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OPTIMAL CONTROL ON SEMILINEAR RETARDED STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS DRIVEN BY POISSON JUMPS IN HILBERT SPACE

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Abstract. This paper deals with an optimal control on semilinear stochastic functional differential equations with Poisson jumps in a Hilbert space. The existence of an optimal control is derived by the solution of proposed system which satisfies weakly sequentially compactness. Also the stochastic maximum principle for the optimal control is established by using spike variation technique of optimal control with a convex control domain in Hilbert space. Finally, an application of retarded type stochastic Burgers equation is given to illustrate the theory.

1. Introduction

In the last decades, many processes in industrial practices have stochastic characterisation and the system have to be modelled in the form of stochastic differential equations (SDEs for short). SDEs have attracted with great interest because of their practical applications in many fields such as mechanical, electrical, control engineering, etc. [2, 6, 12, 14, 18, 25]. Nowadays, more literatures drawn towards the optimal control problem (OCP for short) for discontinuous stochastic systems or stochastic systems with random jumps. For instance, the price dynamics of financial instruments exhibits jumps which can be adequately captured solely by the process satisfying Ito-type SDEs (see [5, 22]). Jumps constitute a key tool in the description of credit risk sensitive instruments. Hence dynamic models that involves random jumps become popular in finance and it is of great interest to find whether the stochastic maximum principle holds also for the pure jump type processes which have been studied in the literatures: Peng [16] first investigated the stochastic maximum principle where the control domain is convex. Shi [20] studied the necessary conditions

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for optimal control of forward backward stochastic systems driven by random jumps with the convex control domain. Recently, Yong [27] derived the problem of finding the maximum principle for the optimal control problem of SDEs in the nonconvex control domain. Also Yong and Zhou [28] obtained maximum principle for the SDEs and control variable occurred in diffusion coefficient with the control domain is nonconvex. The retarded or delayed system makes more complicated to deal with the system not only for infinite dimensional case but also in the presence of noise and jumps. The stochastic maximum principle for OCPs of delay systems involving continuous and impulse controls in finite dimensional space are discussed in [29]. OCPs modelled by stochastic delay differential equations and its applications are studied in [19].

The occurrence of Poisson jumps leads to a new and crucial phenomena which have applications in the shot-noise processes that are used to modelled the Catastrophe insurance and network self-interface in an ad hoc network (see [8]). The optimal control and maximum principle of SDEs with jumps in infinite dimensional spaces are few (see [1, 9, 13, 15, 24, 30, 31]) and it is worth emphasizing that the presence of noise (stochastic) term and perturbation in the dynamical models that often leads to qualitatively new types of behaviour of the processes [26]. Zhou [31] derived the infinite horizon OCP in which the controlled state dynamics is governed by a stochastic evolution equations in Hilbert space and a cost functional contains a quadratic growth. In [30], authors studied the OCP for stochastic evolution equations in Hilbert space. Maximum principle for controlled stochastic evolution equations are focussed in [1]. From the above motivation, to the best of authors knowledge there is no work in the literature for a retarded SDEs driven by Brownian motion and compensated Poisson random measure. Our contribution in this paper is to derive the stochastic maximum principle for the optimal control of retarded SDEs with Poisson jumps in infinite dimensional spaces. Moreover, the existence of the optimal control for the proposed system are also studied. Here the control domain is assumed to be convex and we derive the stochastic maximum principle by the virtue of spike variation technique.

Throughout this paper, let H be a complex Hilbert space and V be another Hilbert space as a dense subspace such that $V \subset H \subset V^*$ by identifying the anti-dual of H with H, where V^* is the dual space of V. Therefore, the corresponding norms satisfies the following:

$$\|p\|_* \leq \|p\| \leq \|p\|_V \qquad \forall p \in V,$$

where the notations $\|\cdot\|$, $\|\cdot\|_V$ and $\|\cdot\|_*$ denote the norms of H, V and V^*, respectively, as usual.

In this paper, the optimal control study of the semilinear retarded stochastic functional differential equation with Poisson jumps as follows:

$$
\begin{aligned}
dx(t) =&[A_0x(t) + \int_{-h}^{0} a(s)A_1x(t+s)ds + f(t,x(t)) + q(t)]dt \\
&+ g(t,x(t))dw(t) + \int_Z e(t,x(t),\eta)\tilde{N}(dt,d\eta), \quad t \in J := [0,T], \\
x(0) =&\phi^0, \quad x(s) = \phi^1(s), \quad d-h \le s < 0,
\end{aligned}
\tag{1.1}
$$

where the state variables $x(\cdot)$ takes the values in the Hilbert space H, A_0 be the bounded linear operator associated with a sesquilinear form defined on the Hilbert space $V \times V$ satisfying Garding's inequality. The operator A_1 is a bounded linear operator from V to V^*. The function $a(\cdot)$ is assumed to be real valued and Holder continuous in $[-h,0]$. That is, for every $s,t \in [-h,0]$

$$
|a(s) - a(t)| \le |s-t|^{\rho} \tag{1.2}
$$

for $0 < \rho < 1$. The history values $x(t+s), -h \le s < 0$ are $L^2([-h,0];H)$-valued stochastic processes. Let $(\Omega, \mathfrak{F}_t, \mathbb{P})$ be a complete probability space. Let K be another separable Hilbert space and suppose that $\{w(t) : t \ge 0\}$ be a K-valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q \ge 0$. We are employing the same notations $\|\cdot\|$ for the norm of $L(K,H)$, where $L(K,H)$ denotes the space of all bounded operators from K into H. Simply as $L(K,H) = L(H)$ if $K = H$ [21]. Let $N(dt,d\eta)$ be the Poisson counting measure induced by the Poisson point process $r(t)$ defined on the complete probability space $(\Omega, \mathfrak{F}_t, \mathbb{P})$.

Let $b : J \times V \to H$ be a nonlinear mapping, we assume that there exists a constant $l_1 > 0$ such that $t \mapsto b(t,x)$ is measurable and

$$
\|b(t,x) - b(t,y)\|^2 \le l_1\|x-y\|^2, \quad b(t,0) = 0, \tag{1.3}
$$

for all $x, y \in V$. For $x \in L^2(J, \mathfrak{F}_t; V)$ and $k \in L^2(0,T)$. We set

$$
f(t,x) = \int_0^t k(t-s)b(s,x(s))ds,
$$

and $g : J \times V \to L_Q(K,H)$, $e : J \times V \times Z \to H$ are Borel measurable functions, where $L_Q(K,H)$ denotes the space of all Q-Hilbert-Schmidt operators from K into H.

The layout of the present paper is as follows. Section 2 gives some basics and the preliminary results. Section 3 describes the existence of an optimal control by utilizing weakly sequentially compactness and the stochastic maximum principle. In Section 4, an application is provided to illustrate the theory.

