GENERIC LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE KAEHLER MANIFOLD WITH A QUARTER-SYMMETRIC METRIC CONNECTION

Dae Ho Jin

Abstract. Jin studied lightlike hypersurfaces of an indefinite Kaehler manifold [6, 8] or indefinite trans-Sasakian manifold [7] with a quarter-symmetric metric connection. Jin also studied generic lightlike submanifolds of an indefinite trans-Sasakian manifold with a quarter-symmetric metric connection [10]. We study generic lightlike submanifolds of an indefinite Kaehler manifold with a quarter-symmetric metric connection.

1. Introduction

A lightlike submanifold M of an indefinite almost complex manifold $\overline{M}$ is called *generic* if there exists a screen distribution $S(TM)$ of M such that

$$(1.1) \qquad J(S(TM)^\perp) \subset S(TM),$$

where $S(TM)^\perp$ is the orthogonal complement of $S(TM)$ in the tangent bundle $T\overline{M}$ of $\overline{M}$. The generic lightlike submanifold was introduced by Jin-Lee [11] and later, studied by several authors [3–5,10,12]. The theory of generic lightlike submanifolds is an extension of that of lightlike hypersurfaces.

A linear connection $\overline{\nabla}$ on a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is said to be a *quarter-symmetric connection* if its torsion tensor $\overline{T}$ satisfies

$$(1.2) \qquad \overline{T}(\overline{X}, \overline{Y}) = \theta(\overline{Y})J\overline{X} - \theta(\overline{X})J\overline{Y},$$

where J is a $(1,1)$-type tensor field and θ is a 1-form associated with a smooth unit vector field ζ by $\theta(X) = \overline{g}(X, \zeta)$. Throughout this paper, we denote by $\overline{X}, \overline{Y}$ and $\overline{Z}$ the smooth vector fields on $\overline{M}$. Furthermore, if $\overline{\nabla}$ is a metric connection, then we say that $\overline{\nabla}$ is a *quarter-symmetric metric connection*. The notion of quarter-symmetric metric connection was introduced Yano-Imai [14]. Recently, Jin extended this notion to indefinite Kaehler manifold or indefinite trans-Sasakian manifold and then, studied the geometry of lightlike hypersurfaces of an indefinite Kaehler manifold [6, 8] or indefinite trans-Sasakian

manifold [7] with a quarter-symmetric metric connection. Also, Jin studied generic lightlike submanifolds of an indefinite trans-Sasakian manifold with a quarter-symmetric metric connection [10].

Remark 1.1. Denote by $\widetilde{\nabla}$ the Levi-Civita connection of an indefinite Kaehler manifold $(\overline{M}, \overline{g}, J)$ with respect to $\overline{g}$. It is known [8] that *a linear connection* $\overline{\nabla}$ *on* $\overline{M}$ *is a quarter-symmetric metric connection if and only if* $\overline{\nabla}$ *sinisfies*

$$(1.3) \qquad \overline{\nabla}_{\overline{X}} \overline{Y} = \widetilde{\nabla}_{\overline{X}} \overline{Y} - \theta(\overline{X}) J \overline{Y}.$$

The object of study of this paper is generic lightlike submanifolds of an indefinite Kaehler manifold $(\overline{M}, \overline{g}, J)$ with a quarter-symmetric metric connection, in which the tensor field J, defined by (1.2), is identical with the indefinite almost complex structure J on $(\overline{M}, \overline{g}, J)$.

In this paper, by the method of [10] but using indefinite Kaehler manifolds instead of indefinite trans-Sasakian manifolds and indefinite complex space forms instead of indefinite generalized Sasakian space forms, we study generic lightlike submanifolds of an indefinite Kaehler manifold and indefinite complex space form with a quarter-symmetric metric connection.

2. Preliminaries

Let $\overline{M} = (\overline{M}, \overline{g}, J)$ be an indefinite Kaeler manifold, where $\overline{g}$ is a semi-Riemannian metric and J is an indefinite almost complex structure such that

$$(2.1) \qquad J^2 \overline{X} = -\overline{X}, \qquad \overline{g}(J\overline{X}, J\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}), \qquad (\widetilde{\nabla}_{\overline{X}} J)\overline{Y} = 0.$$

Replacing the Levi-Civita connection $\widetilde{\nabla}$ by the quarter-symmetric metric connection $\overline{\nabla}$ given by (1.3), the third equation of (2.1) is reduced to

$$(2.2) \qquad (\overline{\nabla}_{\overline{X}} J)\overline{Y} = 0.$$

Let (M, g) be an m-dimensional lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$ of dimension $(m + n)$. Then the radical distribution $Rad(TM)$ of M, defined by $Rad(TM) = TM \cap TM^{\perp}$, is a subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank $r \, (1 \leq r \leq \min\{m, n\})$. In case $1 < r < \min\{m, n\}$, we say that M is an r-*lightlike submanifold* [2] of $\overline{M}$. In this case, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$ respectively, which are called the *screen* and *co-screen* distributions of M [2], such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \; TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Also denote by $(2.1)_i$ the i-th equation of (2.1). We use the same notations for any others. Let X, Y, Z and W be the vector fields on M, unless otherwise specified. We use the following range of indices:

$$i, j, k, \ldots \; \in \{1, \ldots, r\}, \qquad a, b, c, \ldots \; \in \{r+1, \ldots, n\}.$$

Let $tr(TM)$ and $ltr(TM)$ be complementary vector bundles to TM in $T\overline{M}_{|M}$ and $TM^{\perp}$ in $S(TM)^{\perp}$ respectively and let $\{N_1, \ldots, N_r\}$ be a lightlike basis of $ltr(TM)_{|\mathcal{U}}$, where $\mathcal{U}$ is a coordinate neighborhood of M, such that

$$\overline{g}(N_i, \xi_j) = \delta_{ij}, \quad \overline{g}(N_i, N_j) = 0,$$

where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $Rad(TM)_{|\mathcal{U}}$. Then we have

$$\begin{aligned} T\overline{M} = & \ TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ = & \ \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \end{aligned}$$

For the rest of this paper, we consider only r-lightlike submanifolds M, with following local quasi-orthonormal field of frames of $\overline{M}$:

$$\{\xi_1, \ldots, \xi_r, \ N_1, \ldots, N_r, \ F_{r+1}, \ldots, F_m, \ E_{r+1}, \ldots, E_n\},$$

where $\{F_{r+1}, \ldots, F_m\}$ and $\{E_{r+1}, \ldots, E_n\}$ are orthonormal bases of $S(TM)$ and $S(TM^{\perp})$, respectively. Denote $\epsilon_a = \overline{g}(E_a, E_a)$. Then $\epsilon_a \delta_{ab} = \overline{g}(E_a, E_b)$.

Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss-Weingarten formulae of M and $S(TM)$ are given respectively by

$$(2.3) \qquad \overline{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^{\ell}(X, Y) N_i + \sum_{a=r+1}^{n} h_a^s(X, Y) E_a,$$

$$(2.4) \qquad \overline{\nabla}_X N_i = -A_{N_i} X + \sum_{j=1}^{r} \tau_{ij}(X) N_j + \sum_{a=r+1}^{n} \rho_{ia}(X) E_a,$$

$$(2.5) \qquad \overline{\nabla}_X E_a = -A_{E_a} X + \sum_{i=1}^{r} \phi_{ai}(X) N_i + \sum_{b=r+1}^{n} \sigma_{ab}(X) E_b;$$

$$(2.6) \qquad \nabla_X PY = \nabla_X^* PY + \sum_{i=1}^{r} h_i^*(X, PY) \xi_i,$$

$$(2.7) \qquad \nabla_X \xi_i = -A_{\xi_i}^* X - \sum_{j=1}^{r} \tau_{ji}(X) \xi_j,$$

where ∇ and ∇^* are induced linear connections on M and $S(TM)$ respectively, h_i^{ℓ} and h_a^s are called the *local second fundamental forms* on M, h_i^* are called the *local screen second fundamental forms* on $S(TM)$. A_{N_i}, A_{E_a} and $A_{\xi_i}^*$ are linear operators on M, and τ_{ij}, ρ_{ia}, ϕ_{ai} and σ_{ab} are 1-forms on M.

For any generic lightlike submanifold M, from (1.1) we show that the distributions $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^{\perp}))$ are vector subbundles of $S(TM)$. Thus there exist two non-degenerate almost complex distributions H_o and H with respect to J, i.e., $J(H_o) = H_o$ and $J(H) = H$, such that

$$\begin{aligned} S(TM) = & \ \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{orth} J(S(TM^{\perp})) \oplus_{orth} H_o, \\ & H = Rad(TM) \oplus_{orth} J(Rad(TM)) \oplus_{orth} H_o. \end{aligned}$$

In this case, the tangent bundle TM of M is decomposed as follows:

$$(2.8) \qquad TM = H \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^{\perp})).$$

Consider $2r$ local null vector fields U_i and V_i, $(n-r)$ local non-null unit vector fields W_a, and their 1-forms u_i, v_i and w_a defined by

$$(2.9) \qquad U_i = -JN_i, \qquad\qquad V_i = -J\xi_i, \qquad\qquad W_a = -JE_a,$$

$$(2.10) \quad u_i(X) = g(X, V_i), \quad v_i(X) = g(X, U_i), \quad w_a(X) = \epsilon_a g(X, W_a).$$

Denote by S the projection morphism of TM on H and by F the tensor field of type $(1,1)$ globally defined on M by $F = J \circ S$. Then JX is expressed as

$$(2.11) \qquad JX = FX + \sum_{i=1}^{r} u_i(X)N_i + \sum_{a=r+1}^{n} w_a(X)E_a.$$

Applying J to (2.11) and using $(2.1)_1$ and (2.9), we have

$$(2.12) \qquad F^2 X = -X + \sum_{i=1}^{r} u_i(X)U_i + \sum_{a=r+1}^{n} w_a(X)W_a.$$

We say that F is the *induced structure tensor field* of J on M.

3. Quarter-symmetric metric connection

Substituting (2.3) into $(\overline{\nabla}_X \overline{g})(Y, Z) = 0$, we obtain

$$(3.1) \qquad (\nabla_X g)(Y, Z) = \sum_{i=1}^{r} \{h_i^{\ell}(X, Y)\eta_i(Z) + h_i^{\ell}(X, Z)\eta_i(Y)\},$$

where η_i is a 1-form such that $\eta_i(X) = \overline{g}(X, N_i)$. Substituting (2.3) and (2.11) into (1.2) and then, comparing the tangent, lightlike transversal and co-screen components of the left and right terms, we have

$$(3.2) \qquad T(X, Y) = \theta(Y)FX - \theta(X)FY,$$

$$(3.3) \qquad h_i^{\ell}(X, Y) - h_i^{\ell}(Y, X) = \theta(Y)u_i(X) - \theta(X)u_i(Y),$$

$$(3.4) \qquad h_a^{s}(X, Y) - h_a^{s}(Y, X) = \theta(Y)w_a(X) - \theta(X)w_a(Y),$$

where T is the torsion tensor with respect to the induced connection ∇.

From the facts that $h_i^{\ell}(X, Y) = \overline{g}(\overline{\nabla}_X Y, \xi_i)$ and $\epsilon_a h_a^{s}(X, Y) = \overline{g}(\overline{\nabla}_X Y, E_a)$, we know that h_i^{ℓ} and h_a^{s} are independent of the choice of $S(TM)$. The local second fundamental forms are related to their shape operators by

$$(3.5) \qquad h_i^{\ell}(X, Y) = g(A_{\xi_i}^{*} X, Y) - \sum_{k=1}^{r} h_k^{\ell}(X, \xi_i)\eta_k(Y),$$

$$(3.6) \qquad \epsilon_a h_a^{s}(X, Y) = g(A_{E_a} X, Y) - \sum_{k=1}^{r} \phi_{ak}(X)\eta_k(Y),$$

$$(3.7) \qquad h_i^{*}(X, PY) = g(A_{N_i} X, PY).$$

Applying $\overline{\nabla}_X$ to $g(\xi_i, \xi_j) = 0$, $\overline{g}(\xi_i, E_a) = 0$, $\overline{g}(N_i, N_j) = 0$, $\overline{g}(N_i, E_a) = 0$ and $\overline{g}(E_a, E_b) = \epsilon \delta_{ab}$, we obtain

$$(3.8) \quad \begin{aligned} & h_i^\ell(X, \xi_j) + h_j^\ell(X, \xi_i) = 0, \qquad h_a^s(X, \xi_i) = -\epsilon_a \phi_{ai}(X), \\ & \eta_j(A_{N_i} X) + \eta_i(A_{N_j} X) = 0, \qquad \overline{g}(A_{E_a} X, N_i) = \epsilon_a \rho_{ia}(X), \\ & \epsilon_b \sigma_{ab} + \epsilon_a \sigma_{ba} = 0 \quad \text{and} \quad h_i^\ell(X, \xi_i) = 0, \quad h_i^\ell(\xi_j, \xi_k) = 0. \end{aligned}$$

