

MEROMORPHIC FUNCTIONS PARTIALLY SHARED VALUES WITH THEIR SHIFTS

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ABSTRACT. We prove some uniqueness theorems of nonconstant meromorphic functions partially sharing values with their shifts. As an application, we obtain a sufficient condition on periodic meromorphic functions. Moreover, some examples are given to illustrate that the conditions are sharp and necessary.

1. Introduction and main results

Throughout this paper, a meromorphic function always means meromorphic in the whole complex plane \mathbb{C} . We assume that the reader is familiar with the fundamental concepts of Nevanlinna's value distribution theory [15] and in particular with the most usual of symbols: $m(r, f)$, $N(r, f)$, $T(r, f)$. Here and in the follows, f is a meromorphic function in the complex plane. Meanwhile, the order $\rho(f)$ and the hyper-order $\rho_2(f)$ of a meromorphic function f are defined in turn as follows:

$$\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ T(r, f)}{\log r}, \quad \rho_2(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ \log^+ T(r, f)}{\log r}.$$

We denote $\mathcal{S}(f)$ as the family of all meromorphic functions α such that $T(r, \alpha) = o(T(r, f))$, where $r \rightarrow \infty$ outside of a possible exceptional set of finite logarithmic measure. Moreover, we also include all constant functions in $\mathcal{S}(f)$, and let $\hat{\mathcal{S}}(f) = \mathcal{S}(f) \cup \{\infty\}$. Given $a \in \hat{\mathcal{S}}(f)$, we say that two meromorphic functions f and g share a IM when $f - a$ and $g - a$ have the same zeros. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that f and g share a CM.

In addition, let $\bar{E}(a, f)$ denote the set of zeros of $f - a$, where a zero is counted only once in the set, and $\bar{E}_k(a, f)$ denote the set of zeros of $f - a$ with

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multiplicity $l \leq k$, where a zero with multiplicity l is counted only once in the set. The reduced counting function corresponding to $\overline{E}_k(a, f)$ are denoted by $\overline{N}_k(r, \frac{1}{f-a})$.

The relations between two nonconstant meromorphic functions f and g have been studied by R. Nevalinna using value distribution theory of meromorphic functions since 1925. The famous results in this direction are Five-Points Theorem and Four-Points Theorem obtained by R. Nevalinna, which say that if two nonconstant meromorphic functions f and g share five distinct values IM, then $f = g$, and if two nonconstant meromorphic functions f and g share four distinct values CM, then $f = T \circ g$, where T is a Möbius map, respectively. Afterwards, there are many results on uniqueness of meromorphic functions concerning shared values or sets. One could see the monograph [15] written by Yang and Yi for details.

In recent decade, R. G. Halburd and R. J. Korhonen [10] and, independently, Y. M. Chiang and S. J. Feng [2] developed a parallel difference version of classical Nevanlinna theory for meromorphic functions of finite order, which had been extended by R. G. Halburd, R. J. Korhonen and K. Tohge [9] to meromorphic functions of hyper-order strictly less than one in 2014. Those powerful tools allow us to deeply investigate the relations between f and $f(z+c)$ [5–7, 12, 13].

Recently, X. M. Li et al. [11] considered uniqueness questions of meromorphic functions sharing four values with their shifts as follows.

Theorem 1.1 ([11]). *Let f be a nonconstant meromorphic function of hyper-order $\rho_2(f) < 1$ and $\eta \in \mathbb{C} \setminus \{0\}$. Suppose that f and $f(z + \eta)$ share $0, 1, c$ IM, and share ∞ CM, where c is a finite value such that $c \neq 0, 1$. Then $f(z) = f(z + \eta)$ for all $z \in \mathbb{C}$.*

In 2016, K. S. Charak, R. J. Korhonen and G. Kumar introduced the notion of partially shared value and proved the following result under an appropriate deficiency assumption.

Theorem 1.2 ([3]). *Let f be a nonconstant meromorphic function of hyper-order $\rho_2(f) < 1$ and $c \in \mathbb{C} \setminus \{0\}$. Let $a_1, a_2, a_3, a_4 \in \hat{S}(f)$ be four distinct periodic functions with period c . If $\delta(a, f) > 0$ for some $a \in \hat{S}(f)$ and*

$$\overline{E}(a_j, f) \subseteq \overline{E}(a_j, f(z+c)), \quad j = 1, 2, 3, 4$$

then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.