2. Preliminaries

In order to prove the main results, the following notations are used:

Let A_0 be the operator associated with a bounded sesquilinear form $\zeta(p,v)$ which is defined by Garding's inequality

$$
\mathrm{Re}\,\zeta(p,p) \ge c\|p\|^2 - c_0|p|^2, \quad c > 0,\ c_0 \ge 0
$$

that is,

$$(A_0 p, v) = -\zeta(u,p), \quad p, v \in V,$$

where $(\cdot,\cdot)$ denotes the duality pairing between V and V^*. Then A_0 is a bounded linear operator from V to V^*, and its realization in H which is the restriction of A_0 to $D(A_0) = \{p \in V; A_0(p) \in H\}$ is denoted by A_0. Here we note that $D(A_0)$ is dense in V. Therefore it is also dense in H. It is known that A_0 generates an analytic semigroup in both H and V^* (refer [9]). Define $W^{m,p}(0,T;V^*)$ the Sobolev space of V^*-valued functions on [0,T] whose distributional derivatives up to m belonging to $L^p(0,T;V^*)$. Let $(\Omega, \mathfrak{F}_t, \mathbb{P})$ be a complete probability space equipped with a normal filtration $\mathfrak{F}_t$, $t \in J$. An H-valued random variable is an $\mathfrak{F}_t$-measurable function $x(t) : \Omega \to H$ and a collection of random variable $S := \{x(t,\omega) : \Omega \to H : t \in J\}$ is called a stochastic process. We write $x(t)$ instead of $x(t,\omega)$ and $x(t) : J \to H$ in the place of S. Let $\beta_n(t)$ $(n = 1, 2, \ldots)$ be a sequence of real valued one dimensional $\mathfrak{F}_t$-adapted Brownian motions mutually independent over $(\Omega, \mathfrak{F}_t, \mathbb{P})$. Set $w(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n(t)e_n,\ t \geq 0$, where $\lambda_n \geq 0\,(n = 1, 2, \ldots)$ are non-negative real numbers and $\{e_n\}\,(n = 1, 2, \ldots)$ is the complete orthonormal basis in K. Let $Q \in L(K,K)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite Tr(Q)= $\sum_{n=1}^{\infty} \lambda_n < \infty$, where Tr denotes the Trace of the operator. Then the above K-valued stochastic process $w(t)$ is called a Q-Wiener process. Let $\psi \in L(K,H)$ and define

$$\|\psi\|_Q^2 = \mathrm{Tr}(\psi Q \psi^*) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\psi e_n\|^2.$$

If $\|\psi\|_Q < \infty$, then ψ is called a Q-Hilbert-Schmidt operator. Let $L_Q(K,H)$ be the space of all Q-Hilbert-Schmidt operators $\psi : K \to H$. The completion $L_Q(K,H)$ of $L(K,H)$ with respect to the topology induced by the norm $\|\cdot\|_Q$, with $\|\psi\|_Q^2 = \langle \psi, \psi \rangle$ is a Hilbert space with the above norm topology [21]. The collection of all $\mathfrak{F}_t$-measurable, square integrable H-valued random variables denoted by $L^2(\Omega, \mathfrak{F}_t, \mathbb{P}; H) = L^2(\Omega, \mathfrak{F}_t; H)$ is a Banach space equipped with the norm:

$$\|x\|_{L^2}^2 = \sup_{t \in [0,T]} \mathbb{E}\|x(t)\|^2,$$

where $\mathbb{E}$ denotes the mathematical expectation operator of the stochastic process with respect to the given probability measure $\mathbb{P}$. Now, $\mathbb{L}(V, V^*)$ is the space of all linear bounded operators from the space V to V^*. Let $U_{ad} = L^2_{\mathfrak{F}_t}(J;Y)$ be the set of all admissible controls. Let $r(t)$ be the stationary $\mathfrak{F}_t$-adapted Poisson point process with a characteristic measure λ in the measurable space $(Z, \mathfrak{B}(Z))$ and $N(dt, d\eta)$ is the Poisson counting measure associated with $r(t)$ and the compensated martingale measure denoted by $\tilde{N}(dt, d\eta) = N(dt, d\eta) - \lambda(d\eta)dt$ that is independent of the Brownian motion (see [4,15,23]). Now, consider the linear retarded functional differential equation associated with (1.1):

$$\begin{aligned} dx(t) =& [A_0x(t) + \int_{-h}^{0} a(s)A_1x(t+s)ds + q(t)]dt, \\ x(0) =& \phi^0, \quad x(s) = \phi^1(s), \quad -h \le s < 0. \end{aligned} \tag{2.1}$$

Let $W(\cdot)$ be the fundamental solution of the linear homogeneous equation associated with (2.1) which is the operator valued function satisfying,

$$\begin{aligned} &W(t) = S(t) + \int_0^t S(t-s)\int_{-h}^0 a(\tau)A_1W(s+\tau)d\tau ds, \qquad t > 0 \\ &W(0) = I, \quad W(s) = 0, \quad -h \le s < 0, \end{aligned} \tag{2.2}$$

where $S(\cdot)$ is the semigroup generated by A_0. Then, $x(t)$ satisfies the integral equation,

$$\begin{aligned} x(t) &= W(t)\phi^0 + \int_{-h}^0 U_t(s)\phi^1(s)ds + \int_0^t W(t-s)q(s)ds, \\ U_t(s) &= \int_{-h}^s W(t-s+\sigma)a(\sigma)A_1 d\sigma. \end{aligned}$$

By the virtue of Theorem 3.3 in [7], we have the following result on the linear equation(2.1).

Proposition 2.1. (i) *Let $F = (D(A_0), H)_{\frac{1}{2},2}$ where $(D(A_0), H)_{\frac{1}{2},2}$ denote the real interpolation space between $D(A_0)$ and H. Let $(\phi^0, \phi^1) \in F \times L^2(-h, 0, \mathfrak{F}_t; D(A_0))$ and $q \in L^2(J, \mathfrak{F}_t; H)$, $T > 0$. Then, there exists a unique solution x of (2.1) belonging to*

$$\mathcal{W}_0(T) \equiv L^2(-h, 0, \mathfrak{F}_t; D(A_0)) \cap W^{1,2}(0, T, \mathfrak{F}_t; H) \subset C([0, T]; F),$$

and satisfying,

$$\mathbb{E}\|x\|^2_{\mathcal{W}_0(T)} \le C_1\Big[\mathbb{E}\|\phi^0\|^2_F + \mathbb{E}\|\phi^1\|^2_{L^2(-h,0,\mathfrak{F}_t;D(A_0))} + \mathbb{E}\|q\|^2_{L^2(J,\mathfrak{F}_t;H)}\Big], \tag{2.3}$$

where C_1 is a constant depending on T.

(ii) *Let $(\phi^0, \phi^1) \in H \times L^2(-h, 0, \mathfrak{F}_t; V)$ and $q \in L^2(J, \mathfrak{F}_t; V^*)$, $T > 0$. Then there exists a unique solution x of (2.1) belonging to*

$$\mathcal{W}_1(T) \equiv L^2(-h, T, \mathfrak{F}_t; V) \cap W^{1,2}(0, T, \mathfrak{F}_t; V^*) \subset C([0, T]; H)$$

and satisfying,

$$\mathbb{E}\|x\|^2_{\mathcal{W}_1(T)} \le C_1\Big[\mathbb{E}|\phi^0|^2 + \mathbb{E}\|\phi^1\|^2_{L^2(-h,0,\mathfrak{F}_t;V)} + \mathbb{E}\|q\|^2_{L^2(J,\mathfrak{F}_t;V^*)}\Big], \tag{2.4}$$

where C_1 is a constant depending on T.

