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b-GENERALIZED DERIVATIONS ON MULTILINEAR POLYNOMIALS IN PRIME RINGS

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Abstract. Let R be a noncommutative prime ring of characteristic different from 2, Q be its maximal right ring of quotients and C be its extended centroid. Suppose that $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over C, $b \in Q$, F a b-generalized derivation of R and d is a nonzero derivation of R such that

$$d([F(f(r)), f(r)]) = 0$$

for all $r = (r_1, \ldots, r_n) \in R^n$. Then one of the following holds:

(1) there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;

(2) there exist $\lambda \in C$ and $p \in Q$ such that $F(x) = \lambda x + px + xp$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued in R.

1. Introduction

Throughout this paper R always denotes an associative prime ring with center $Z(R)$. A ring R is said to be a prime ring if for any $a, b \in R$, $aRb = 0$ implies $a = 0$ or $b = 0$. Q denotes the maximal right ring of quotients of R. Then $C = Z(Q)$ is called the extended centroid of R. It is well known that when R is a prime ring, then Q is also a prime ring and C is a field. We refer the reader to the book [1] for details. The commutator of x and y is denoted by $[x, y]$ and defined by $[x, y] = xy - yx$ for $x, y \in R$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Evidently, for some $a, b \in R$, the map $F(x) = ax + xb$ for all $x \in R$ is an example of generalized derivation which is called as inner generalized derivation of R.

For a subset S of R, a mapping $f : S \to R$ is called commuting (centralizing) on S if $[f(x), x] = 0$ (resp. $[f(x), x] \in Z(R)$) for all $x \in S$. Posner [19] initiated

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the study of commuting and centralizing maps. Posner [19] proved that a prime ring must be commutative, if it possesses a nonzero centralizing derivation. Since then many authors investigated commuting and centralizing maps in different directions.

In [13], Lee and Lee proved that if R is a prime ring, I a nonzero ideal of R and d is a nonzero derivation of R such that $[d(f(r)), f(r)] \in Z(R)$ for all $r = (r_1, \ldots, r_n) \in I^n$, then $f(x_1, \ldots, x_n)$ is central-valued on R, except when char $(R) = 2$ and R satisfies $s_4(x_1, x_2, x_3, x_4)$.

Recently, De Filippis and Di Vincenzo (see [5]) studied the situation when $\delta([d(f(r)), f(r)]) = 0$ for all $r = (r_1, \ldots, r_n) \in R^n$, where d and δ are two derivations of R. The statement of De Filippis and Di Vincenzo's theorem is the following:

Theorem A ([5, Theorem 1]). *Let K be a noncommutative ring with unity, R a prime K-algebra of characteristic different from 2, d and δ two nonzero derivations of R and $f(r_1, \ldots, r_n)$ a multilinear polynomial over K. If*

$$\delta([d(f(r)), f(r)]) = 0$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then $f(r_1, \ldots, r_n)$ is central-valued on R.

Then, De Filippis and Di Vincenzo [6] studied above result replacing derivation d with a generalized derivation F of R. More precisely, authors proved the following:

Theorem B. *Let R be a prime algebra over a commutative ring K with unity, and let $f(x_1, \ldots, x_n)$ be a multilinear polynomial over K, not central valued on R. Suppose that d is a nonzero derivation of R and F is a nonzero generalized derivation of R such that*

$$d([F(f(r)), f(r)]) = 0$$

for all $r = (r_1, \ldots, r_n) \in R^n$. If the characteristic of R is different from 2, then one of the following holds:

(1) *There exists $\lambda \in C$, the extended centroid of R such that $F(x) = \lambda x$ for all $x \in R$;*
(2) *There exist $a \in U$, the Utumi quotient ring of R, and $\lambda \in C$ such that $F(x) = ax + xa + \lambda x$ for all $x \in R$, and $f(x_1, \ldots, x_n)^2$ is central valued on R.*

Our motivation in the present paper is to consider F as a b-generalized derivation of R. Let $b \in Q$. An additive map $G : R \to Q$ is called a b-generalized derivation of R if $g(xy) = g(x)y + bxd(y)$ holds for all $x, y \in R$, where $d : R \to Q$ is an additive map. It is proved in [11] that if R is a prime ring and $b \neq 0$, then the associated map d must be a derivation of R. Evidently, a generalized derivation is a 1-generalized derivation. For some $a, b, c \in Q$, the map $F(x) = ax + bxc \in Q$ is an example of b-generalized derivation of R, which we call as inner b-generalized derivation of R. The b-generalized derivations appeared canonically in [3] and were introduced and studied recently in [11, 15, 17].

More precisely, we prove the following theorem.