Applying $\overline{\nabla}_X$ to (2.9) and (2.11) by turns and using $(2.2) \sim (2.5)$, (2.7), $(2.9) \sim (2.11)$ and $(3.5) \sim (3.8)_2$, we have

$$(3.9) \quad \begin{aligned} & h_j^\ell(X, U_i) = h_i^*(X, V_j), \qquad \epsilon_a h_i^*(X, W_a) = h_a^s(X, U_i), \\ & h_j^\ell(X, V_i) = h_i^\ell(X, V_j), \qquad \epsilon_a h_i^\ell(X, W_a) = h_a^s(X, V_i), \\ & \epsilon_b h_b^s(X, W_a) = \epsilon_a h_a^s(X, W_b), \end{aligned}$$

$$(3.10) \quad \nabla_X U_i = F(A_{N_i} X) + \sum_{j=1}^r \tau_{ij}(X) U_j + \sum_{a=r+1}^n \rho_{ia}(X) W_a,$$

$$(3.11) \quad \nabla_X V_i = F(A_{\xi_i}^* X) - \sum_{j=1}^r \tau_{ji}(X) V_j + \sum_{j=1}^r h_j^\ell(X, \xi_i) U_j$$
$$- \sum_{a=r+1}^n \epsilon_a \phi_{ai}(X) W_a,$$

$$(3.12) \quad \nabla_X W_a = F(A_{E_a} X) + \sum_{i=1}^r \phi_{ai}(X) U_i + \sum_{b=r+1}^n \sigma_{ab}(X) W_b,$$

$$(3.13) \quad (\nabla_X F)(Y) = \sum_{i=1}^r u_i(Y) A_{N_i} X + \sum_{a=r+1}^n w_a(Y) A_{E_a} X$$
$$- \sum_{i=1}^r h_i^\ell(X, Y) U_i - \sum_{a=r+1}^n h_a^s(X, Y) W_a.$$

4. Recurrent and Lie recurrent submanifolds

Recently, Jin [10] studied recurrent and Lie recurrent generic lightlike submanifolds of an indefinite trans-Sasakian manifold with a quarter-symmetric metric connection. We now follow the general scheme in [10]. Using the method of [10], we study recurrent and Lie recurrent generic lightlike submanifolds of an indefinite Kaehler manifold with a quarter-symmetric metric connection.

Definition. We say that a lightlike submanifold M of $\overline{M}$ is called
   (1) *irrotational* [13] if $\overline{\nabla}_X \xi_i \in \Gamma(TM)$ for all $i \in \{1, \ldots, r\}$,
   (2) *solenoidal* [12] if A_{E_a} and A_{N_i} are $S(TM)$-valued,
   (3) *statical* [12] if M is both irrotational and solenoidal.

Remark 4.1. From (2.3) and $(3.8)_2$, the item (1) is equivalent to

$$(4.1) \qquad h_j^\ell(X, \xi_i) = 0, \qquad h_a^s(X, \xi_i) = \phi_{ai}(X) = 0.$$

By using $(3.8)_4$, the item (2) is equivalent to

$$(4.2) \qquad \eta_j(A_{N_i} X) = 0, \qquad \rho_{ia}(X) = \eta_i(A_{E_a} X) = 0.$$

Denote by λ_{ij}, μ_{ia}, ν_{ia}, κ_{ab} and χ_{ij} the 1-forms on M such that

$$
\begin{aligned}
& \lambda_{ij}(X) = h_i^\ell(X, U_j) = h_j^*(X, V_i), \qquad \kappa_{ab}(X) = \epsilon_a h_a^s(X, W_b), \\
(4.3) \quad & \mu_{ia}(X) = h_i^\ell(X, W_a) = \epsilon_a h_a^s(X, V_i), \qquad \chi_{ij}(X) = h_i^\ell(X, V_j), \\
& \nu_{ai}(X) = h_i^*(X, W_a) = \epsilon_a h_a^s(X, U_i).
\end{aligned}
$$

Definition. The structure tensor field F of the generic lightlike submanifold M is said to be *recurrent* [9,10] if there exists a 1-form ϖ on M such that

$$(\nabla_X F)Y = \varpi(X)FY.$$

Theorem 4.2. *Let M be a generic lightlike submanifold of an indefinite Kaehler manifold with a quarter-symmetric metric connection. If F is recurrent, then*
   (1) *F is parallel with respect to the induced connection ∇ on M,*
   (2) *M is statical,*
   (3) *$J(ltr(TM))$, $J(S(TM^\perp))$ and H are parallel distributions on M, and*
   (4) *M is locally a product manifold $M_r \times M_{n-r} \times M^\sharp$, where M_r, M_{n-r} and $M^\sharp$ are leaves of $J(ltr(TM))$, $J(S(TM^\perp))$ and H respectively.*

Proof. (1) From the above definition and (3.13), we obtain

$$
\begin{aligned}
(4.4) \qquad \varpi(X)FY = {} & \sum_{i=1}^r u_i(Y) A_{N_i} X + \sum_{a=r+1}^n w_a(Y) A_{E_a} X \\
& - \sum_{i=1}^r h_i^\ell(X, Y) U_i - \sum_{a=r+1}^n h_a^s(X, Y) W_a.
\end{aligned}
$$

Replacing Y by ξ_j to this equation and using the fact that $F\xi_j = -V_j$, we get

$$(4.5) \qquad \varpi(X)V_j = \sum_{k=1}^r h_k^\ell(X, \xi_j) U_k + \sum_{b=r+1}^n h_b^s(X, \xi_j) W_b.$$

Taking the scalar product with U_j, V_i and W_a by turns, we obtain

$$\varpi = 0, \qquad h_i^\ell(X, \xi_j) = 0, \qquad h_a^s(X, \xi_j) = \phi_{aj}(X) = 0,$$

respectively. As $\varpi = 0$, F is parallel with respect to the connection ∇.

   (2) As $h_i^\ell(X, \xi_j) = 0$ and $h_a^s(X, \xi_j) = 0$, by (4.1) M is irrotational. Also, M is solenoidal. In fact, taking the scalar product with N_j to (4.4), we have

$$\sum_{i=1}^r u_i(Y) \overline{g}(A_{N_i} X, N_j) + \sum_{a=r+1}^n w_a(Y) \overline{g}(A_{E_a} X, N_j) = 0.$$

Taking $Y = U_i$ and $Y = W_a$ by turns, we get (4.2). Thus M is statical.