Naturally, we are interesting to know what happens if the condition in Theorem 1.2 “partially shared value $\overline{E}(a, f) \subseteq \overline{E}(a, f(z+c))$ ” is replaced by “truncated partially shared value $\overline{E}_k(a, f) \subseteq \overline{E}_k(a, f(z+c))$, k is a positive integer”. However, we find that Theorem 1.2 can not be valid for each $k \in \mathbb{Z}^+$ even if f and $f(z + \eta)$ share some $a \in \{a_1, a_2, a_3, a_4\}$ CM. Indeed, we can find it as follows.

Example 1.3. Let $f(z) = \frac{2e^z}{e^{2z}+1}$, $c = \pi i$, $a_1 = 1$, $a_2 = -1$, $a_3 = 0$, $a_4 = \infty$ and $k = 1$. Then, $f(z + \pi i) = -\frac{2e^z}{e^{2z}+1}$, f and $f(z + \pi i)$ share a_3, a_4 CM, and satisfy $\overline{E}_1(a_j, f) = \overline{E}_1(a_j, f(z + c)) = \emptyset$, $j = 1, 2$, but $f(z) \not\equiv f(z + c)$.

In present paper we shall prove the following results under partially shared values.

Theorem 1.4. *Let f be a nonconstant meromorphic function of hyper-order $\rho_2(f) < 1$ and $c \in \mathbb{C} \setminus \{0\}$. Let k_1, k_2 be two positive integers, and let $a_1, a_2 \in \mathcal{S}(f) \setminus \{0\}$, $a_3, a_4 \in \hat{\mathcal{S}}(f)$ be four distinct periodic functions with period c such that f and $f(z + c)$ share a_3, a_4 CM and*

$$\overline{E}_{k_j}(a_j, f) \subseteq \overline{E}_{k_j}(a_j, f(z + c)), \quad j = 1, 2.$$

If $\Theta(0, f) + \Theta(\infty, f) > \frac{2}{k+1}$, where $k := \min\{k_1, k_2\}$, then $f(z) = f(z + c)$ for all $z \in \mathbb{C}$.

The above example shows that the condition “ $\Theta(0, f) + \Theta(\infty, f) > \frac{2}{k+1}$ ” in Theorem 1.4 is necessary and sharp since $k_1 = k_2 = 1$ and $\Theta(0, f) + \Theta(\infty, f) = 1$.

As the consequence of Theorem 1.4, we have:

Theorem 1.5. *Let f be a nonconstant meromorphic function of hyper-order $\rho_2(f) < 1$, $\Theta(\infty, f) = 1$ and $c \in \mathbb{C} \setminus \{0\}$. Let $a_1, a_2, a_3 \in \mathcal{S}(f)$ be three distinct periodic functions with period c such that f and $f(z + c)$ share a_3 CM and*

$$\overline{E}_k(a_j, f) \subseteq \overline{E}_k(a_j, f(z + c)), \quad j = 1, 2.$$

If $k \geq 2$, then $f(z) = f(z + c)$ for all $z \in \mathbb{C}$.

The number “ $k = 2$ ” is sharp. Consider $f(z) = \sin z$ and $c = \pi$, then we know that $f(z + c)$ and f share 0 CM, $\overline{E}_1(1, f) = \overline{E}_1(1, f(z + c)) = \emptyset$ and $\overline{E}_1(-1, f) = \overline{E}_1(-1, f(z + c)) = \emptyset$, but $f(z) \not\equiv f(z + c)$.

Naturally, we are interesting to find what happens in details when $k = 1$. Indeed, we obtain the following result.

Theorem 1.6. *Let f be a nonconstant meromorphic function of hyper-order $\rho_2(f) < 1$ with $\Theta(\infty, f) = 1$, $a_1, a_2, a_3 \in \mathcal{S}(f)$ be three distinct periodic functions with period c such that f and $f(z + c)$ share a_3 CM and*

$$\overline{E}_1(a_j, f) \subseteq \overline{E}_1(a_j, f(z + c)), \quad j = 1, 2.$$

Then $f(z) = f(z + c)$ or $f(z) = -f(z + c)$ for all $z \in \mathbb{C}$. Moreover, the later occurs only if $a_1 + a_2 = 2a_3$.

Remark 1.7. The result “ $f(z) \equiv -f(z + c)$ ” could be occurred, just as the above example $f(z) = \sin z$ and $c = \pi$. Moreover, we give the following example to show that the assumption “ $\Theta(\infty, f) = 1$ ” in Theorem 1.6 cannot be omitted. Let $f(z) = \frac{1}{\sin z}$ and $c = \frac{\pi}{2}$, and let $a_1 = 1, a_2 = -1$, then f and $f(z + \frac{\pi}{2})$ share 0 CM, $\overline{E}_1(a_j, f) = \overline{E}_1(a_j, f(z + \frac{\pi}{2})) = \emptyset$, $j = 1, 2$, and $\Theta(\infty, f) = 0$. However, $f(z) \not\equiv f(z + \frac{\pi}{2})$ and $f(z) \not\equiv -f(z + \frac{\pi}{2})$.