Theorem 1.1. *Let R be a noncommutative prime ring of characteristic different from 2, Q be its maximal right ring of quotients and C be its extended centroid. Suppose that $f(x_1,\dots,x_n)$ be a noncentral multilinear polynomial over C, $b \in Q$, F a b-generalized derivation of R and d is a nonzero derivation of R such that*

$$d([F(f(r)), f(r)]) = 0$$

for all $r = (r_1,\dots,r_n) \in R^n$. Then one of the following holds:

1. *there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;*
2. *there exist $\lambda \in C$ and $p \in Q$ such that $F(x) = \lambda x + px + xp$ for all $x \in R$ with $f(x_1,\dots,x_n)^2$ is central valued in R.*

As an application of above theorem, we have the following corollary which is a generalization of particular result of [4].

Corollary 1.2. *Let R be a noncommutative prime ring of characteristic different from 2, Q be its maximal right ring of quotients and C be its extended centroid. Suppose that $f(x_1,\dots,x_n)$ be a noncentral multilinear polynomial over C, $b \in Q$, F a b-generalized derivation of R and d is a nonzero derivation of R such that*

$$[F(f(r)), f(r)] \in C$$

for all $r = (r_1,\dots,r_n) \in R^n$. Then one of the following holds:

1. *there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;*
2. *there exist $\lambda \in C$ and $p \in Q$ such that $F(x) = \lambda x + px + xp$ for all $x \in R$ with $f(x_1,\dots,x_n)^2$ is central valued in R.*

Let σ be an automorphism of R. σ is said to be inner automorphism of R, if there exists an invertible element $p \in Q$ such that $\sigma(x) = pxp^{-1}$ for all $x \in R$. If σ is not inner, we say σ as an outer automorphism of R. An additive map $d : R \to R$ is called a σ-derivation, if $d(xy) = d(x)y + \sigma(x)d(y)$ holds for all $x, y \in R$. For some $a \in Q$, $d(x) = ax - \sigma(x)a$ is an example of σ-derivation, which is called as inner σ-derivation. An additive map $G : R \to R$ is called a generalized σ-derivation, if there exists a σ-derivation d such that $G(xy) = G(x)y + \sigma(x)d(y)$ holds for all $x, y \in R$. Note that generalized 1_R-derivation is called as generalized derivation, where 1_R denotes the identity automorphism of R. Generally, generalized σ-derivation is called as generalized skew derivation. If for some invertible $b \in Q$, $\sigma(x) = bxb^{-1}$ for all $x \in R$, and d is inner σ-derivation of R, then $G(xy) = G(x)y + \sigma(x)d(y) = G(x)y + bxb^{-1}(ay - byb^{-1}a) = G(x)y + bx(b^{-1}ay - yb^{-1}a) = G(x)y + bx[b^{-1}a, y]$ for all $x, y \in R$, is nothing but a b-generalized derivation of R with associated derivation $d(x) = [b^{-1}a, x]$ for all $x \in R$. It is very easy to prove that any generalized σ-derivation of R with associated σ-derivation d, where $\sigma(x) = bxb^{-1}$ for all $x \in R$ and $b \in Q$ is an inner automorphism, is a b-generalized derivation of R with the associated map $b^{-1}d$.

Thus as an application of Theorem 1.1, we have the following corollary.

Corollary 1.3. *Let R be a noncommutative prime ring of characteristic different from 2, Q be its maximal right ring of quotients, C be its extended centroid and $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over C. Suppose that F is a generalized σ-derivation of R with σ an inner automorphism of R and d is a nonzero derivation of R such that*

$$d([F(f(r)), f(r)]) = 0$$

for all $r = (r_1, \ldots, r_n) \in R^n$. Then one of the following holds:

(1) *there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;*
(2) *there exist $\lambda \in C$ and $p \in Q$ such that $F(x) = \lambda x + px + xp$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued in R.*

Similarly, following the corollary also holds.

Corollary 1.4. *Let R be a noncommutative prime ring of characteristic different from 2, Q be its maximal right ring of quotients, C be its extended centroid and $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over C. Suppose that F is a generalized σ-derivation of R with σ an inner automorphism of R such that*

$$[F(f(r)), f(r)] \in C$$

for all $r = (r_1, \ldots, r_n) \in R^n$. Then one of the following holds:

(1) *there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;*
(2) *there exist $\lambda \in C$ and $p \in Q$ such that $F(x) = \lambda x + px + xp$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued in R.*

2. The case of inner b-generalized derivation

First we consider the case when F is the inner b-generalized derivation and d is inner derivation of R. Let $F(x) = ax + bxq$ for all $x \in R$ and $d(x) = [c, x]$ for all $x \in R$, for some $a, b, c, q \in Q$. Then by our hypothesis, we have

$$[c, [ar + brq, r]] = 0$$

for all $r \in f(R)$. This can be re-written as

$$car^2 + cbrqr - crar - crbrq - ar^2c - brqrc + rarc + rbrqc = 0$$

for all $r \in f(R)$.