(3) Taking the scalar product with U_j to (4.4), we get

$$\sum_{i=1}^{r} u_i(Y)g(A_{N_i}X, U_j) + \sum_{a=r+1}^{n} w_a(Y)g(A_{E_a}X, U_j) = 0.$$

Taking $Y = U_k$ and $Y = W_b$ to this equation by turns, we obtain

$$(4.6) \qquad h_i^*(X, U_j) = \overline{g}(A_{N_i}X, U_j) = 0, \qquad \nu_{ai}(X) = \overline{g}(A_{E_a}X, U_i) = 0.$$

Taking the scalar product with V_j and W_b to (4.4) by turns, we have

$$(4.7) \qquad h_i^\ell(X, Y) = \sum_{j=1}^{r} \lambda_{ij}(X)u_j(Y) + \sum_{a=r+1}^{n} \mu_{ia}(X)w_a(Y),$$

$$\epsilon_a h_a^s(X, Y) = \sum_{b=r+1}^{n} \kappa_{ba}(X)w_b(Y),$$

by (3.6), (3.7), (4.3) and $(4.6)_2$. Replacing Y by V_j to $(4.7)_{1,2}$, we have

$$(4.8) \qquad \chi_{ij}(X) = h_i^\ell(X, V_j) = 0, \qquad \mu_{ia}(X) = h_a^s(X, V_i) = 0.$$

Taking $Y = U_j$ and $Y = W_b$ to (4.4) and using (4.3), $(4.6)_2$ and $(4.8)_2$, we get

$$(4.9) \qquad A_{N_i}X = \sum_{j=1}^{r} \lambda_{ji}(X)U_j, \qquad A_{E_a}X = \sum_{b=r+1}^{n} \epsilon_b \kappa_{ba}(X)W_b.$$

Using (3.5), (4.1), $(4.8)_2$ and the non-degenerateness of $S(TM)$, $(4.7)_1$ reduces

$$(4.10) \qquad\qquad A_{\xi_i}^*X = \sum_{j=1}^{r} \lambda_{ij}(X)V_j.$$

Applying F to $(4.9)_{1,2}$, we have $F(A_{N_i}X) = 0$ and $F(A_{E_a}X) = 0$. Substituting these results into (3.10) and (3.12), we obtain

$$(4.11) \qquad \nabla_X U_i = \sum_{j=1}^{r} \tau_{ij}(X)U_j, \qquad \nabla_X W_a = \sum_{b=r+1}^{n} \sigma_{ab}(X)W_b.$$

It follow that $J(ltr(TM))$ and $J(S(TM^\perp))$ are parallel distributions on M with respect to the induced connection ∇ on M, that is,

$$\nabla_X U_i \in \Gamma(J(ltr(TM))), \qquad \nabla_X W_a \in \Gamma(J(S(TM^\perp))).$$

Applying F to (4.10), we get $F(A_{\xi_i}^*X) = \sum_{j=1}^{r} \lambda_{ij}(X)\xi_j$. Thus we have

$$(4.12) \qquad\qquad \nabla_X V_i = \sum_{j=1}^{r} \{\lambda_{ij}(X)\xi_j - \tau_{ji}(X)V_j\}.$$

By directed calculations from (2.3), (4.8), $(4.11)_2$ and (4.12), we see that $g(\nabla_X Y, V_i) = 0$ and $g(\nabla_X Y, W_a) = 0$ for all $X \in \Gamma(TM)$ and $Y \in \Gamma(H)$. Thus

$$\nabla_X Y \in \Gamma(H), \quad \forall X \in \Gamma(TM), \ \forall Y \in \Gamma(H).$$

Thus H is also a parallel distribution on M with respect to ∇.

(4) As $J(ltr(TM))$, $J(S(TM^\perp))$ and H are parallel distributions and satisfy the decomposition form (2.8), by the decomposition theorem of de Rham [1], M is locally a product manifold $M_r \times M_{n-r} \times M^\sharp$, where M_r, M_{n-r} and $M^\sharp$ are leaves of $J(ltr(TM))$, $J(S(TM^\perp))$ and H respectively. □

Definition. The structure tensor field F of M is said to be *Lie recurrent* or *L-recurrent* [9] if there exists a 1-form ϑ on M such that

$$(\mathcal{L}_X F)Y = \vartheta(X)FY.$$

In particular, if $\vartheta = 0$, that is, $\mathcal{L}_X F = 0$, then F is called *Lie parallel*, where $\mathcal{L}_X$ denotes the Lie derivative on M with respect to X, that is,

$$(\mathcal{L}_X F)Y = [X, FY] - F[X, Y].$$

Theorem 4.3. *Let M be a generic lightlike submanifold of an indefinite Kaehler manifold with a quarter-symmetric metric connection. If F is L-recurrent, then*

(1) *F is Lie parallel,*

(2) *τ_{ij} and ρ_{ia} are satisfied $\tau_{ij} \circ F = 0$ and $\rho_{ia} \circ F = 0$. Moreover,*

$$\tau_{ij}(X) = \sum_{k=1}^{r} u_k(X) g(A_{N_k} V_j, N_i).$$

Proof. (1) Using (2.12), (3.2) and (3.13), we get

$$
\begin{aligned}
(4.13) \quad \vartheta(X)FY = {} & -\nabla_{FY}X + F\nabla_Y X - \theta(Y)X - \theta(FY)FX \\
& + \sum_{i=1}^{r} u_i(Y) A_{N_i} X + \sum_{a=r+1}^{n} w_a(Y) A_{E_a} X \\
& - \sum_{i=1}^{r} \{h_i^\ell(X,Y) - \theta(Y)u_i(X)\} U_i \\
& - \sum_{a=r+1}^{n} \{h_a^s(X,Y) - \theta(Y)w_a(X)\} W_a.
\end{aligned}
$$

Let $\alpha_i = \theta(\xi_i)$. Taking $Y = \xi_j$ and $Y = V_j$ to (4.13) respectively, we have

$$
\begin{aligned}
(4.14) \quad -\vartheta(X)V_j = {} & \nabla_{V_j}X + F\nabla_{\xi_j}X - \alpha_j X + \theta(V_j)FX \\
& - \sum_{i=1}^{r} \{h_i^\ell(X,\xi_j) - \alpha_j u_i(X)\} U_i \\
& - \sum_{a=r+1}^{n} \{h_a^s(X,\xi_j) - \alpha_j w_a(X)\} W_a,
\end{aligned}
$$

$$(4.15) \quad \vartheta(X)\xi_j = -\nabla_{\xi_j}X + F\nabla_{V_j}X - \theta(V_j)X - \alpha_j FX$$

$$- \sum_{i=1}^{r} \{h_i^\ell(X, V_j) - \theta(V_j) u_i(X)\} U_i$$

$$- \sum_{a=r+1}^{n} \{h_a^s(X, V_j) - \theta(V_j) w_a(X)\} W_a.$$

Taking the scalar product with U_i to (4.14) and then, taking the scalar product with N_i to (4.15), we obtain respectively

$$-\vartheta(X)\delta_{ij} = g(\nabla_{V_j} X, U_i) - \overline{g}(\nabla_{\xi_j} X, N_i) - \alpha_j v_i(X) - \theta(V_j)\eta_i(X),$$
$$\vartheta(X)\delta_{ij} = g(\nabla_{V_j} X, U_i) - \overline{g}(\nabla_{\xi_j} X, N_i) - \alpha_j v_i(X) - \theta(V_j)\eta_i(X).$$

Comparing these two equations, we get $\vartheta = 0$. Thus F is Lie parallel.