It is easy to obtain the consequence of Theorem 1.6 as follows.

Corollary 1.8. *Under the assumptions of Theorem 1.6, f is a periodic function with period c or $2c$.*

Remark 1.9. The restrictions on the growth of f in our above results are necessary and sharp by the following examples.

Example 1.10. Let $f(z) = e^{\sin z}$ and $c = \pi$. Then, we obtain that f and $f(z+c)$ share $0, 1, -1, \infty$ CM, and $\rho_2(f) = 1$, but $f(z) \not\equiv f(z+c)$ and $f(z) \not\equiv -f(z+c)$.

Furthermore, according to the result obtained by Ozawa (see [14]), for an arbitrary number $\sigma \in (1, \infty)$, there exists a periodic entire function $D(z)$ with period $c \neq 0$ such that $\rho(D) = \sigma$. Let

$$f(z) = e^{D(z) \cdot \sin \frac{\pi z}{c}}.$$

Then, we can see that f and $f(z+c)$ share $0, 1, -1, \infty$ CM and that $\rho_2(f) = \sigma \in (1, \infty)$ since $\rho(D) > \rho(\sin \frac{\pi z}{c}) = 1$. However, $f(z+c) = e^{-D(z) \cdot \sin \frac{\pi z}{c}}$, so $f(z) \not\equiv f(z+c)$ and $f(z) \not\equiv -f(z+c)$.

2. Some lemmas

Here and throughout this paper, we denote $S(r, *)$ as the quality $o(T(r, *))$ as $r \rightarrow \infty$ outside of a possible exceptional set of finite logarithmic measure.

Lemma 2.1 ([15]). *Suppose that h is a nonconstant entire function and $f = e^h$, then $\rho_2(f) = \rho(h)$.*

Lemma 2.2 ([15]). *Let f be a meromorphic function. If*

$$g = \frac{af + b}{cf + d},$$

where $a, b, c, d \in \mathcal{S}(f)$ and $ad - bc \not\equiv 0$, then

$$T(r, g) = T(r, f) + S(r, f).$$

Lemma 2.3 ([16]). *Let f be a nonconstant meromorphic function, $a_j \in \hat{\mathcal{S}}(f)$, $j = 1, 2, \dots, q$ ($q \geq 3$). Then for any positive real number ε , we have*

$$(q - 2 - \varepsilon)T(r, f) \leq \sum_{j=1}^q \bar{N} \left(r, \frac{1}{f - a_j} \right), \quad r \notin E,$$

where $E \subset [0, +\infty)$ and satisfies $\int_E d \log \log r < \infty$.

The first difference analogue of the lemma on the logarithmic derivative was proved in [2, 8]. The following is its extension to the case of hyper-order < 1 , see [9].

Lemma 2.4 ([9]). *Let f be a nonconstant meromorphic function and $c \in \mathbb{C}$. If f is of finite order, then*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) = O\left(\frac{\log r}{r} T(r, f)\right)$$

for all r outside of a set E with zero logarithmic density. If the hyper-order ρ_2 of f is less than one, then for each $\varepsilon > 0$, we have

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) = o\left(\frac{T(r, f)}{r^{1-\rho_2-\varepsilon}}\right)$$

for all r outside of a set of finite logarithmic measure.

Lemma 2.5 ([9]). *Let $T : [0, +\infty) \rightarrow [0, +\infty)$ be a non-decreasing continuous function, and let $s \in (0, +\infty)$. If the hyper-order of T is strictly less than one, i.e.,*

$$\limsup_{r \rightarrow \infty} \frac{\log^+ \log^+ T(r)}{\log r} = \rho_2 < 1,$$

then

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^{1-\rho_2-\varepsilon}}\right),$$

where $\varepsilon > 0$ and $r \rightarrow \infty$ outside of a set of finite logarithmic measure.