We investigate this generalized polynomial identity in prime ring. In all that follows, let R be a prime ring with extended centroid C, char $(R) \neq 2$ and $c \notin C$. Moreover, we assume that $f(x_1, \ldots, x_n)$ is a multilinear polynomial over C which is not central valued on R.

Lemma 2.1. *If $b \in C$, then either $a, bq \in C$ or $a - bq \in C$ with $f(x_1, \dots, x_n)^2$ is central valued in R.*

Proof. If $b \in C$, then our hypothesis becomes

$$[c, [ar + rbq, r]] = 0$$

for all $r \in f(R)$. In this case by [6], one of the following holds: (i) $a, bq \in C$; (ii) $a - bq \in C$ with $f(x_1, \dots, x_n)^2$ is central valued. □

Lemma 2.2. *If $q \in C$, then $a + bq \in C$.*

Proof. If $q \in C$, then our hypothesis becomes

$$[c, [(a + bq)r, r]] = 0,$$

that is,

$$[c, [a + bq, r]r] = 0$$

for all $r \in f(R)$. In this case by [8, Corollary 2.9], $a + bq \in C$. □

Lemma 2.3 ([6, Lemma 1]). *Let C be an infinite field and $m \geq 2$. If $A_1, \dots, A_k$ are not scalar matrices in $M_m(C)$, then there exists some invertible matrix $P \in M_m(C)$ such that any matrices $PA_1P^{-1}, \dots, PA_kP^{-1}$ have all non-zero entries.*

Proposition 2.4. *Let $R = M_m(C)$, $m \geq 2$, be the ring of all $m \times m$ matrices over the infinite field C, $f(x_1, \dots, x_n)$ a non-central multilinear polynomial over C and $a, b, c, q \in R$. If*

$$car^2 + cbrqr - crar - crbrq - ar^2c - brqrc + rarc + rbrqc = 0$$

for all $r \in f(R)$, then either b or c or q are central.

Proof. By our assumption R satisfies the generalized polynomial identity

$$\begin{aligned} &caf(r_1, \dots, r_n)^2 + cbf(r_1, \dots, r_n)qf(r_1, \dots, r_n) \\ &- cf(r_1, \dots, r_n)af(r_1, \dots, r_n) - cf(r_1, \dots, r_n)bf(r_1, \dots, r_n)q \\ &- af(r_1, \dots, r_n)^2c - bf(r_1, \dots, r_n)qf(r_1, \dots, r_n)c \\ &+ f(r_1, \dots, r_n)af(r_1, \dots, r_n)c + f(r_1, \dots, r_n)bf(r_1, \dots, r_n)qc = 0. \end{aligned} \tag{1}$$

We assume first that $b \notin Z(R)$, $c \notin Z(R)$ and $q \notin Z(R)$. Now we shall show that this case leads to a contradiction.

Since $b \notin Z(R)$, $c \notin Z(R)$ and $q \notin Z(R)$, by Lemma 2.3 there exists a C-automorphism ϕ of $M_m(C)$ such that $\phi(b)$, $\phi(c)$ and $\phi(q)$ have all non-zero entries. Clearly R must satisfies the condition

$$\begin{aligned} &\phi(ca)f(r_1, \dots, r_n)^2 + \phi(cb)f(r_1, \dots, r_n)\phi(q)f(r_1, \dots, r_n) \\ &- \phi(c)f(r_1, \dots, r_n)\phi(a)f(r_1, \dots, r_n) \\ &- \phi(c)f(r_1, \dots, r_n)\phi(b)f(r_1, \dots, r_n)\phi(q) \\ &- \phi(a)f(r_1, \dots, r_n)^2\phi(c) - \phi(b)f(r_1, \dots, r_n)\phi(q)f(r_1, \dots, r_n)\phi(c) \end{aligned}$$

$$+ f(r_1,\dots,r_n)\phi(a)f(r_1,\dots,r_n)\phi(c)$$
$$+ f(r_1,\dots,r_n)\phi(b)f(r_1,\dots,r_n)\phi(qc) = 0. \tag{2}$$

