\beta - n), \Re(\alpha + \eta)\}$  and  $n = [\Re(\alpha)] + 1$ , then*

$$(38) \quad (D_-^{\alpha, \beta, \eta} t^{\sigma-1})(x) = \frac{\Gamma(1 - \sigma - \beta)\Gamma(1 - \sigma + \alpha + \eta)}{\Gamma(1 - \sigma)\Gamma(1 - \sigma + \eta - \beta)} x^{\sigma+\beta-1}.$$

*In particular, we have*

$$(39) \quad (D_-^\alpha t^{\sigma-1})(x) = \frac{\Gamma(1 - \sigma + \alpha)}{\Gamma(1 - \sigma)} x^{\sigma-\alpha-1} \quad (\Re(\alpha) > 0, \Re(\sigma) < 1 + \Re(\alpha) - n)$$

*and*

$$(40) \quad (D_{\eta, \alpha}^- t^{\sigma-1})(x) = \frac{\Gamma(1 - \sigma + \alpha + \eta)}{\Gamma(1 - \sigma - \eta)} x^{\sigma-1} \quad (\Re(\alpha) > 0, \Re(\sigma) < 1 + \Re(\alpha + \eta) - n).$$



**Theorem 3.2.** Let  $\alpha, \beta, \eta, \sigma, \nu, \omega \in \mathbb{C}$  be such that  $\min\{\Re(p), \Re(q)\} > 0$ ,  $\Re(\nu) > -1$ ,  $\Re(\alpha) \geq 0$  and  $\Re(\sigma + \nu) > -\min\{0, \Re(\alpha + \beta + \eta)\}$ . Then

$$(41) \quad \left( D_{0+}^{\alpha, \beta, \eta} \left\{ t^{\sigma-1} J_{\nu, p, q}(\omega t) \right\} \right) (x) \\ = \sqrt{\pi} x^{\sigma+\nu+\beta-1} \frac{\left(\frac{\omega}{2}\right)^\nu}{\Gamma(\nu+1)} \\ \times F_{p, q} \left[ \begin{matrix} 1, \frac{1}{2}; \\ \nu+1; \end{matrix} -\frac{\omega^2 x^2}{4} \right] * {}_2\Psi_3 \left[ \begin{matrix} (\sigma+\nu, 2), (\sigma+\nu+\alpha+\beta+\eta, 2); \\ (\frac{1}{2}, 1), (\sigma+\nu+\beta, 2), (\sigma+\nu+\eta, 2); \end{matrix} -\frac{\omega^2 x^2}{4} \right],$$

where  $*$  denotes the Hadamard product in (18) and whose left-sided hypergeometric fractional derivative is assumed to be convergent.

*Proof.* Applying the fractional differential operator (3) to (15) and conducting term-by-term fractional differentiation, which is valid under the given conditions here, with the aid of (35), we have

$$(42) \quad \left( D_{0+}^{\alpha, \beta, \eta} \left\{ t^{\sigma-1} J_{\nu, p, q}(\omega t) \right\} \right) (x) \\ = \frac{\left(\frac{\omega}{2}\right)^\nu}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{B\left(k+\frac{1}{2}, \nu+\frac{1}{2}; p, q\right)}{B\left(\frac{1}{2}, \nu+\frac{1}{2}\right) k!} \left(-\frac{\omega^2}{4}\right)^k \left( D_{0+}^{\alpha, \beta, \eta} t^{\sigma+\nu+2k-1} \right) (x) \\ = x^{\sigma+\nu+\beta-1} \frac{\left(\frac{\omega}{2}\right)^\nu}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{(1)_k B\left(k+\frac{1}{2}, \nu+\frac{1}{2}; p, q\right)}{B\left(\frac{1}{2}, \nu+\frac{1}{2}\right) k!} \\ \times \frac{\Gamma(\sigma+\nu+2k)\Gamma(\sigma+\nu+\alpha+\beta+\eta+2k)}{\left(\frac{1}{2}\right)_k \Gamma(\sigma+\nu+\beta+2k)\Gamma(\sigma+\nu+\eta+2k) k!} \left(-\frac{\omega^2 x^2}{4}\right)^k.$$