(2) Taking the scalar product with N_i to (4.14) such that $X = W_a$ and using (3.4), (3.6), (3.8)$_4$ and (3.12), we get $h_a^s(U_i, V_j) = \rho_{ia}(\xi_j)$. On the other hand, taking the scalar product with W_a to (4.15) such that $X = U_i$ and using (3.10), we have $h_a^s(U_i, V_j) = -\rho_{ia}(\xi_j)$. Thus $\rho_{ia}(\xi_j) = 0$ and $h_a^s(U_i, V_j) = 0$.

Taking the scalar product with U_i to (4.14) such that $X = W_a$ and using (3.4), (3.6), (3.8)$_{2,4}$ and (3.12), we get $\epsilon_a \rho_{ia}(V_j) = \phi_{aj}(U_i)$. On the other hand, taking the scalar product with W_a to (4.14) such that $X = U_i$ and using (3.8)$_2$ and (3.10), we get $\epsilon_a \rho_{ia}(V_j) = -\phi_{aj}(U_i)$. Thus $\rho_{ia}(V_j) = 0$ and $\phi_{aj}(U_i) = 0$.

Taking the scalar product with V_i to (4.14) such that $X = W_a$ and using (3.3), (3.4), (3.8)$_2$, (3.9)$_4$ and (3.12), we get $\phi_{ai}(V_j) = -\phi_{aj}(V_i)$. On the other hand, taking the scalar product with W_a to (4.14) such that $X = V_i$ and using (3.8)$_2$ and (3.11), we have $\phi_{ai}(V_j) = \phi_{aj}(V_i)$. Thus $\phi_{ai}(V_j) = 0$.

Taking the scalar product with W_a to (4.14) such that $X = \xi_i$ and using (2.7), (3.5) and (3.8)$_2$, we get $h_i^\ell(V_j, W_a) = \phi_{ai}(\xi_j)$. On the other hand, taking the scalar product with V_i to (4.15) such that $X = W_a$ and using (3.3) and (3.12), we have $h_i^\ell(V_j, W_a) = -\phi_{ai}(\xi_j)$. Thus $\phi_{ai}(\xi_j) = 0$ and $h_i^\ell(V_j, W_a) = 0$. Summarizing the above results, we obtain

$$(4.16) \quad \rho_{ia}(\xi_j) = 0, \ \rho_{ia}(V_j) = 0, \ \phi_{ai}(U_j) = 0, \ \phi_{ai}(V_j) = 0, \ \phi_{ai}(\xi_j) = 0,$$
$$h_a^s(U_i, V_j) = h_j^\ell(U_i, W_a) = 0, \quad h_i^\ell(V_j, W_a) = h_a^s(V_j, V_i) = 0.$$

Taking the scalar product with N_i to (4.13) and using (3.8)$_4$, we have

$$(4.17) \qquad -\overline{g}(\nabla_{FY} X, N_i) + \overline{g}(\nabla_Y X, U_i) - \theta(Y)\eta_i(X) - \theta(FY)v_i(X)$$
$$+ \sum_{k=1}^{r} u_k(Y)\overline{g}(A_{N_k} X, N_i) + \sum_{a=r+1}^{n} \epsilon_a w_a(Y)\rho_{ia}(X) = 0.$$

Replacing X by V_j to (4.17) and using (3.5), (3.11) and (4.16)$_2$, we have

$$(4.18) \qquad h_j^\ell(FX, U_i) + \tau_{ij}(X) + \delta_{ij}\theta(FX) = \sum_{k=1}^{r} u_k(X)\overline{g}(A_{N_k} V_j, N_i).$$

Replacing X by ξ_j to (4.17) and using (2.7), (3.5) and $(4.16)_1$, we have

$$(4.19) \qquad h_j^\ell(X, U_i) + \delta_{ij}\theta(X) = \sum_{k=1}^r u_k(X)\overline{g}(A_{N_k}\xi_j, N_i) + \tau_{ij}(FX).$$

Taking $X = U_k$ to (4.19), we have

$$(4.20) \qquad h_i^*(U_k, V_j) = h_j^\ell(U_k, U_i) = \overline{g}(A_{N_k}\xi_j, N_i) - \delta_{ij}\theta(U_k).$$

Replacing X by U_i to (4.13) and using (2.12), (3.3), (3.4), (3.7), $(3.9)_{1,2}$ and (3.10), we obtain

$$(4.21) \qquad \sum_{k=1}^r u_k(Y)A_{N_k}U_i + \sum_{a=r+1}^n w_a(Y)A_{E_a}U_i - \theta(Y)U_i$$
$$+ \theta(U_i)\{\sum_{i=1}^r u_j(Y)U_j + \sum_{a=r+1}^n w_a(Y)W_a\} - A_{N_i}Y$$
$$- F(A_{N_i}FY) - \sum_{j=1}^r \tau_{ij}(FY)U_j - \sum_{a=r+1}^n \rho_{ia}(FY)W_a = 0.$$

Taking the scalar product with V_j to (4.21) and using (3.7), $(3.8)_3$, $(3.9)_1$, $(4.16)_6$ and (4.20), we get

$$h_j^\ell(X, U_i) + \delta_{ij}\theta(X) = -\sum_{k=1}^r u_k(X)\overline{g}(A_{N_k}\xi_j, N_i) - \tau_{ij}(FX).$$

Comparing this equation with (4.19), we obtain

$$h_j^\ell(X, U_i) + \delta_{ij}\theta(X) = 0, \quad \tau_{ij}(FX) + \sum_{k=1}^r u_k(X)\overline{g}(A_{N_k}\xi_j, N_i) = 0.$$

Replacing X by U_h to the second equation, we get $\overline{g}(A_{N_k}\xi_j, N_i) = 0$. Thus,

$$(4.22) \qquad \tau_{ij}(FX) = 0, \qquad h_j^\ell(X, U_i) + \delta_{ij}\theta(X) = 0.$$