Here e_{kl} denotes the usual matrix unit with 1 in (k,l)-entry and zero elsewhere. Since $f(x_1,\dots,x_n)$ is not central, by [14] (see also [16]), there exist $u_1,\dots,u_n \in M_m(C)$ and $0 \neq \gamma \in C$ such that $f(u_1,\dots,u_n) = \gamma e_{kl}$, with $k \neq l$. Moreover, since the set $\{f(r_1,\dots,r_n) : r_1,\dots,r_n \in M_m(C)\}$ is invariant under the action of all C-automorphisms of $M_m(C)$, then for any $i \neq j$ there exist $r_1,\dots,r_n \in M_m(C)$ such that $f(r_1,\dots,r_n) = \gamma e_{ij}$, where $0 \neq \gamma \in C$. Hence by (2) we have

$$\begin{aligned} &\phi(cb)e_{ij}\phi(q)e_{ij} - \phi(c)e_{ij}\phi(a)e_{ij} - \phi(c)e_{ij}\phi(b)e_{ij}\phi(q) \\ &- \phi(b)e_{ij}\phi(q)e_{ij}\phi(c) + e_{ij}\phi(a)e_{ij}\phi(c) + e_{ij}\phi(b)e_{ij}\phi(qc) = 0 \end{aligned} \tag{3}$$

and then left and right multiplying by e_{ij}, it follows $2e_{ij}\phi(c)e_{ij}\phi(b)e_{ij}\phi(q)e_{ij} = 0$, which is a contradiction, since $\phi(b)$, $\phi(c)$ and $\phi(q)$ have all non-zero entries. Thus we conclude that either b or c or q are central. □

Proposition 2.5. *Let $R = M_m(C)$, $m \geq 2$ be the ring of all matrices over the field C with char $(R) \neq 2$ and $f(x_1,\dots,x_n)$ a non-central multilinear polynomial over C and $a,b,c,q \in R$. If*

$$car^2 + cbrqr - crar - crbrq - ar^2c - brqrc + rarc + rbrqc = 0$$

for all $r \in f(R)$, then either b or c or q are central.

Proof. If one assumes that C is infinite, then the conclusions follow by Proposition 2.4.

Now let C be finite and K be an infinite field which is an extension of the field C. Let $\overline{R} = M_m(K) \cong R \otimes_C K$. Notice that the multilinear polynomial $f(x_1,\dots,x_n)$ is central-valued on R if and only if it is central-valued on $\overline{R}$. Consider the generalized polynomial

$$\begin{aligned} P(r_1,\dots,r_n) = {} & caf(r_1,\dots,r_n)^2 + cbf(r_1,\dots,r_n)qf(r_1,\dots,r_n) \\ & - cf(r_1,\dots,r_n)af(r_1,\dots,r_n) - cf(r_1,\dots,r_n)bf(r_1,\dots,r_n)q \\ & - af(r_1,\dots,r_n)^2c - bf(r_1,\dots,r_n)qf(r_1,\dots,r_n)c \\ & + f(r_1,\dots,r_n)af(r_1,\dots,r_n)c + f(r_1,\dots,r_n)bf(r_1,\dots,r_n)qc \\ = {} & 0 \end{aligned} \tag{4}$$

which is a generalized polynomial identity for R.

Moreover, it is a multi-homogeneous of multi-degree $(2,\dots,2)$ in the indeterminates $x_1,\dots,x_n$.

Hence the complete linearization of $P(x_1,\dots,x_n)$ is a multilinear generalized polynomial $\Theta(x_1,\dots,x_n,y_1,\dots,y_n)$ in $2n$ indeterminates, moreover

$$\Theta(x_1,\dots,x_n,x_1,\dots,x_n) = 2^n P(x_1,\dots,x_n).$$

Clearly the multilinear polynomial $\Theta(x_1,\dots,x_n,y_1,\dots,y_n)$ is a generalized polynomial identity for R and $\overline{R}$ too. Since $char(C) \neq 2$ we obtain $P(r_1,\dots,r_n) = 0$ for all $r_1,\dots,r_n \in \overline{R}$ and then conclusion follows from Proposition 2.4. □

Corollary 2.6. *Let $R = M_m(C)$, $m \geq 2$ be the ring of all matrices over the field C with char $(R) \neq 2$ and $a, b, c, q \in R$. If*

$$car^2 + cbrqr - crar - crbrq - ar^2c - brqrc + rarc + rbrqc = 0$$

for all $r \in R$, then either b or c or q are central.

Above corollary can be rewritten as

Corollary 2.7. *Let $R = M_m(C)$, $m \geq 2$ be the ring of all matrices over the field C with char $(R) \neq 2$ and $a_1, a_2, a_3, a_4, a_5, a_6, a_7 \in R$. If*

$$a_1r^2 + a_2ra_3r - a_5ra_4r - a_5ra_6ra_3 - a_4r^2a_5 - a_6ra_3ra_5 + ra_4ra_5 + ra_6ra_7 = 0$$

for all $r \in R$, then either a_3 or a_5 or a_6 are central.