Lemma 2.2. *Let $x \in L^2(J, \mathfrak{F}_t; V)$, $T > 0$. Then $f(\cdot, x) \in L^2(J, \mathfrak{F}_t; H)$ and*

$$\mathbb{E}\|f(\cdot, x)\|^2_{L^2(J,\mathfrak{F}_t;H)} \le l_1^2\|k\|^2_{L^2(0,T)}T\|x\|^2_{L^2(J,\mathfrak{F}_t;V)}.$$

Moreover, if $x_1, x_2 \in L^2(J, \mathfrak{F}_t; V)$*, then*

$$\mathbb{E}\|f(\cdot, x_1) - f(\cdot, x_2)\|^2_{L^2(J,\mathfrak{F}_t;H)} \leq l_1^2 \|k\|^2_{L^2(0,T)} T \|x_1 - x_2\|^2_{L^2(J,\mathfrak{F}_t;V)}.$$

Proof. Let $x \in L^2(J, \mathfrak{F}_t; V)$ $T > 0$ and since $f : J \times V \to H$, $f(\cdot, x) \in L^2(J, \mathfrak{F}_t; H)$. Now by using (1.3) and the Holder inequality we get,

$$\begin{aligned} \mathbb{E}\|f(\cdot,x)\|^2_{L^2(J,\mathfrak{F}_t;H)} &\leq \int_0^T \mathbb{E}\Big\| \int_0^t k(t-s) b(s, x(s)) ds \Big\|^2 dt \\ &\leq \|k\|^2_{L^2(0,T)} \int_0^T \mathbb{E} \int_0^t l_1^2 \|x(s)\|^2 ds dt \\ &\leq l_1^2 \|k\|^2_{L^2(0,T)} \int_0^T \int_0^t \mathbb{E}\|x(s)\|^2 ds dt \\ &\leq T l_1^2 \|k\|^2_{L^2(0,T)} \Big[\sup_{s \in J} \mathbb{E}\|x(s)\|^2 \Big] \\ &= T l_1^2 \|k\|^2_{L^2(0,T)} \|x\|^2_{L^2(J,\mathfrak{F}_t;V)}. \end{aligned} \tag{2.5}$$

Now consider,

$$\begin{aligned} \mathbb{E}\|f(\cdot,x_1) - f(\cdot,x_2)\|^2_{L^2(J,\mathfrak{F}_t;H)} &\leq \|k\|^2_{L^2(0,T)} \int_0^T \mathbb{E} \int_0^t l_1^2 \|x_1(s) - x_2(s)\|^2 ds dt \\ &\leq l_1^2 \|k\|^2_{L^2(0,T)} \int_0^T \int_0^t \mathbb{E}\|x_1(s) - x_2(s)\|^2 ds dt \\ &\leq T l_1^2 \|k\|^2_{L^2(0,T)} \Big[\sup_{s \in J:=[0,T]} \mathbb{E}\|x_1(s) - x_2(s)\|^2 \Big] \\ &= T l_1^2 \|k\|^2_{L^2(0,T)} \|x_1 - x_2\|^2_{L^2(J,\mathfrak{F}_t;V)}. \end{aligned}$$ □

In the virtue of Lemma 2.2, using maximal regularity for more general retarded parabolic system from Theorem 3.1 in [10] we establish for the following result on the solvability of (1.1).

Proposition 2.3. *Suppose that the inequality* (1.3) *is satisfied. Then for any* $(\phi^0, \phi^1) \in H \times L^2(-h, 0, \mathfrak{F}_t; V)$ *and* $q \in L^2(J, \mathfrak{F}_t; V^*)$, $T > 0$ *the solution* x *of* (1.1) *exists and is unique in* $L^2(-h, T, \mathfrak{F}_t; V) \cap W^{1,2}(0, T, \mathfrak{F}_t; V^*)$ *and satisfies for a constant* C_2 *depending on* T *such that,*

$$\begin{aligned} &\mathbb{E}\|x\|^2_{L^2(-h,T,\mathfrak{F}_t;V) \cap W^{1,2}(0,T,\mathfrak{F}_t;V^*)} \\ &\leq C_2 \Big[1 + \mathbb{E}|\phi^0|^2_H + \mathbb{E}\|\phi^1\|^2_{L^2(-h,0,\mathfrak{F}_t;V)} + \mathbb{E}\|q\|^2_{L^2(J,\mathfrak{F}_t;V^*)} \Big]. \end{aligned} \tag{2.6}$$

Lemma 2.4 (see [6])**.** *Let* $G : J \times \Omega \to L_Q(K, H)$ *be a strongly measurable mapping such that* $\int_0^T \mathbb{E}\|G(t)\|^2_{L_Q(K,H)} dt < \infty$*. Then*

$$\mathbb{E}\Big\| \int_0^t G(s) dw(s) \Big\|^2 \leq L_G \int_0^t \mathbb{E}\|G(s)\|^2 ds \quad \forall t \in J, \tag{2.7}$$

where L_G *is a constant involving* T*.*

3. Main results

In this section, we will show that there exists an optimal control for the proposed system and also the stochastic maximum principle.

Assume that $D(A_0) \subset V$ is compact. Now consider the nonlinear stochastic control system corresponding to (1.1) with all its assumptions defined in Sec. 1 as follows:

$$\begin{aligned} dx(t) &= [A_0 x(t) + \int_{-h}^{0} a(s)A_1 x(t+s)ds + f(t,x(t)) + Bu(t)]dt \\ &\quad + g(t,x(t))dw(t) + \int_Z e(t,x(t),\eta)\tilde{N}(dt,d\eta), \quad t \in J. \\ x(0) &= \phi^0, \quad x(s) = \phi^1(s), \quad -h \le s < 0, \end{aligned} \tag{3.1}$$

The control space will be modelled by a Banach space Y. Let the controller B is a bounded linear operator from Y to H. Choose a bounded subset U of Y and we say that U is a control set. Suppose that an admissible control $u \in L^2_{\mathfrak{F}_t}(J;Y)$ is $\mathfrak{F}_t$-measurable, square integrable Y-valued random variable satisfying $u(t) \in U$ for almost all t. Let $x(t;u)$ be a solution of (3.1) associated with the nonlinear functions f, g and e and a control u at the time t. The solution $x(t;u)$ of (3.1) for each admissible control u is called a trajectory corresponding to u. We need the following hypotheses.

- $(H1)$ The nonlinear functions g and e satisfies the Lipschitz condition, there exists a positive constants $M_g, M_e > 0$ such that

$$\|g(s,x_1(s)) - g(s,x_2(s))\|^2 \le M_g \|x_1 - x_2\|^2 \ \forall x_1\,, x_2 \in L^2(J,\mathfrak{F}_t;V)$$

$$\int_Z \|e(s,x_1(s),\eta) - e(s,x_2(s),\eta)\|^2 \lambda(d\eta) \le M_e \|x_1 - x_2\|^2 \ \forall x_1, x_2 \in L^2(J,\mathfrak{F}_t;V).$$

- $(H2)$ We assume that $W(t)$ is uniformly bounded. That is there is a constant $M > 0$ such that

$$\|W(t)\|^2 \le M \quad \forall t > 0.$$

- $(H3)$ For our convenience, take $l_2 = \|B\|^2$.