Taking $X = FY$ to $(4.22)_2$, we obtain

$$h_j^\ell(FX, U_i) + \delta_{ij}\theta(FY) = 0.$$

From this equation and (4.18), we see that

$$(4.23) \qquad \tau_{ij}(X) = \sum_{k=1}^r u_k(X)\overline{g}(A_{N_k}V_j, N_i).$$

Replacing Y by W_b to (4.21), we have

$$A_{N_i}W_a + \theta(W_a)U_i = A_{E_a}U_i + \theta(U_i)W_a.$$

Taking the product with U_j and using (3.4), (3.6), (3.7) and $(3.9)_2$, we get

$$(4.24) \qquad h_i^*(W_a, U_j) = \epsilon_a h_a^s(U_i, U_j) = \epsilon_a h_a^s(U_j, U_i) = h_i^*(U_j, W_a).$$

Taking the scalar product with W_a to (4.21), we have

$$
\begin{aligned}
\epsilon_a \rho_{ia}(FY) = {} & -h_i^*(Y, W_a) + \epsilon_a \theta(U_i) w_a(Y) \\
& + \sum_{k=1}^{r} u_k(Y) h_k^*(U_i, W_a) + \sum_{b=r+1}^{n} \epsilon_b w_b(Y) h_b^s(U_i, W_a).
\end{aligned}
$$

Taking the scalar product with U_i to (4.13) and then, taking $X = W_a$ and using (3.4), (3.6), (3.7), $(3.8)_4$, $(3.9)_2$, (3.12) and (4.24), we obtain

$$
\begin{aligned}
\epsilon_a \rho_{ia}(FY) = {} & h_i^*(Y, W_a) - \epsilon_a \theta(U_i) w_a(Y) \\
& - \sum_{k=1}^{r} u_k(Y) h_k^*(U_i, W_a) - \sum_{b=r+1}^{n} \epsilon_b w_b(Y) h_b^s(U_i, W_a).
\end{aligned}
$$

Comparing the last two equations, we obtain $\rho_{ia}(FY) = 0$. $\qquad\square$

5. Indefinite complex space forms

Denote by $\overline{R}$, R and R^* the curvature tensor of the quarter-symmetric metric connection $\overline{\nabla}$ on $\overline{M}$, and the curvature tensors of the induced connections ∇ and ∇^* on M and $S(TM)$, respectively. Using the Gauss-Weingarten formulae, we obtain the Gauss equations for M and $S(TM)$, respectively :

$$
\begin{aligned}
(5.1) \qquad \overline{R}(X,Y)Z = {} & R(X,Y)Z \\
& + \sum_{i=1}^{r} \{ h_i^\ell(X,Z) A_{N_i} Y - h_i^\ell(Y,Z) A_{N_i} X \} \\
& + \sum_{a=r+1}^{n} \{ h_a^s(X,Z) A_{E_a} Y - h_a^s(Y,Z) A_{E_a} X \} \\
& + \sum_{i=1}^{r} \{ (\nabla_X h_i^\ell)(Y,Z) - (\nabla_Y h_i^\ell)(X,Z) \\
& + \sum_{j=1}^{r} [\tau_{ji}(X) h_j^\ell(Y,Z) - \tau_{ji}(Y) h_j^\ell(X,Z)] \\
& + \sum_{a=r+1}^{n} [\phi_{ai}(X) h_a^s(Y,Z) - \phi_{ai}(Y) h_a^s(X,Z)] \\
& - \theta(X) h_i^\ell(FY,Z) + \theta(Y) h_i^\ell(FX,Z) \} N_i \\
& + \sum_{a=r+1}^{n} \{ (\nabla_X h_a^s)(Y,Z) - (\nabla_Y h_a^s)(X,Z) \\
& + \sum_{i=1}^{r} [\rho_{ia}(X) h_i^\ell(Y,Z) - \rho_{ia}(Y) h_a^s(X,Z)]
\end{aligned}
$$

$$+ \sum_{b=r+1}^{n} [\sigma_{ba}(X)h_b^s(Y,Z) - \sigma_{ba}(Y)h_b^s(X,Z)]$$
$$- \theta(X)h_a^s(FY,Z) + \theta(Y)h_a^s(FX,Z)\}E_a,$$

(5.2) $R(X,Y)PZ = R^*(X,Y)PZ$
$$+ \sum_{i=1}^{r} \{h_i^*(X,PZ)A_{\xi_i}^*Y - h_i^*(Y,PZ)A_{\xi_i}^*X\},$$
$$+ \sum_{i=1}^{r} \{(\nabla_X h_i^*)(Y,PZ) - (\nabla_Y h_i^*)(X,PZ)$$
$$+ \sum_{j=1}^{r} [h_j^*(X,PZ)\tau_{ij}(Y) - h_j^*(Y,PZ)\tau_{ij}(X)]$$
$$- \theta(X)h_i^*(FY,PZ) + \theta(Y)h_i^*(FX,PZ)\}\xi_i.$$

Definition. An indefinite complex space form $\overline{M}(c)$ is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature c such that

(5.3) $\widetilde{R}(\overline{X},\overline{Y})\overline{Z} = \dfrac{c}{4}\{\overline{g}(\overline{Y},\overline{Z})\overline{X} - \overline{g}(\overline{X},\overline{Z})\overline{Y}$
$$+ \overline{g}(J\overline{Y},\overline{Z})J\overline{X} - \overline{g}(J\overline{X},\overline{Z})J\overline{Y} + 2\overline{g}(\overline{X},J\overline{Y})J\overline{Z}\},$$

where $\widetilde{R}$ is the curvature tensor of the Levi-Civita connection $\widetilde{\nabla}$ on $\overline{M}$.