Lemma 2.8. *Let R be a primitive ring, which is isomorphic to a dense ring of linear transformations of a vector space V over C, such that $dim_C V = \infty$. Let $a_1, a_2, a_3, a_4, a_5, a_6, a_7 \in R$. If*

$$a_1r^2 + a_2ra_3r - a_5ra_4r - a_5ra_6ra_3 - a_4r^2a_5 - a_6ra_3ra_5 + ra_4ra_5 + ra_6ra_7 = 0$$

for all $x \in R$, then either a_3 or a_5 or a_6 are central

Proof. We assume that a_3, a_5 and a_6 are noncentral central. Since V is infinite dimensional over C, for any $e = e^2 \in Soc(R)$, we have $eRe \cong M_k(C)$ with $k = \dim_C Ve$. Since $a_3 \notin C$, $a_5 \notin C$ and $a_6 \notin C$, they do not centralize the nonzero ideal $Soc(R)$ of R, so $a_3h_0 \neq h_0a_3$, $a_5h_1 \neq h_1a_5$ and $a_6h_2 \neq h_2a_6$ for some $h_0, h_1, h_2 \in Soc(R)$. By Litoff's theorem [12, p. 280] there exists an idempotent $e \in Soc(R)$ such that $h_0, h_1, h_2, h_0a_3, a_3h_0, h_1a_5, a_5h_1, h_2a_6, a_6h_2$ are all in eRe. We have $eRe \cong M_k(C)$ where $k = \dim_C Ve$. Since R satisfies GPI $e(a_1(ere)^2 + a_2erea_3ere - a_5erea_4ere - a_5erea_6erea_3 - a_4(ere)^2a_5 - a_6erea_3erea_5 + erea_4erea_5 + erea_6erea_7)e = 0$, the subring eRe satisfies the GPI

$$ea_1er^2 + ea_2erea_3er - ea_5erea_4er - ea_5erea_6erea_3e - ea_4er^2ea_5e$$
$$- ea_6erea_3erea_5e + rea_4erea_5e + rea_6erea_7e = 0.$$

Then by above finite dimensional case, we conclude that either $ea_3e \in Z(eRe)$ or $ea_5e \in Z(eRe)$ or $ea_6e \in Z(eRe)$. Then

$$a_3h_0 = ea_3h_0 = ea_3eh_0 = h_0ea_3e = h_0a_3e = h_0a_3,$$

$$a_5h_1 = ea_5h_1 = ea_5eh_1 = h_1ea_5e = h_1a_5e = h_1a_5,$$

and

$$a_6h_2 = ea_6h_2 = ea_6eh_2 = h_2ea_6e = h_2a_6e = h_2a_6,$$

All the cases lead to the contradiction. □

Lemma 2.9. *Let R be a noncommutative prime ring of characteristic different from 2, Q be its maximal right ring of quotients, C be its extended centroid and $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over C. Suppose for some*

$a, b, c, q \in Q$ that $F(x) = ax + bxq$ for all $x \in R$ and $d(x) = [c, x]$ for all $x \in R$ with $c \notin C$. If

$$d([F(f(r)), f(r)]) = 0$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:

(i) *$b \in C$ and $a, bq \in C$;*
(ii) *$b \in C$, $a - bq \in C$ with $f(x_1, \ldots, x_n)^2$ is central valued in R;*
(iii) *$q \in C$, $a + bq \in C$.*

Proof. If $b \in C$ or $q \in C$, then result follows by Lemma 2.1 and Lemma 2.2 respectively. Thus we assume that $b \notin C$ and $q \notin C$.

By hypothesis, we have

$$\Psi(x_1, \ldots, x_n) = [c, [af(x_1, \ldots, x_n) + bf(x_1, \ldots, x_n)q, f(x_1, \ldots, x_n)]] = 0 \tag{5}$$

for all $x_1, \ldots, x_n \in R$. Since R and Q satisfy same generalized polynomial identities (see [2]), Q satisfies $\Psi(x_1, \ldots, x_n) = 0$. Since $c \notin C$, $b \notin C$ and $q \notin C$, $\Psi(x_1, \ldots, x_n)$ is a non-trivial GPI for Q. By the well known Martindale's theorem [18], Q is then a primitive ring with nonzero socle and with C as its associated division ring. Then, by Jacobson's theorem [9, p. 75], Q is isomorphic to a dense ring of linear transformations of a vector space V over C. Assume first that V is finite dimensional over C, that is, $\dim_C V = m$. By density of R, we have $R \cong M_m(C)$. Since $f(r_1, \ldots, r_n)$ is not central valued on R, R must be noncommutative and so $m \geq 2$. In this case, by Proposition 2.5, we get that b or q or c are in C, a contradiction.