3. Proof of Theorem 1.4

We suppose that $a_1, a_2, a_3, a_4 \in \mathbb{C}$. By Lemma 2.1 and the assumption that $f(z)$ and $f(z+c)$ share a_3, a_4 CM, we have

$$(1) \quad \frac{(f(z) - a_3)(f(z+c) - a_4)}{(f(z) - a_4)(f(z+c) - a_3)} = e^{Q(z)},$$

where $Q(z)$ is an entire function with $\rho(Q) < 1$. According to Lemma 2.4, we know that

$$T(r, e^Q) = S(r, f).$$

Together this with Lemma 2.2, it follows that

$$(2) \quad T(r, f(z+c)) = T(r, f) + S(r, f).$$

For each $z_0 \in \overline{E}_k(a_1, f) \cup \overline{E}_k(a_2, f)$, from (1) we deduce that $e^{Q(z_0)} = 1$. For convenience, we set $\alpha := e^{Q(z)}$ and

$$S(r) := S(r, f(z+c)) = S(r, f).$$

If $e^{Q(z)} \neq 1$, then we can deduce that

$$(3) \quad \overline{N}_{k_1}\left(r, \frac{1}{f-a_1}\right) \leq N\left(r, \frac{1}{\alpha-1}\right) \leq T(r, \alpha) + O(1) = S(r)$$

and

$$(4) \quad \overline{N}_{k_2}\left(r, \frac{1}{f-a_2}\right) \leq N\left(r, \frac{1}{\alpha-1}\right) \leq T(r, \alpha) + O(1) = S(r).$$

Without loss of generality, we suppose that $a_3, a_4 \in \mathcal{S}(f) \setminus \{0\}$, by Lemma 2.3 for $\varepsilon \in \left(0, \frac{1}{3}(\Theta(0, f) + \Theta(\infty, f) - \frac{2}{k+1})\right)$, we have

$$(4 - \varepsilon)T(r, f) \leq \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f}\right) + \sum_{j=1}^4 \bar{N}\left(r, \frac{1}{f - a_j}\right) + S(r, f),$$

combine (3) and (4), it then follows that

$$(2 - \varepsilon)T(r, f) \leq \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f}\right) + \sum_{j=1}^2 \bar{N}_{(k_j+1)}\left(r, \frac{1}{f - a_j}\right) + S(r, f).$$

Moreover, we have

$$(2 - \varepsilon)T(r, f) \leq \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f}\right) + \frac{2}{k+1}T(r, f) + S(r, f),$$

it yields that

$$\Theta(0, f) + \Theta(\infty, f) \leq \frac{2}{k+1},$$

which is a contradiction.

Therefore, we have $e^{Q(z)} \equiv 1$, that is ,

$$\frac{(f(z) - a_3)(f(z+c) - a_4)}{(f(z) - a_4)(f(z+c) - a_3)} \equiv 1,$$

this implies that $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.

The proof of Theorem 1.4 completes.

4. Proof of Theorem 1.6

Assume that $Q(z)$ is the canonical product of the poles of f , by Lemma 2.4, we obtain

$$(5) \quad m\left(r, \frac{Q(z+c)}{Q(z)}\right) = S(r, f).$$

Note that $\Theta(\infty, f) = 1$, from (5) we have

$$(6) \quad T\left(r, \frac{Q(z+c)}{Q(z)}\right) = S(r, f).$$

By Lemma 2.1 and the assumption that $f(z)$ and $f(z+c)$ share a_3 CM, we have

$$(7) \quad \frac{f(z+c) - a_3}{f(z) - a_3} = e^{H(z)} \cdot \frac{Q(z)}{Q(z+c)},$$

where $H(z)$ is an entire function with $\rho(H) < 1$. According to Lemma 2.4, we know that

$$T\left(r, e^{H(z)} \cdot \frac{Q(z)}{Q(z+c)}\right) = S(r, f).$$

Together this with Lemma 2.2, it follows that

$$(8) \quad T(r, f(z+c)) = T(r, f) + S(r, f).$$

For convenience, we set

$$f_c(z) := f(z+c), \quad \alpha := e^{H(z)} \cdot \frac{Q(z)}{Q(z+c)},$$

and

$$S(r) := S(r, f(z+c)) = S(r, f).$$

If $f(z) \not\equiv f(z+c)$, that is $\alpha(z) \not\equiv 1$. By (7) and the assumptions of Theorem 1.6, one can see that

$$(9) \quad \bar{N}_1 \left(r, \frac{1}{f-a_1} \right) \leq N \left(r, \frac{1}{\alpha-1} \right) \leq T(r, \alpha) + O(1) = S(r)$$

and

$$(10) \quad \bar{N}_1 \left(r, \frac{1}{f-a_2} \right) \leq N \left(r, \frac{1}{\alpha-1} \right) \leq T(r, \alpha) + O(1) = S(r).$$