Let $\mathcal{F}, \mathcal{B}, \mathcal{G}$ and $\mathcal{E}$ be the Nemitsky operators corresponding to the mappings f, B, g and e, which are defined by,

$$(\mathcal{F}u)(\cdot) = f(\cdot, x_u(\cdot)), \qquad (\mathcal{B}u)(\cdot) = Bu(\cdot),$$
$$(\mathcal{G}u)(\cdot) = g(\cdot, x_u(\cdot)), \quad \text{and} \quad (\mathcal{E}u,\eta)(\cdot) = e(\cdot, x_u(\cdot), \eta) \text{ respectively.}$$

Then, the solution of (3.1) can be written as (see [9, 17])

$$\begin{aligned} x(t;u) &= x(t;\phi) + \int_0^t W(t-s)\{f(s,x(s)) + Bu(s)\}ds \\ &\quad + \int_0^t W(t-s)g(s,x(s))dw(s) + \int_0^t W(t-s)\int_Z e(s,x(s),\eta)\tilde{N}(dt,d\eta) \end{aligned}$$

$$= x(t;\phi) + \int_0^t W(t-s)((\mathcal{F}+\mathcal{B})u)(s)ds$$
$$+ \int_0^t W(t-s)(\mathcal{G}u)(s)dw(s) + \int_0^t W(t-s)\int_Z (\mathcal{E}u,\eta)(s)\tilde{N}(dt,d\eta),$$

where

$$x(t;\phi) = W(t)\phi^0 + \int_{-h}^0 U_t(s)\phi^1(s)ds.$$

Let $\mathcal{Z}$ be a real Hilbert space. Let $C(t)$ be a bounded function from H to $\mathcal{Z}$ for each t and be continuous in $t \in J$. Let $y \in L^2(J, \mathfrak{F}_t; \mathcal{Z})$. Suppose that there exists no admissible control which satisfies $C(t)x(t;u) = y(t)$ for almost all t. Then, we define the cost function as follows:

$$\mathcal{J}(u) = \mathbb{E}\Big[\frac{1}{2}\int_0^T \|C(t)x(t;u) - y(t)\|^2 dt\Big]. \tag{3.2}$$

Let $u \in L^2_{\mathfrak{F}_t}(J;Y)$. Then, it is well known that

$$\lim_{h\to 0} h^{-1}\mathbb{E}\int_0^h \|u(t+s) - u(t)\|_Y ds = 0 \tag{3.3}$$

for almost all points of $t \in [0,T]$.

Remark 3.1. The point t, which permits (3.3) to hold, is called a Lebesgue point of u.

Lemma 3.2. *If the hypotheses* $(H1)$-$(H3)$ *and Lemma 2.4 holds. Let* x_u *be the solution of* (1.1) *corresponding to* u. *Then the mapping* $u \mapsto x_u$ *is compact from* $L^2_{\mathfrak{F}_t}(J;Y)$ *to* $L^2(J,\mathfrak{F}_t;V)$.

Proof. We define the solution mapping S from $L^2_{\mathfrak{F}_t}(J;Y)$ to $L^2(J,\mathfrak{F}_t;V)$ by

$$(Su)(t) = x_u(t), \qquad \text{where } u \in L^2_{\mathfrak{F}_t}(J;Y).$$

Using equations (2.3), (2.6) and Lemma 2.2, we have that,

$$\mathbb{E}\|Su\|^2_{L^2(J,\mathfrak{F}_t;D(A_0))\cap W^{1,2}(J,\mathfrak{F}_t;H)} = \mathbb{E}\|x_u\|^2_{L^2(J,\mathfrak{F}_t;D(A_0))\cap W^{1,2}(J,\mathfrak{F}_t;H)}$$
$$\leq C_1\Big[\mathbb{E}\|\phi^0\|^2_F + \mathbb{E}\|\phi^1\|^2_{L^2(J,\mathfrak{F}_t;D(A_0))} + \Big\{\mathbb{E}\|(\mathcal{F}+\mathcal{B})u\|^2_{L^2(J,\mathfrak{F}_t;H)}$$
$$+ L_g\mathbb{E}\|\mathcal{G}u\|^2_{L^2(J,\mathfrak{F}_t;H)} + \mathbb{E}\int_Z \|(\mathcal{E}u,\eta)\|^2_{L^2(J,\mathfrak{F}_t;H)}\lambda(d\eta)\Big\}\Big]$$
$$\leq C_1\Big[\mathbb{E}\|\phi^0\|^2_F + \mathbb{E}\|\phi^1\|^2_{L^2(J,\mathfrak{F}_t;D(A_0))} + [l_1^2\|k\|^2_{L^2(0,T)}T\|x\|^2_{L^2(J,\mathfrak{F}_t;V)}]$$
$$+ \|B\|^2\mathbb{E}\|u\|^2_{L^2_{\mathfrak{F}_t}(J;Y)} + L_gM_g\|x\|^2_{L^2(J,\mathfrak{F}_t;V)} + M_e\|x\|^2_{L^2(J,\mathfrak{F}_t;V)}\Big]$$
$$\leq C_1\Big[\mathbb{E}\|\phi^0\|^2_F + \mathbb{E}\|\phi^1\|^2_{L^2(J,\mathfrak{F}_t;D(A_0))}$$
$$+ \Big\{l_1^2\|k\|^2_{L^2(0,T)}T + L_gM_g + M_e\Big\}\|x\|^2_{L^2(J,\mathfrak{F}_t;V)} + \|B\|^2\mathbb{E}\|u\|^2_{L^2_{\mathfrak{F}_t}(J;Y)}\Big]$$

$$\leq C_1\Big[\mathbb{E}\|\phi^0\|_F^2 + \mathbb{E}\|\phi^1\|^2_{L^2(J,\mathfrak{F}_t;D(A_0))} + \Big\{l_1^2\|k\|^2_{L^2(0,T)}T + L_g M_g + M_e\Big\}\Big(C_2[1+\mathbb{E}\|\phi^0\|_H^2 + \mathbb{E}\|\phi^1\|^2_{L^2(-h,0,\mathfrak{F}_t;V)} + \|B\|^2\mathbb{E}\|u\|^2_{L^2_{\mathfrak{F}_t}(J;Y)}]\Big) + \|B\|^2\mathbb{E}\|u\|^2_{L^2_{\mathfrak{F}_t}(J;Y)}\Big].$$

Hence, if u is bounded in $L^2_{\mathfrak{F}_t}(J;Y)$, then so is x_u in $L^2(J,\mathfrak{F}_t;D(A_0))\cap W^{1,2}(J,\mathfrak{F}_t;H)$. Noting that $D(A_0)$ is compactly embedded in V, the embedding

$$L^2(J,\mathfrak{F}_t;D(A_0))\cap W^{1,2}(J,\mathfrak{F}_t;H)\subset L^2(J,\mathfrak{F}_t;V)$$

is also compact in the view of Theorem 2 of Aubin [3]. Hence the mapping $u\mapsto S_u = x_u$ is compact from $L^2_{\mathfrak{F}_t}(J;Y)$ to $L^2(J,\mathfrak{F}_t;V)$. □

Theorem 3.3. *Let U be a bounded closed convex subset of Y. All the hypotheses of Lemma 3.2 are satisfied then there exist an optimal control for the cost function* (3.2).