If V is infinite dimensional over C, then by Lemma 2 in [20], the set $f(Q)$ is dense on R. Then by hypothesis, Q satisfies

$$[c, [ar + brq, r]] = 0, \tag{6}$$

which gives

$$car^2 + cbrqr - crar - crbrq - ar^2c - brqrc + rarc + rbrqc = 0.$$

Then by Lemma 2.8, we conclude that either $b \in C$ or $q \in C$ or $c \in C$, which leads to a contradiction. □

3. Result on *b*-generalized derivations

Lemma 3.1. *Let R be a noncommutative prime ring of characteristic different from 2, Q be its maximal right ring of quotients and C be its extended centroid. Suppose that $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over C, $b \in Q$, F a b-generalized derivation of R and $c \in R - C$ such that*

$$[c, [F(f(r)), f(r)]] = 0$$

for all $r = (r_1, \ldots, r_n) \in R^n$. Then one of the following holds:

(i) *there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;*
(ii) *there exist $\lambda \in C$ and $p \in Q$ such that $F(x) = \lambda x + px + xp$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued in R.*

Proof. By [11, Theorem 2.3], there exist a derivation $d : R \to Q$ and $a \in Q$ such that $F(x) = ax + bd(x)$ for all $x \in R$. By assumption,

$$[c, [af(r) + bd(f(r)), f(r)]] = 0$$

for all $r = (r_1, \dots, r_n) \in R^n$.

If d is an inner derivation, that is $d(x) = [p, x]$ for all $x \in R$ and for some $p \in Q$, then $F(x) = (a + bp)x - bxp$ for all $x \in R$ and hence by Lemma 2.9, we have:

(i) $a + bp, b, bp \in C$. In this case $F(x) = ax$ for all $x \in R$, where $a \in C$.
(ii) $b \in C$, $a + 2bp \in C$ and $f(x_1, \dots, x_n)^2$ is central valued in R. Let $a + 2bp = \lambda \in C$. Then $F(x) = \lambda x - bpx - xbp$ for all $x \in R$.
(iii) $p \in C$ and $a \in C$. In this case also $F(x) = ax$ for all $x \in R$, where $a \in C$.

Next assume that d is an outer derivation of R. It is well know that any derivation of R can be uniquely extended to a derivation of Q (see [14, Lemma 2]). By hypothesis, we have

$$[c, [af(r) + bd(f(r)), f(r)]] = 0$$

for all $r = (r_1, \dots, r_n) \in R^n$, which gives

$$[c, [af(r_1, \dots, r_n) + bf^d(r_1, \dots, r_n) + b\sum_i f(r_1, \dots, d(r_i), \dots, r_n), f(r_1, \dots, r_n)]] = 0$$

for all $r_1, \dots, r_n \in Q$ by [1, Theorem 6.4.4]. By Kharchenko's Theorem [10], Q satisfies

$$[c, [af(r_1, \dots, r_n) + bf^d(r_1, \dots, r_n) + b\sum_i f(r_1, \dots, s_i, \dots, r_n), f(r_1, \dots, r_n)]] = 0.$$

In particular, Q satisfies the blended component

$$[c, [b\sum_i f(r_1, \dots, s_i, \dots, r_n), f(r_1, \dots, r_n)]] = 0. \tag{7}$$

Assuming $s_1 = r_1$ and $s_2 = \cdots = s_n = 0$, Q satisfies

$$[c, [bf(r_1, \dots, r_n), f(r_1, \dots, r_n)]] = 0 \tag{8}$$

that is

$$[c, [b, f(r_1, \dots, r_n)]f(r_1, \dots, r_n)] = 0. \tag{9}$$

By [8, Corollary 2.9], since $f(r_1, \dots, r_n)$ is noncentral valued in R and $c \notin C$, we have $b \in C$. Then (7) yields

$$[c, [\sum_i f(r_1, \dots, s_i, \dots, r_n), f(r_1, \dots, r_n)]] = 0. \tag{10}$$

Replacing s_i with $[q, r_i]$ for some $q \notin C$, we get from above relation that Q satisfies

$$[c, [[q, f(r_1, \ldots, \ldots, \ldots, r_n)], f(r_1, \ldots, r_n)]] = 0. \tag{11}$$

By [5, Theorem 1], either $c \in C$ or $q \in C$, a contradiction. □

Theorem 3.2. *Let R be a noncommutative prime ring of characteristic different from 2, Q be its maximal right ring of quotients and C be its extended centroid. Suppose that $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over C, $b \in Q$, F a b-generalized derivation of R and d is a nonzero derivation of R such that*

$$d([F(f(r)), f(r)]) = 0$$

for all $r = (r_1, \ldots, r_n) \in R^n$. Then one of the following holds:

(i) *there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;*
(ii) *there exist $\lambda \in C$ and $p \in Q$ such that $F(x) = \lambda x + px + xp$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued in R.*

Proof. By [11, Theorem 2.3], there exist a derivation $\delta : R \to Q$ and $a \in Q$ such that $F(x) = ax + b\delta(x)$ for all $x \in R$. If d is inner derivation of R, then result follows by Lemma 3.1. Thus we assume that d is outer derivation of R. By hypothesis R satisfies

$$d([af(r_1, \ldots, r_n) + b\delta(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)]) = 0. \tag{12}$$

Since any derivation of R can be uniquely extended to a derivation of Q (see [14, Lemma 2]), by [14] this differential identity is also satisfied by Q.
Case-I: Assume that d and δ are C-dependent modulo inner derivations of Q, say $\alpha d + \beta\delta = ad_q$, where $\alpha, \beta \in C$, $q \in Q$ and $ad_q(x) = [q, x]$ for all $x \in Q$.