By directed calculations from (1.2) and (1.3), we see that

(5.4) $\overline{R}(\overline{X},\overline{Y})\overline{Z} = \widetilde{R}(\overline{X},\overline{Y})\overline{Z} - \{(\overline{\nabla}_X\theta)(Y) - (\overline{\nabla}_Y\theta)(X)$
$$+ \theta(Y)\theta(JX) - \theta(X)\theta(JY)\}JZ.$$

Taking the scalar product with ξ_i and N_i to (5.4) by turns and substituting (5.1) and (5.3) into the left-right terms and using $(3.8)_4$ and (5.2), we get

(5.5) $(\nabla_X h_i^\ell)(Y,Z) - (\nabla_Y h_i^\ell)(X,Z)$
$$+ \sum_{k=1}^{r} \{\tau_{ki}(X)h_k^\ell(Y,Z) - \tau_{ki}(Y)h_k^\ell(X,Z)\}$$
$$+ \sum_{a=r+1}^{n} \{\phi_{ai}(X)h_a^s(Y,Z) - \phi_{ai}(Y)h_a^s(X,Z)\}$$
$$- \theta(X)h_i^\ell(FY,Z) + \theta(Y)h_i^\ell(FX,Z)$$
$$+ \{(\overline{\nabla}_X\theta)(Y) - (\overline{\nabla}_Y\theta)(X) + \theta(Y)\theta(JX) - \theta(X)\theta(JY)\}u_i(Z)$$
$$= \dfrac{c}{4}\{u_i(X)\overline{g}(JY,Z) - u_i(Y)\overline{g}(JX,Z) + 2u_i(Z)\overline{g}(X,JY)\},$$

(5.6) $(\nabla_X h_i^*)(Y,PZ) - (\nabla_Y h_i^*)(X,PZ)$

$$+ \sum_{k=1}^{r} \{\tau_{ik}(Y)h_k^*(X, PZ) - \tau_{ik}(X)h_k^*(Y, PZ)\}$$

$$+ \sum_{k=1}^{r} \{h_k^{\ell}(X, PZ)\eta_i(A_{N_k}Y) - h_k^{\ell}(Y, PZ)\eta_i(A_{N_k}X)\}$$

$$+ \sum_{a=r+1}^{n} \epsilon_a\{\rho_{ia}(Y)h_a^s(X, PZ) - \rho_{ia}(X)h_a^s(Y, PZ)\}$$

$$- \theta(X)h_i^*(FY, PZ) + \theta(Y)h_i^*(FX, PZ)$$

$$+ \{(\overline{\nabla}_X\theta)(Y) - (\overline{\nabla}_Y\theta)(X) + \theta(Y)\theta(JX) - \theta(X)\theta(JY)\}v_i(PZ)$$

$$= \frac{c}{4}\{g(Y, PZ)\eta_i(X) - g(X, PZ)\eta_i(Y)$$

$$+ v_i(X)\overline{g}(JY, PZ) - v_i(Y)\overline{g}(JX, PZ) + 2v_i(PZ)\overline{g}(X, JY)\}.$$

Theorem 5.1. *Let M be a generic lightlike submanifold of an indefinite complex space form $\overline{M}(c)$ with a quarter-symmetric metric connection. If one of the following four conditions is satisfied;*

(1) *F is recurrent,*

(2) *F is Lie recurrent,*

(3) *$U_i s$ are parallel with respect to the induced connection ∇, or*

(4) *$W_a s$ are parallel with respect to ∇ and $A_{N_i}\xi_i$ belong to $S(TM)$,*

then $c = 0$ and $\overline{M}(c)$ is flat. Furthermore, in case (3) M is solenoidal.

Proof. (1) By Theorem 4.1, we show that M is solenoidal, *i.e.*, $\eta_j(A_{N_i}X) = 0$ and $\rho_{ia} = 0$, and the equations (4.9) and (4.11) are satisfied. Taking the scalar product with U_j to $(4.9)_1$ and using (3.7), we obtain

$$h_i^*(X, U_j) = 0.$$

Applying ∇_X to $h_i^*(Y, U_j) = 0$ and using $(4.11)_1$, we obtain

$$(\nabla_X h_i^*)(Y, U_j) = 0.$$

Taking $PZ = U_j$ to (5.6) and using the last two equations, we have

$$\frac{c}{4}\{v_j(Y)\eta_i(X) - v_j(X)\eta_i(Y) + v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)\} = 0.$$

Taking $X = \xi_i$ and $Y = V_j$ to this equation, we have $c = 0$. Thus $\overline{M}(c)$ is flat.

(2) Replacing Y by U_j to (3.3) and using $(4.22)_2$, we have

$$h_i^{\ell}(U_j, X) = -\theta(U_j)u_i(X).$$

From this equation and (2.10), we obtain

$$h_i^{\ell}(U_i, F(A_{N_j}\xi_j)) = -\theta(U_i)u_i(F(A_{N_j}\xi_j)) = 0.$$

Replacing X by ξ_j and Y by $F(A_{N_j}U_i)$ to (3.3), we obtain

$$h_i^{\ell}(\xi_j, F(A_{N_j}U_i)) = h_i^{\ell}(F(A_{N_j}U_i), \xi_j).$$

Taking $Y = U_j$ to (4.21), we have

$$A_{N_j} U_i + \theta(U_i) U_j = A_{N_i} U_j + \theta(U_j) U_i.$$

Applying F to this equation, we get $F(A_{N_j} U_i) = F(A_{N_i} U_j)$. Thus $F(A_{N_j} U_i)$ is symmetric with respect to i and j. From this result and $(3.8)_1$, we obtain

$$h_i^\ell(\xi_j, F(A_{N_j} U_i)) = h_i^\ell(F(A_{N_j} U_i), \xi_j) = 0.$$

By Theorem 4.2, the equations (4.16), (4.21) and (4.22) are satisfied. Applying ∇_X to $h_i^\ell(Y, U_j) = -\delta_{ij}\theta(Y)$ and using (2.3) and (3.10), we have

$$(\nabla_X h_i^\ell)(Y, U_j)$$

$$= -\delta_{ij}\{(\overline{\nabla}_X \theta)(Y) + \sum_{k=1}^{r} \beta_k h_k^\ell(X, Y) + \sum_{a=r+1}^{n} \gamma_a h_a^s(X, Y)\}$$

$$- h_i^\ell(Y, F(A_{N_j} X)) + \theta(Y)\tau_{ji}(X) - \sum_{a=r+1}^{n} \rho_{ja}(X) h_i^\ell(Y, W_a).$$

Substituting this equation and $(4.22)_2$ into (5.5) such that $Z = U_j$ and using (2.11), (3.2)$\sim$(3.4), $(3.9)_4$ and (3.10), we get

$$h_i^\ell(X, F(A_{N_j} Y)) - h_i^\ell(Y, F(A_{N_j} X))$$

$$+ \sum_{a=r+1}^{n} \epsilon_a\{\rho_{ja}(Y) h_a^s(X, V_i) - \rho_{ja}(X) h_a^s(Y, V_i)\}$$

$$+ \sum_{a=r+1}^{n} \{\phi_{ai}(X) h_a^s(Y, U_j) - \phi_{ai}(Y) h_a^s(X, U_j)\}$$

$$= \frac{c}{4}\{u_i(Y)\eta_j(X) - u_i(X)\eta_j(Y) + 2\delta_{ij}\overline{g}(X, JY)\}.$$

Taking $Y = U_i$ and $X = \xi_j$ and using (3.4), $(3.8)_2$ and $(4.16)_{1,3,4,5}$, we get

$$(5.7) \qquad h_i^\ell(U_i, F(A_{N_j}\xi_j)) - h_i^\ell(\xi_j, F(A_{N_j} U_i)) = \frac{3}{4}c.$$

From the above results and (5.7), we have $c = 0$. Thus $\overline{M}(c)$ is flat.