Proof. Let $\{u_n\}$ in U be a minimizing sequence of $L^2_{\mathfrak{F}_t}(J;Y)$ such that

$$\inf_{u\in U}\mathcal{J}(u) = \lim_{n\to\infty}\mathcal{J}(u_n).$$

Since U is bounded and weakly closed, then there exists a subsequence denoted by $\{u_n\}$ again and there exists a $\hat{u}\in U$ such that

$$u_n\to\hat{u} \text{ weakly in } L^2_{\mathfrak{F}_t}(J;Y). \tag{3.4}$$

Since U is a closed convex subset of Y and by using Mazur's theorem as an important consequence of the Hahn-Banach theorem, there exist a $f_0\in Y^*$ and $c\in(-\infty,\infty)$ such that $f_0(u)\leq c\ \forall\, u\in U$. Thus $\hat{u}$ is admissible. Let s be a Lebesgue point of $\hat{u}$ and put

$$\omega_{\epsilon,n} = \frac{1}{\epsilon}\int_s^{s+\epsilon} u_n(t)dt,$$

for each $t>0$ and n. Then, $f_0(\omega_{\epsilon,n})\leq c$ and we have (3.4),

$$\omega_{\epsilon,n}\to\omega_\epsilon = \frac{1}{\epsilon}\int_s^{s+\epsilon}\hat{u}(t)dt \quad \text{weakly as } n\to\infty.$$

By letting $\epsilon\to 0$, it holds that $\omega_\epsilon\to\hat{u}(s)$ and $f_0(\hat{u})\leq c$, so that $\hat{u}(s)\in U$. Also we have,

$$x_n(t) = x(t;\phi) + \int_0^t W(t-s)((\mathcal{F}+\mathcal{B})u_n)(s)ds + \int_0^t W(t-s)(\mathcal{G}u_n)(s)dw(s) + \int_0^t W(t-s)\int_Z(\mathcal{E}u_n,\eta)(s)\tilde{N}(dt,d\eta),$$

where

$$x(t;\phi) = W(t)\phi^0 + \int_{-h}^0 U_t(s)\phi^1(s)ds,$$

it follows from the Proposition 2.3 that $\{x_n(t)\}$ is bounded. By applying the Eberlein-Smulian theorem we obtain $\{x_n(t)\}$ is weakly sequentially compact. From (1.3) and Lemma 3.2, we say that $\mathcal{F}$ is a compact operator from $L^2_{\mathfrak{F}_t}(J;Y)$ to $L^2(J,\mathfrak{F}_t;H)$ and hence it holds $\mathcal{F}u_n \to \mathcal{F}\hat{u}$ strongly in $L^2(J,\mathfrak{F}_t;H)$. Similarly, $\mathcal{B}$, $\mathcal{G}$, $\mathcal{E}$ and $W(t)$ are strongly continuous on J, then $\mathcal{B}u_n \to \mathcal{B}\hat{u}$, $\mathcal{G}u_n \to \mathcal{G}\hat{u}$, $\mathcal{E}(u_n,\eta) \to \mathcal{E}(\hat{u},\eta)$ strongly in $L^2(J,\mathfrak{F}_t;H)$. Then, we have that

$$x_n(t) \to x(t;\hat{u}) \quad \text{weakly in } H,$$

where

$$\begin{aligned} x(t;\hat{u}) = x(t;\phi) &+ \int_0^t W(t-s)((\mathcal{F}+\mathcal{B})\hat{u})(s)ds \\ &+ \int_0^t W(t-s)(\mathcal{G}\hat{u})(s)dw(s) + \int_0^t W(t-s)\int_Z (\mathcal{E}\hat{u},\eta)(s)\tilde{N}(dt,d\eta). \end{aligned}$$

Hence, we have that

$$\inf \mathcal{J}(u) \le \mathcal{J}(\hat{u}) \le \lim_{n\to\infty} \inf \mathcal{J}(u_n) = \inf \mathcal{J}(u).$$

Hence, this $\hat{u}$ is an optimal control. □

Theorem 3.4. *Let the hypotheses* $(H1)$-$(H3)$ *and* (1.3) *be satisfied and let* $\hat{u}$ *be an optimal control. Then the equality*

$$\max_{v\in U} \mathbb{E}\langle v, B^*z(s)\rangle = \mathbb{E}\langle \hat{u}(s), B^*z(s)\rangle$$

holds, where

$$z(s) = \int_s^T W^*(t-s)C^*(t)\{y(t) - C(t)x(t;\hat{u})\}dt.$$

Here, $z(s)$ *satisfies the following transposed system* (*see the details in* [9, 11]):

(3.5)

$$z'(t)+A_0^*z(t)+\int_{-h}^0 a(s)A_1^*z(t-s)ds+C^*(t)\{y(t)-C(t)x(t;\hat{u})\} = 0 \ a.e\ \forall t\in J.$$

$$z(T) = 0,\ z(s) = 0\ a.e\ s\in(T, T+h],$$

in the weak sense.

Proof. Let $\hat{u}$ be an optimal control and let $\hat{x}(t) = x(t;\hat{u})$. For $\epsilon > 0$, choose $v \in L^2_{\mathfrak{F}_t}(J;U)$ so that $\mathbb{E}\|\hat{u}-v\|^2_{L^2_{\mathfrak{F}_t}(J;U)} < \epsilon$. Let t_0 be a Lebesgue point of $\hat{u}$ and v.

For $t_0 < t_0+\epsilon < T$. Put

$$u(t) = \begin{cases} v & \text{if } \ t_0 < t < t_0+\epsilon \\ \hat{u} & \text{otherwise.} \end{cases} \tag{3.6}$$

Then, u is an admissible control. Let $x(t) = x(t;u)$. Then

$$\mathbb{E}\|x(t) - \hat{x}(t)\|^2 = \mathbb{E}\|x(t) - x(t)\|^2 = 0$$

for $0 \leq t \leq t_0$, since $\hat{u}(t) = u(t)$ for $0 \leq t \leq t_0$.