Subcase-i: *Let $\alpha \neq 0$.*
Then $d(x) = \lambda\delta(x) + [c, x]$ for all $x \in Q$, where $\lambda = -\beta\alpha^{-1}$ and $c = \alpha^{-1}q$. Then d can not be inner derivation of Q. From (12), we obtain

$$\lambda\delta([af(r) + b\delta(f(r)), f(r)]) + [c, [af(r) + b\delta(f(r)), f(r)]] = 0 \tag{13}$$

that is,

$$\begin{aligned} &\lambda[af(r) + b\delta(f(r)), \delta(f(r))] \\ &+ \lambda[\delta(a)f(r) + a\delta(f(r)) + \delta(b)\delta(f(r)) + b\delta^2(f(r)), f(r)] \\ &+ [c, [af(r) + b\delta(f(r)), f(r)]] = 0 \end{aligned} \tag{14}$$

for all $r = (r_1, \ldots, r_n) \in Q^n$. Let $f^{\delta}(r_1, \ldots, r_n)$ and $f^{\delta^2}(r_1, \ldots, r_n)$ be the polynomials obtained from $f(r_1, \ldots, r_n)$ replacing each coefficients α_σ with $\delta(\alpha_\sigma)$ and $\delta^2(\alpha_\sigma)$ respectively. Then we have

$$\delta(f(r_1, \ldots, r_n)) = f^{\delta}(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, \delta(r_i), \ldots, r_n)$$

and

$$\delta^2(f(r_1,\dots,r_n)) = f^{\delta^2}(r_1,\dots,r_n) + 2\sum_i f^{\delta}(r_1,\dots,\delta(r_i),\dots,r_n)$$
$$+ \sum_i f(r_1,\dots,\delta^2(r_i),\dots,r_n)$$
$$+ \sum_{i\neq j} f(r_1,\dots,\delta(r_i),\dots,\delta(r_j),\dots,r_n).$$

By applying Kharchenko's Theorem [10] to (14), we can replace $\delta(f(r_1,\dots,r_n))$ with $f^{\delta}(r_1,\dots,r_n) + \sum_i f(r_1,\dots,y_i,\dots,r_n)$ and $\delta^2(f(r_1,\dots,r_n))$ with

$$f^{\delta^2}(r_1,\dots,r_n) + 2\sum_i f^{\delta}(r_1,\dots,y_i,\dots,r_n)$$
$$+ \sum_i f(r_1,\dots,t_i,\dots,r_n) + \sum_{i\neq j} f(r_1,\dots,y_i,\dots,y_j,\dots,r_n)$$

in (14) and then Q satisfies blended component

$$\lambda[b\sum_i f(r_1,\dots,t_i,\dots,r_n), f(r_1,\dots,r_n)] = 0. \tag{15}$$

In particular, for $t_2 = \dots = t_n = 0$ and $t_1 = r_1$, Q satisfies

$$\lambda[bf(r_1,\dots,r_n), f(r_1,\dots,r_n)] = 0 \tag{16}$$

which is

$$[\lambda b, f(r_1,\dots,r_n)]f(r_1,\dots,r_n) = 0. \tag{17}$$

By [7], it yields $\lambda b \in C$.

Replacing t_i with $[q, r_i]$ for some $q \notin C$ in (15) and then using $\lambda b \in C$, we have that Q satisfies

$$[\lambda bq, f(r_1,\dots,r_n)]_2 = 0. \tag{18}$$

By [13, Theorem], this implies $\lambda bq \in C$. Since $q \notin C$, we conclude that $\lambda b = 0$. This implies $\lambda = 0$ or $b = 0$. Both case leads to a contradiction.

Subcase-ii: *Let* $\alpha = 0$.

Then $\delta(x) = [c, x]$ for all $x \in Q$, where $c = \beta^{-1}q$. From (12), we obtain

$$d([af(r) + b[c, f(r)], f(r)]) = 0 \tag{19}$$

that is

$$\begin{aligned}&[af(r) + b[c, f(r)], d(f(r))] + [d(a)f(r) + ad(f(r)), f(r)]\\ &+ [d(b)[c, f(r)] + b[d(c), f(r)] + b[c, d(f(r))], f(r)] = 0\end{aligned} \tag{20}$$

for all $r = (r_1,\dots,r_n) \in Q^n$.