Expressing the last summation in (42) in terms of the Hadamard product (18) together with the two functions (19) and (20) yields the desired formula (41).  $\square$

**Theorem 3.3.** Let  $\alpha, \beta, \eta, \sigma, \nu, \omega \in \mathbb{C}$  be such that  $\min\{\Re(p), \Re(q)\} > 0$ ,  $\Re(\nu) > -1$ ,  $\Re(\alpha) \geq 0$  and  $\Re(\sigma - \nu) < 1 + \min\{\Re(-\beta - n), \Re(\alpha + \eta)\}$ ,  $n = [\Re(\alpha)] + 1$ . Then

$$(43) \quad \left( D_{-}^{\alpha, \beta, \eta} \left\{ t^{\sigma-1} J_{\nu, p, q} \left( \frac{\omega}{t} \right) \right\} \right) (x) \\ = \sqrt{\pi} x^{\sigma-\nu+\beta-1} \frac{\left(\frac{\omega}{2}\right)^\nu}{\Gamma(\nu+1)} F_{p, q} \left[ \begin{matrix} 1, \frac{1}{2}; \\ \nu+1; \end{matrix} -\frac{\omega^2}{4x^2} \right] \\ * {}_2\Psi_3 \left[ \begin{matrix} (1-\sigma+\nu-\beta, 2), (1-\sigma+\nu+\alpha+\eta, 2); \\ (\frac{1}{2}, 1), (1-\sigma+\nu, 2), (1-\sigma+\nu+\eta-\beta, 2); \end{matrix} -\frac{\omega^2}{4x^2} \right],$$

where  $*$  denotes the Hadamard product in (18) and whose left-sided hypergeometric fractional derivative is assumed to be convergent.

*Proof.* The proof would run parallel to that of Theorem 3.2. We omit the details.  $\square$

Four interesting special cases of the results in Theorems 3.2 and 3.3 are given in the following corollaries, without their proofs.

**Corollary 3.4.** *Let  $\alpha, \sigma, \nu, \omega \in \mathbb{C}$  be such that  $\min\{\Re(p), \Re(q)\} > 0$ ,  $\Re(\nu) > -1$  and  $\Re(\alpha) \geq 0$ ,  $\Re(\sigma + \nu) > 0$ . Then*

$$(44) \quad \begin{aligned} & (D_{0+}^{\alpha} \{t^{\sigma-1} J_{\nu, p, q}(\omega t)\}) (x) \\ &= \sqrt{\pi} x^{\sigma+\nu-\alpha-1} \frac{\left(\frac{\omega}{2}\right)^{\nu}}{\Gamma(\nu+1)} \\ & \quad \times F_{p, q} \left[ \begin{matrix} 1, \frac{1}{2}; \\ \nu+1; \end{matrix} -\frac{\omega^2 x^2}{4} \right] * {}_1\Psi_2 \left[ \begin{matrix} (\sigma+\nu, 2); \\ (\frac{1}{2}, 1), (\sigma+\nu-\alpha, 2); \end{matrix} -\frac{\omega^2 x^2}{4} \right], \end{aligned}$$

where  $*$  denotes the Hadamard product in (18) and whose left-sided hypergeometric fractional derivative is assumed to be convergent.

**Corollary 3.5.** *Let  $\alpha, \eta, \sigma, \nu, \omega \in \mathbb{C}$  be such that  $\min\{\Re(p), \Re(q)\} > 0$ ,  $\Re(\nu) > -1$ ,  $\Re(\alpha) \geq 0$  and  $\Re(\sigma + \nu) > -\Re(\eta + \alpha)$ . Then*

$$(45) \quad \begin{aligned} & (D_{\eta, \alpha}^+ \{t^{\sigma-1} J_{\nu, p, q}(\omega t)\}) (x) \\ &= \sqrt{\pi} x^{\sigma+\nu-1} \frac{\left(\frac{\omega}{2}\right)^{\nu}}{\Gamma(\nu+1)} \\ & \quad \times F_{p, q} \left[ \begin{matrix} 1, \frac{1}{2}; \\ \nu+1; \end{matrix} -\frac{\omega^2 x^2}{4} \right] * {}_1\Psi_2 \left[ \begin{matrix} (\sigma+\nu+\alpha+\eta, 2); \\ (\frac{1}{2}, 1), (\sigma+\nu+\eta, 2); \end{matrix} -\frac{\omega^2 x^2}{4} \right], \end{aligned}$$

where  $*$  denotes the Hadamard product in (18) and whose left-sided hypergeometric fractional derivative is assumed to be convergent.