Now, if $t_0 \leq t \leq T$, then

$$
\begin{aligned}
&\mathbb{E}\|x(t) - \hat{x}(t)\|^2 \\
&= \mathbb{E}\Big\| \int_{t_0}^{t_0+\epsilon} W(t-s)\Big\{[f(s,x(s)) - f(s,\hat{x}(s))] + B(v - \hat{u}(s))\Big\}ds \\
&\quad + \int_{t_0}^{t_0+\epsilon} W(t-s)[g(s,x(s)) - g(s,\hat{x}(s))]dw(s) \\
&\quad + \int_{t_0}^{t_0+\epsilon} W(t-s)\int_Z [e(s,x(s),\eta) - e(s,\hat{x}(s),\eta)]\tilde{N}(ds,d\eta)\Big\|^2. \qquad (3.7)
\end{aligned}
$$

If $t_0 < t < t_0 + \epsilon$, then

$$
\begin{aligned}
\mathbb{E}\|x(t) - \hat{x}(t)\|^2 = \mathbb{E}\Big\| &\int_{t_0}^{t} W(t-s)\Big\{[f(s,x(s)) - f(s,\hat{x}(s))] + B(v - \hat{u}(s))\Big\}ds \\
&+ \int_{t_0}^{t} W(t-s)[g(s,x(s)) - g(s,\hat{x}(s))]dw(s) \\
&+ \int_{t_0}^{t} W(t-s)\int_Z [e(s,x(s),\eta) - e(s,\hat{x}(s),\eta)]\tilde{N}(ds,d\eta)\Big\|^2. \qquad (3.8)
\end{aligned}
$$

Let $t_0 \leq t \leq T$. Let us put

$$
\begin{aligned}
q(t) &= \int_{t_0}^{t_0+\epsilon} W(t-s)B(v - \hat{u}(s))ds, \\
\mathbb{E}\|q(t)\|^2 &= \mathbb{E}\Big\| \int_{t_0}^{t_0+\epsilon} W(t-s)B(v - \hat{u}(s))ds\Big\|^2 \\
&\leq M\int_{t_0}^{t_0+\epsilon} \mathbb{E}\|B(v - \hat{u}(s))\|^2 ds \leq Ml_2 \int_{t_0}^{t_0+\epsilon} \mathbb{E}\|(v - \hat{u}(s))\|^2 ds \\
&\leq \epsilon^2 M l_2. \qquad (3.9)
\end{aligned}
$$

From Lemma 2.4, $(H1)$-$(H3)$, (3.7) and (3.9) it follows that,

$$
\begin{aligned}
&\mathbb{E}\|x(t) - \hat{x}(t)\|^2 \\
&\leq 4\mathbb{E}\Big\| \int_{t_0}^{t_0+\epsilon} W(t-s)\{f(s,x(s)) - f(s,\hat{x}(s))\}ds\Big\|^2 \\
&\quad + 4\mathbb{E}\|q(t)\|^2 + 4\mathbb{E}\Big\| \int_{t_0}^{t_0+\epsilon} W(t-s)\{g(s,x(s)) - g(s,\hat{x}(s))\}dw(s)\Big\|^2 \\
&\quad + 4\mathbb{E}\Big\| \int_{t_0}^{t_0+\epsilon} W(t-s)\int_Z \{e(s,x(s),\eta) - e(s,\hat{x}(s),\eta)\}\tilde{N}(ds,d\eta)\Big\|^2 \\
&\leq 4M\int_{t_0}^{t_0+\epsilon} \mathbb{E}\|f(s,x(s)) - f(s,\hat{x}(s))\|^2 ds
\end{aligned}
$$

$$
\begin{aligned}
&\quad + 4\epsilon^2 M l_2 + 4ML_g \int_{t_0}^{t_0+\epsilon} \mathbb{E}\|g(s,x(s)) - g(s,\hat{x}(s))\|^2 ds \\
&\quad + 4M \int_{t_0}^{t_0+\epsilon} \int_Z \mathbb{E}\|e(s,x(s),\eta) - e(s,\hat{x}(s),\eta)\|^2 \lambda(d\eta) ds \\
&\leq 4Ml_1^2 \|k\|^2_{L^2(0,T)} \int_{t_0}^{t_0+\epsilon} \mathbb{E}\|x(s) - \hat{x}(s)\|^2 ds + 4\epsilon^2 M l_2 \\
&\quad + 4ML_g M_g \int_{t_0}^{t_0+\epsilon} \mathbb{E}\|x(s) - \hat{x}(s)\|^2 ds + 4MM_e \int_{t_0}^{t_0+\epsilon} \mathbb{E}\|x(s) - \hat{x}(s)\|^2 ds \\
&\leq \Big[4Ml_1^2 \|k\|^2_{L^2(0,T)} + 4ML_g M_g + 4MM_e \Big] \int_{t_0}^{t_0+\epsilon} \mathbb{E}\|x(s) - \hat{x}(s)\|^2 ds + 4\epsilon^2 M l_2 \\
&\leq 4MC \int_{t_0}^{t_0+\epsilon} \mathbb{E}\|x(s) - \hat{x}(s)\|^2 ds + 4\epsilon^2 M l_2,
\end{aligned}
$$

where $C = l_1^2 \|k\|^2_{L^2(0,T)} + L_g M_g + M_e$. By using Gronwall's inequality,

$$
\begin{aligned}
\mathbb{E}\|x(t) - \hat{x}(t)\|^2 &\leq 4\epsilon^2 M l_2 \exp\Big(\int_{t_0}^{t_0+\epsilon} 4MC ds \Big) \\
&= 4\epsilon^2 M l_2 \exp(4MC\epsilon)
\end{aligned}
$$

$$
\Rightarrow \mathbb{E}\|x(t) - \hat{x}(t)\|^2 \leq \epsilon c_2 l_2 \exp(c_2 C) \tag{3.10}
$$

for some positive constant $c_2 = 4M\epsilon$. Clearly, it holds the inequality (3.10) in case where $0 \leq t < t_0 + \epsilon$. Since $\hat{u}$ is optimal, we have that

$$
\begin{aligned}
0 &\leq \frac{1}{\epsilon}(\mathcal{J}(u) - \mathcal{J}(\hat{u})) \\
&= \frac{1}{\epsilon}\Big[\frac{1}{2}\mathbb{E}\Big(\int_0^T \|C(t)x(t) - y(t)\|^2 dt - \int_0^T \|C(t)\hat{x}(t) - y(t)\|^2 dt \Big) \Big] \\
&= \frac{1}{2\epsilon}\mathbb{E}\Big[\int_0^T \langle C(t)x(t) - y(t), C(t)x(t) - y(t) \rangle dt \\
&\qquad - \int_0^T \langle C(t)\hat{x}(t) - y(t), C(t)\hat{x}(t) - y(t) \rangle dt \Big] \\
&= \frac{1}{2\epsilon}\mathbb{E}\Big[\int_0^T \{ \langle C(t)x(t), C(t)x(t) \rangle - \langle C(t)x(t), y(t) \rangle - \langle y(t), C(t)x(t) \rangle \\
&\qquad + \langle y(t), y(t) \rangle - \langle C(t)\hat{x}(t), C(t)\hat{x}(t) \rangle \\
&\qquad + \langle C(t)\hat{x}(t), y(t) \rangle + \langle y(t), C(t)\hat{x}(t) \rangle - \langle y(t), y(t) \rangle \} dt \Big].
\end{aligned}
$$