Since

$$d(f(r_1,\dots,r_n)) = f^{d}(r_1,\dots,r_n) + \sum_i f(r_1,\dots,d(r_i),\dots,r_n)$$

by Kharchenko's Theorem [10], we can replace $d(f(r_1,\dots,r_n))$ by $f^d(r_1,\dots,r_n)$ $+\sum_i f(r_1,\dots,y_i,\dots,r_n)$ in (20) and then Q satisfies blended component

$$\begin{aligned}&[af(r_1,\dots,r_n)+b[c,f(r_1,\dots,r_n)],\sum_i f(r_1,\dots,y_i,\dots,r_n)]\\ &+[a\sum_i f(r_1,\dots,y_i,\dots,r_n),f(r_1,\dots,r_n)]\\ &+[b[c,\sum_i f(r_1,\dots,y_i,\dots,r_n)],f(r_1,\dots,r_n)]=0.\end{aligned}\tag{21}$$

In particular, for $y_1=r_1$ and $y_2=\dots=y_n=0$, we have

$$2[af(r)+b[c,f(r)],f(r)]=0\tag{22}$$

for all $r=(r_1,\dots,r_n)\in Q^n$. Since char $(R)\neq 2$, this can be written as

$$[(a+bc)f(r)-bf(r)c,f(r)]=0\tag{23}$$

for all $r=(r_1,\dots,r_n)\in Q^n$.

By Lemma 2.9, one of the following holds: (i) $a+bc,b,bc\in C$, that is $a,b,bc\in C$. In this case $F(x)=ax+b\delta(x)=ax+b[c,x]=ax$ for all $x\in R$, which is our conclusion (1). (ii) $b,a+2bc\in C$ and $f(r_1,\dots,r_n)^2$ is central valued. In this case $F(x)=ax+b\delta(x)=ax+b[c,x]=ax+[bc,x]=(a+bc)x-x(bc)=(a+2bc)x-bcx-xbc$ for all $x\in R$. This gives conclusion (2). (iii) $c,a\in C$. In this case $F(x)=ax+b\delta(x)=ax+b[c,x]=ax$ for all $x\in R$ which is conclusion (1).

<u>Case-II</u>: Assume next that d and δ are C-independent modulo inner derivations of Q.

From (12) we have

$$\begin{aligned}&[af(r)+b\delta(f(r)),d(f(r))]\\ &+([d(a)f(r)+ad(f(r))+d(b)\delta(f(r))+b(d\delta)(f(r)),f(r)])=0\end{aligned}\tag{24}$$

for all $r=(r_1,\dots,r_n)\in Q^n$. By applying Kharchenko's theorem [10] to (24), we can replace $d(f(r_1,\dots,r_n))$ with $f^d(r_1,\dots,r_n)+\sum_i f(r_1,\dots,y_i,\dots,r_n)$, $\delta(f(r_1,\dots,r_n))$ with $f^\delta(r_1,\dots,r_n)+\sum_i f(r_1,\dots,s_i,\dots,r_n)$ and $d\delta(f(r_1,\dots,r_n))$ with

$$\begin{aligned}&f^{d\delta}(r_1,\dots,r_n)+\sum_i f^\delta(r_1,\dots,s_i,\dots,r_n)+\sum_i f^d(r_1,\dots,y_i,\dots,r_n)\\ &+\sum_i f(r_1,\dots,t_i,\dots,r_n)+\sum_i f(r_1,\dots,y_i,\dots,s_j,\dots,r_n)\end{aligned}$$

in (24) and then Q satisfies blended component

$$[b\sum_i f(r_1,\dots,t_i,\dots,r_n),f(r_1,\dots,r_n)]=0.\tag{25}$$

Replacing $t_1 = r_1$ and $t_2 = \cdots = t_n = 0$ in (25), we have

$$[b, f(r_1, \ldots, r_n)]f(r_1, \ldots, r_n) = 0$$

for all $r_1, \ldots, r_n \in Q$. By [7], this yields $b \in C$. Since $b \neq 0$, again (25) yields

$$[\sum_i f(r_1, \ldots, t_i, \ldots, r_n), f(r_1, \ldots, r_n)] = 0 \tag{26}$$

for all $r_1, \ldots, r_n \in Q$.

Let $q \notin C$ be an element of Q. Then replacing t_i with $[q, r_i]$, we have that

$$[\sum_{i=0}^{n} f(r_1, \ldots, [q, r_i], \ldots, r_n), f(r_1, \ldots, r_n)] = 0$$

which gives,

$$[q, f(r_1, \ldots, r_n)]_2 = 0$$

for all $r_1, \ldots, r_n \in R$ implying $f(r_1, \ldots, r_n)$ is central-valued on R [13, Theorem], a contradiction. □

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