**Corollary 3.6.** *Let  $\alpha, \sigma, \nu, \omega \in \mathbb{C}$  be such that  $\min\{\Re(p), \Re(q)\} > 0$ ,  $\Re(\nu) > -1$ ,  $\Re(\alpha) \geq 0$ ,  $\Re(\sigma - \nu) < \Re(\alpha) - [\Re(\alpha)]$ . Then*

$$(46) \quad \begin{aligned} & \left( D_-^{\alpha} \left\{ t^{\sigma-1} J_{\nu, p, q} \left( \frac{\omega}{t} \right) \right\} \right) (x) \\ &= \sqrt{\pi} x^{\sigma-\nu-\alpha-1} \frac{\left(\frac{\omega}{2}\right)^{\nu}}{\Gamma(\nu+1)} \\ & \quad \times F_{p, q} \left[ \begin{matrix} 1, \frac{1}{2}; \\ \nu+1; \end{matrix} -\frac{\omega^2}{4x^2} \right] * {}_1\Psi_2 \left[ \begin{matrix} (1-\sigma+\nu+\alpha, 2); \\ (\frac{1}{2}, 1), (1-\sigma+\nu, 2); \end{matrix} -\frac{\omega^2}{4x^2} \right], \end{aligned}$$

where  $*$  denotes the Hadamard product in (18) and whose left-sided hypergeometric fractional derivative is assumed to be convergent.

**Corollary 3.7.** *Let  $\alpha, \eta, \sigma, \nu, \omega \in \mathbb{C}$  be such that  $\min\{\Re(p), \Re(q)\} > 0$ ,  $\Re(\nu) > -1$ ,  $\Re(\alpha) \geq 0$  and  $\Re(\sigma - \nu) < \Re(\alpha + \eta) - [\Re(\alpha)]$ . Then*

$$(47) \quad \begin{aligned} & \left( D_{\eta, \alpha}^- \left\{ t^{\sigma-1} J_{\nu, p, q} \left( \frac{\omega}{t} \right) \right\} \right) (x) \\ &= \sqrt{\pi} x^{\sigma-\nu-1} \frac{\left(\frac{\omega}{2}\right)^{\nu}}{\Gamma(\nu+1)} \end{aligned}$$

$$\times F_{p,q} \left[ \begin{matrix} 1, \frac{1}{2}; \\ \nu + 1; \end{matrix} -\frac{\omega^2}{4x^2} \right] * {}_1\Psi_2 \left[ \begin{matrix} (1 - \sigma + \nu + \alpha + \eta, 2); \\ (\frac{1}{2}, 1), (1 - \sigma + \nu + \eta, 2); \end{matrix} -\frac{\omega^2}{4x^2} \right],$$

where  $*$  denotes the Hadamard product in (18) and whose left-sided hypergeometric fractional derivative is assumed to be convergent.

### Concluding remarks

The results presented in Theorems 2.2-3.3 together with corollaries are sure to be new and potentially useful, mainly because they are expressed in terms of the Hadamard product with two known functions. At least, what seemingly complicated resulting series are expressed in terms of two known functions *means* that certain properties involved in the complicated resulting series can be revealed via those of the two known functions in their respective Hadamard product.

**Acknowledgments.** The authors would like to express their deep-felt thanks for the reviewer's very helpful comments.

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JUNESANG CHOI  
DEPARTMENT OF MATHEMATICS  
DONGGUK UNIVERSITY  
GYEONGJU 38066, KOREA  
*E-mail address:* junesang@mail.dongguk.ac.kr

RAKESH K. PARMAR  
DEPARTMENT OF MATHEMATICS  
GOVERNMENT COLLEGE OF ENGINEERING AND TECHNOLOGY  
BIKANER 334004, RAJASTHAN STATE, INDIA  
*E-mail address:* rakeshparmar27@gmail.com