Since $\mathcal{Z}$ is a real Hilbert space,

$$
\begin{aligned}
0 &\leq \frac{1}{\epsilon}(\mathcal{J}(u) - \mathcal{J}(\hat{u})) \\
&= \frac{1}{2\epsilon}\mathbb{E}\Big[\int_0^T \{ \langle C(t)x(t), C(t)x(t) \rangle - 2\langle C(t)x(t), y(t) \rangle + 2\langle C(t)\hat{x}(t), y(t) \rangle
\end{aligned}
$$

$$
\begin{aligned}
&\qquad - \langle C(t)\hat{x}(t), C(t)\hat{x}(t)\rangle\}dt\Big] \\
&= \frac{1}{2\epsilon}\mathbb{E}\Big[\int_0^T \{\langle C(t)x(t), C(t)x(t)\rangle - 2\langle C(t)x(t), y(t)\rangle \\
&\qquad + 2\langle C(t)\hat{x}(t), y(t)\rangle - \langle C(t)\hat{x}(t), C(t)\hat{x}(t)\rangle \\
&\qquad + 2\langle C(t)x(t), C(t)\hat{x}(t)\rangle - 2\langle C(t)x(t), C(t)\hat{x}(t)\rangle \\
&\qquad + \langle C(t)\hat{x}(t), C(t)\hat{x}(t)\rangle - \langle C(t)\hat{x}(t), C(t)\hat{x}(t)\rangle\}dt\Big] \\
&= \frac{1}{2\epsilon}\mathbb{E}\Big[\int_0^T \{\langle 2C(t)x(t), C(t)\hat{x}(t)\rangle - 2\langle C(t)x(t), y(t)\rangle \\
&\qquad + 2\langle C(t)\hat{x}(t), y(t)\rangle - \langle 2C(t)\hat{x}(t), C(t)\hat{x}(t)\rangle\}dt\Big] \\
&\quad + \frac{1}{2\epsilon}\mathbb{E}\Big[\int_0^T \{\langle C(t)x(t), C(t)x(t)\rangle - \langle C(t)x(t), C(t)\hat{x}(t)\rangle \\
&\qquad - \langle C(t)\hat{x}(t), C(t)x(t)\rangle + \langle C(t)\hat{x}(t), C(t)\hat{x}(t)\rangle\}dt\Big] \\
&= \frac{1}{\epsilon}\mathbb{E}\Big[\int_0^T \langle C(t)(x(t)-\hat{x}(t)), C(t)\hat{x}(t) - y(t)\rangle dt\Big] \\
&\quad + \frac{1}{2\epsilon}\mathbb{E}\Big[\int_0^T \|C(t)(x(t)-\hat{x}(t))\|^2 dt\Big]. \\
&= I_A + I_B. \qquad (3.11)
\end{aligned}
$$

From (3.10), it follows that

$$\lim_{\epsilon\to 0} I_B = 0. \tag{3.12}$$

The first term of (3.11) can be represented as,

$$
\begin{aligned}
I_A &= \frac{1}{\epsilon}\mathbb{E}\int_{t_0}^T \langle C(t)(x(t)-\hat{x}(t)), C(t)\hat{x}(t) - y(t)\rangle dt \\
&= \frac{1}{\epsilon}\mathbb{E}\int_{t_0}^{t_0+\epsilon} \langle C(t)(x(t)-\hat{x}(t)), C(t)\hat{x}(t) - y(t)\rangle dt \\
&\quad + \frac{1}{\epsilon}\mathbb{E}\int_{t_0+\epsilon}^{T} \langle C(t)(x(t)-\hat{x}(t)), C(t)\hat{x}(t) - y(t)\rangle dt \\
&= I_1 + I_2.
\end{aligned}
$$

Considering (3.10), it holds that

$$\lim_{\epsilon\to 0} I_1 = 0. \tag{3.13}$$

Therefore, for $\epsilon \to 0$, (3.11) becomes that,

$$0 \leq \lim_{\epsilon\to 0} I_2.$$

Consider,

$$\mathbb{E}\|f(s,x(s))-f(s,\hat{x}(s))\|^2 \le l_1^2\|k\|^2\int_0^T \mathbb{E}\|x(t)-\hat{x}(t)\|^2dt \le l_1^2\|k\|^2T[\epsilon l_2c_2\exp(c_2C)].$$

Then, we have that,

$$\begin{aligned}&\mathbb{E}\Big\|\int_{t_0}^{t_0+\epsilon}W(t-s)[f(s,x(s))-f(s,\hat{x}(s))]ds\Big\|^2\\ &\le M\int_{t_0}^{t_0+\epsilon}\mathbb{E}\|f(s,x(s))-f(s,\hat{x}(s))\|^2ds\\ &\le Ml_1^2\|k\|^2\int_{t_0}^{t_0+\epsilon}\mathbb{E}\|x(s)-\hat{x}(s)\|^2ds\\ &\le Ml_1^2\|k\|^2T[\epsilon c_2l_2\exp(c_2C)](t_0+\epsilon-t_0)\\ &\le \epsilon^2c_2Tl_1^2\|k\|^2Ml_2\exp(c_2C). \qquad (3.14)\end{aligned}$$

Also we have that,

$$\begin{aligned}&\mathbb{E}\Big\|\int_{t_0}^{t_0+\epsilon}W(t-s)[g(s,x(s))-g(s,\hat{x}(s))]dw(s)\Big\|^2\\ &\le ML_gM_g\int_{t_0}^{t_0+\epsilon}\mathbb{E}\|x(s)-\hat{x}(s)\|^2ds\\ &\le ML_gM_g[\epsilon c_2l_2\exp(c_2C)](t_0+\epsilon-t_0)\\ \Rightarrow\ &\mathbb{E}\Big\|\int_{t_0}^{t_0+\epsilon}W(t-s)[g(s,x(s))-g(s,\hat{x}(s))]dw(s)\Big\|^2 \qquad (3.15)\\ &\le \epsilon^2c_2Ml_2L_gM_g\exp(c_2C),\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\Big\|\int_{t_0}^{t_0+\epsilon}W(t-s)\int_Z[e(s,x(s),\eta)-e(s,\hat{x}(s),\eta)]\tilde{N}(ds,d\eta)\Big\|^2\\ &\le MM_e\int_{t_0}^{t_0+\epsilon}\mathbb{E}\|x(s)-\hat{x}(s)\|^2ds\\ &\le \epsilon^2c_2Ml_2M_e\exp(c_2C). \qquad (3.16)\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbb{E}\|x(t)-\hat{x}(t)\|^2\\ =\ &\lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big[\mathbb{E}\Big\|\int_{t_0}^{t_0+\epsilon}W(t-s)\Big\{[f(s,x(s))-f(s,\hat{x}(s))]+B(v-\hat{u}(s))\Big\}ds\\ &+\int_{t_0}^{t_0+\epsilon}W(t-s)[g(s,x(s))-g(s,\hat{x}(s))]dw(s)\end{aligned}$$

$$+\int_{t_0}^{t_0+\epsilon} W(t-s)\int_Z [e(s,x(s),\eta)-e(s,\hat{x}(s),\eta)]\tilde{N}(ds,d\eta)\Big\|^2\Big]$$

$$\leq \mathbb{E}\|W(t-t_0)B(v-\hat{u})(t_0)\|^2. \tag{3.17}$$

Thus, as in (3.11) we have that,

$$0 \leq \lim_{\epsilon\to 0} I_2 \leq \int_{t_0}^{T} \mathbb{E}\langle C(t)W(t-t_0)B(v-\hat{u})(t_0), C(t)\hat{x}(t)-y(t)\rangle dt$$