(3) If U_i is parallel with respect to ∇, then, taking the scalar product with U_j, W_a and N_j to (3.10) with $\nabla_X U_i = 0$ by turns, we get respectively

$$(5.8) \qquad \eta_j(A_{N_i} X) = 0, \qquad \rho_{ia} = 0, \qquad h_i^*(X, U_j) = 0.$$

From (4.2) and $(5.8)_{1,2}$, we see that M is solenoidal.

Applying ∇_X to $h_i^*(Y, U_j) = 0$ and using the fact that $\nabla_X U_j = 0$, we have

$$(\nabla_X h_i^*)(Y, U_j) = 0.$$

Substituting (5.8) and the last equation into (5.6) with $PZ = U_j$, we obtain

$$\frac{c}{4}\{v_j(Y)\eta_i(X) - v_j(X)\eta_i(Y) + v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)\} = 0.$$

Taking $X = \xi_i$ and $Y = V_j$ to this equation, we have $c = 0$. Thus $\overline{M}(c)$ is flat.

(4) If W_a is parallel with respect to ∇, then, taking the scalar product with V_i, U_i and N_i to (3.12) with $\nabla_X W_a = 0$ by turns, we get respectively

(5.9) $\qquad \phi_{ai} = 0, \qquad \rho_{ia} = 0, \qquad h_i^*(X, W_a) = 0.$

Applying ∇_X to $h_i^*(Y, W_a) = 0$ and using the fact that $\nabla_X W_a = 0$, we have

$$(\nabla_X h_i^*)(Y, W_a) = 0.$$

Substituting this equation and (5.9) into (5.6) with $PZ = W_a$, we get

$$\sum_{k=1}^r \{h_a^s(X, V_k)\eta_i(A_{N_k} Y) - h_a^s(Y, V_k)\eta_i(A_{N_k} X)\}$$
$$= \frac{c}{4}\{w_a(Y)\eta_i(X) - w_a(X)\eta_i(Y)\}.$$

Taking $X = \xi_i$ and $Y = W_a$ and using (3.4) and $(3.8)_{2,3}$, we obtain

$$h_a^s(W_a, V_k)\eta_k(A_{N_i}\xi_i) = \frac{c}{4}.$$

Assume that $A_{N_i}\xi_i$ belong to $S(TM)$. Then we have $c = 0$. $\qquad\square$

Definition. The lightlike submanifold M is called *screen totally umbilical* [2] if there exist smooth functions γ_i on a coordinate neighborhood $\mathcal{U}$ such that

(5.10) $\qquad h_i^*(X, PY) = \gamma_i g(X, Y).$

Theorem 5.2. *Let M be a screen totally umbilical generic lightlike submanifold of an indefinite complex space form $\overline{M}(c)$ with a quarter-symmetric metric connection. If M is irrotational or solenoidal, then $\overline{M}(c)$ is flat.*

Proof. From $(3.9)_{1,2}$ and (5.10), we have

$$h_j^\ell(X, U_i) = \gamma_i u_j(X), \qquad h_a^s(X, U_i) = \gamma_i w_a(X).$$

Taking $X = \xi_k$ to these two equations, we obtain

(5.11) $\qquad h_j^\ell(\xi_k, U_i) = 0, \qquad h_a^s(\xi_k, U_i) = 0.$

Taking $X = \xi_k$ and $Y = U_j$ to (3.3) and (3.4) and using $(5.11)_{1,2}$, we have

$$h_j^\ell(U_i, \xi_k) = \delta_{ij}\alpha_k, \qquad h_a^s(U_i, \xi_k) = 0.$$

Taking $j = k$ to the first equation and using $(3.8)_6$, we get $\alpha_i = 0$ for all i.

Applying ∇_Z to (5.10) and using (3.1), we obtain

$$(\nabla_X h_i^*)(Y, PZ) = (X\gamma_i)g(Y, PZ) + \sum_{k=1}^r \gamma_i h_k^\ell(X, PZ)\eta_k(Y).$$

Substituting the last equation into (5.6), we have

$$\{X\gamma_i - \sum_{k=1}^r \gamma_k \tau_{ik}(X)\}g(Y, PZ) - \{Y\gamma_i - \sum_{k=1}^r \gamma_k \tau_{ik}(Y)\}g(X, PZ)$$

$$+ \gamma_i \{ \sum_{k=1}^{r} [h_k^\ell(X, PZ)\eta_k(Y) - h_k^\ell(Y, PZ)\eta_k(X)]$$
$$+ g(FX, PZ)\theta(Y) - g(FY, PZ)\theta(X) \}$$
$$+ \sum_{a=r+1}^{n} \epsilon_a \{ \rho_{ia}(Y)h_a^s(X, PZ) - \rho_{ia}(X)h_a^s(Y, PZ) \}$$
$$+ \sum_{k=1}^{r} \{ h_k^\ell(X, PZ)\eta_i(A_{N_k}Y) - h_k^\ell(Y, PZ)\eta_i(A_{N_k}X) \}$$
$$+ \{ (\overline{\nabla}_X\theta)(Y) - (\overline{\nabla}_Y\theta)(X) + \theta(Y)\theta(JX) - \theta(X)\theta(JY) \} v_i(PZ)$$
$$= \frac{c}{4} \{ g(Y, PZ)\eta_i(X) - g(X, PZ)\eta_i(Y)$$
$$+ v_i(X)\overline{g}(JY, PZ) - v_i(Y)\overline{g}(JX, PZ) + 2v_i(PZ)\overline{g}(X, JY) \}.$$

Taking $X = V_i$, $Y = U_j$ and $PZ = \xi_j$ and using (5.11), we obtain

$$\sum_{k=1}^{r} \{ h_k^\ell(V_i, \xi_j)\eta_i(A_{N_k}U_j) - \sum_{a=r+1}^{n} \rho_{ia}(U_j)\phi_{aj}(V_i) \} = \frac{c}{4}.$$

Therefore, if M is irrotational or solenoidal, then $\overline{M}(c)$ is flat. $\qquad\square$

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Dae Ho Jin
Department of Mathematics
Dongguk University
Kyongju 780-714, Korea
E-mail address: jindh@dongguk.ac.kr