that is, from (3.11) to (3.13), it follows that

$$\int_s^T \mathbb{E}\langle C(t)W(t-s)B(v-\hat{u})(s), C(t)\hat{x}(t)-y(t)\rangle dt \geq 0.$$

which holds for all $v \in U$ and for every Lebesgue point s of $\hat{u}$. Hence, we

$$\int_s^T \mathbb{E}\langle (v-\hat{u})(s), B^*W^*(t-s)C^*(t)[C(t)\hat{x}(t)-y(t)]\rangle dt \geq 0$$
$$\Rightarrow \mathbb{E}\langle (v-\hat{u})(s), B^*z(s)\rangle \leq 0,$$

where

$$z(s) = \int_s^T W^*(t-s)C^*(t)\{y(t)-C(t)\hat{x}(t)\}dt.$$

Here $z(s)$ is a solution in the weak sense of the equation (3.5). Hence

$$\mathbb{E}[\langle v, B^*(s)\rangle - \langle \hat{u}(s), B^*(s)\rangle] \leq 0,$$
$$\mathbb{E}[\langle v, B^*(s)\rangle] \leq \mathbb{E}[\langle \hat{u}(s), B^*(s)\rangle].$$

Thus, the stochastic maximum principle holds for every $v \in U$. □

4. Example

This example concerns with an optimal control of stochastic Burgers equation. Consider the retarded stochastic type Burgers equation with Poisson jumps,

$$\partial z(t,x) = \Big[\gamma \frac{\partial^2}{\partial x^2} z(t,x) + \int_{-h}^{0} a(s)A_1 z(t+s)ds + F(t,z(t,x)) + p(x)\nu(t)\Big]\partial t$$
$$+ G(t,z(t,x))\partial w(t) + \int_Z \eta E(t,(z(t,x))N(ds,d\eta), \tag{4.1}$$
$$0 \leq x \leq \pi,\ t \in J=[0,T],$$

with viscosity $\gamma > 0$, the Dirichlet boundary conditions

$$z(t,0) = z(t,\pi) = 0, \quad t \geq 0, \tag{4.2}$$

and the initial condition

$$z(t,x) = \phi(t,x), \quad t \in [-h,0) \quad 0 \leq x \leq \pi. \tag{4.3}$$

Let $(\Omega, \mathfrak{F}_t, \mathbb{P})$ be the complete probability space and $H = L^2([0,\pi])$. Let w be the H-valued Wiener process and $N(dt,d\eta)$ be the Poisson counting measure and the compensated martingale measure is given by $\tilde{N}(dt, d\eta) = N(dt, d\eta) - \lambda(d\eta)dt$. Let $A_0 : D(A_0) \subseteq H \to V$ be an operator defined by

$$A_0\xi = \gamma \frac{\partial^2}{\partial x^2}\xi,$$

with the domain

$$D(A_0) = \Big\{\xi \in H : \xi \text{ and } \Big(\frac{d}{dx}\Big)\xi \text{ are absolutely continuous}, \Big(\frac{\partial^2}{\partial x^2}\Big)\xi \in H,\ \xi(0) = \xi(\pi) = 0\Big\}.$$

Further, A_0 has a discrete spectrum, the eigen values are $-n^2$, $n = 1, 2, 3, \ldots$ with the corresponding eigen vectors, $e_n(s)= \sqrt{\frac{2}{\pi}}\sin(ns)$ and the set $\{e_n : n = 1, 2, 3, \ldots\}$ is an orthonormal basis of H. Then,

$$A_0\xi = \sum_{n=1}^{\infty} -n^2\langle \xi, e_n\rangle e_n, \quad \xi \in H.$$

Thus A_0 generates a compact semigroup $S(t)$, $t > 0$ in H and is given by

$$S(t)\xi = \sum_{n=1}^{\infty} e^{-n^2}\langle \xi, e_n\rangle e_n, \quad \xi \in H.$$

Let $z(t)(x) = z(t, x)$. Let $f : J \times V \to H$ be defined by

$$f(t, z(t))(x) = F(t, z(t)(x)), \quad 0 \le x \le \pi, (t, z(t)) \in J \times H.$$

Let $g : J \times V \to L_Q(H)$ be defined by

$$g(t, z(t))(x) = G(t, z(t)(x)), \quad 0 \le x \le \pi, (t, z(t)) \in J \times H.$$

Let $e : J \times V \times Z \to H$ be defined by

$$e(t, z(t)(x), (\eta)) = \eta E(t, z(t)(x)), \quad 0 \le x \le \pi, (t, z(t)) \in J \times H.$$

Let B be a continuous linear operator from Y to H. Put $B\nu(t) = p(x)\nu$. We take the functions $\nu : Sx([0, \pi]) \to R$ such that $\nu \in L^2(Sx([0, \pi]))$ as the controls. Set $U(t) = \{\nu \in Y : \|\nu\|_Y \le \tau\}$ where $\tau \in L^2(J, R^+)$. We restrict the admissible controls U_{ad} to be all the $\nu \in L^2(Sx([0, \pi]))$, such that $\|\nu(\cdot, t)\|^2 \le \tau(t)$ $a.e$ (see [15]). The solution of (4.1) is given as $z(t, x)$. We consider the cost function:

(4.4)

$$\mathcal{J}(z, \nu) = \mathbb{E}\Big[\int_0^T \int_{[0,\pi]} \|e^{tx}z(t, x) - r(t, x)\|^2 dxdt + \int_0^T \int_{[0,\pi]} \|\nu(t, x)\|^2 dxdt\Big],$$

with respect to the system (4.1), where $r(t, x)$ is a given desired velocity profile with a suitable control $\nu(t, x)$ and $e^{tx} = C(t)$ be a bounded function in $\mathcal{Z}$. The aim of the cost functional is that to get a final velocity field to be close to $r(t, x)$.

Thus, with this choice of A_0, B, f, g and e the problem (4.1) can be written in the abstract form of (1.1) with the cost function (3.2). Hence, all the hypotheses in Theorem 3.3 are satisfied. That is, there exists an admissible control $\hat{\nu} \in U_{ad}$ such that $\mathcal{J}(z, \hat{\nu}) \leq \mathcal{J}(z, \nu) \, \forall \, \nu \in U_{ad}$. Also, the nonlinear functions such as f, g and e satisfies the Lipschitz conditions and let $\hat{\nu}$ be the optimal control then

$$\max_{v \in U} \mathbb{E}\langle v, B^* z_1(s) \rangle = \mathbb{E}\langle \hat{\nu}(s), B^* z_1(s) \rangle,$$

where the function $z_1(s)$ satisfies the corresponding transposed system of (4.1)-(4.3) as in the Theorem 3.4.

5. Conclusion

This paper has been investigated about the optimal control on semilinear stochastic functional differential equations with Poisson jumps in Hilbert space by using the construction of the fundamental solution operator. We proved the existence of an optimal control which is obtained by the solution of the proposed system should satisfies the weakly sequentially compactness and Mazur's theorem. Further the stochastic maximum principle for the optimal control have been formulated and proved for the proposed system by setting a spike variation technique in the optimal control. Finally, an example is given to illustrate this theory. In the future, the authors are interested to study the optimal control results on multi-valued fractional stochastic partial differential equations with Poisson jumps in Hilbert space using variational inequalities.

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