Using Lemma 2.5, from (9) and (10) we have

$$(11) \quad \begin{aligned} \bar{N}_1 \left(r, \frac{1}{f_c-a_1} \right) &\leq \bar{N}_1 \left(r+|c|, \frac{1}{f-a_1} \right) \\ &= \bar{N}_1 \left(r, \frac{1}{f-a_1} \right) + S(r) = S(r) \end{aligned}$$

and

$$(12) \quad \begin{aligned} \bar{N}_1 \left(r, \frac{1}{f_c-a_2} \right) &\leq \bar{N}_1 \left(r+|c|, \frac{1}{f-a_2} \right) \\ &= \bar{N}_1 \left(r, \frac{1}{f-a_2} \right) + S(r) = S(r). \end{aligned}$$

Moreover, from (7), we can deduce that

$$(13) \quad N \left(r, \frac{1}{f_c-a_1} \right) = N \left(r, \frac{1}{f - \frac{(\alpha-1)a_3+a_1}{\alpha}} \right) + S(r)$$

and

$$(14) \quad N \left(r, \frac{1}{f_c-a_2} \right) = N \left(r, \frac{1}{f - \frac{(\alpha-1)a_3+a_2}{\alpha}} \right) + S(r).$$

Next, we deal with the following three cases.

Case 1. Assume that $\frac{(\alpha-1)a_3+a_1}{\alpha} \not\equiv a_2$. Since $\frac{(\alpha-1)a_3+a_1}{\alpha} \not\equiv a_1$ and $\Theta(\infty, f) = 1$, by Lemma 2.3 for $\varepsilon \in (0, \frac{1}{2})$, it follows from (7), (9), (10), (11) and (13) that

$$(2-\varepsilon)T(r, f)$$

$$\begin{aligned}
&\leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f-a_1}\right) + \overline{N}\left(r, \frac{1}{f-a_2}\right) + \overline{N}\left(r, \frac{1}{f-\frac{(\alpha-1)a_3+a_1}{\alpha}}\right) + S(r) \\
&\leq \overline{N}_{(2)}\left(r, \frac{1}{f-a_1}\right) + \overline{N}_{(2)}\left(r, \frac{1}{f-a_2}\right) + \overline{N}_{(2)}\left(r, \frac{1}{f_c-a_1}\right) + S(r) \\
&\leq \frac{1}{2}T(r, f) + \frac{1}{2}T(r, f) + \frac{1}{2}T(r, f_c) + S(r) \\
&\leq \frac{3}{2}T(r, f) + S(r),
\end{aligned}$$

which is a contradiction.

Case 2. Assume that $\frac{(\alpha-1)a_3+a_2}{\alpha} \neq a_1$. Since $\frac{(\alpha-1)a_3+a_2}{\alpha} \neq a_2$ and $\Theta(\infty, f) = 1$, by Lemma 2.3 for $\varepsilon \in (0, \frac{1}{2})$, it follows from (7), (9), (10), (12) and (14) that

$$\begin{aligned}
&(2-\varepsilon)T(r, f) \\
&\leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f-a_1}\right) + \overline{N}\left(r, \frac{1}{f-a_2}\right) + \overline{N}\left(r, \frac{1}{f-\frac{(\alpha-1)a_3+a_2}{\alpha}}\right) + S(r) \\
&\leq \overline{N}_{(2)}\left(r, \frac{1}{f-a_1}\right) + \overline{N}_{(2)}\left(r, \frac{1}{f-a_2}\right) + \overline{N}_{(2)}\left(r, \frac{1}{f_c-a_2}\right) + S(r) \\
&\leq \frac{1}{2}T(r, f) + \frac{1}{2}T(r, f) + \frac{1}{2}T(r, f_c) + S(r) \\
&\leq \frac{3}{2}T(r, f) + S(r),
\end{aligned}$$

which is also a contradiction.

Case 3. Assume that $\frac{(\alpha-1)a_3+a_2}{\alpha} \equiv a_1$ and $\frac{(\alpha-1)a_3+a_1}{\alpha} \equiv a_2$. Therefore, we get $\alpha \equiv -1$, so that $2a_3 = a_1 + a_2$.

The proof of Theorem 1.6 completes.

5. Applications

As we known, the periodic functions and elliptic functions have found a wide utilization in many fields [1, 4, 5, 14, 17]. They are also widely used to be considered as the solutions of classic differential equations, including the complex Kuramoto-Sivashinsky equation, the k order Briot-Bouquet equations and so on. Therefore, it is interesting and important to study the sufficient conditions for periodicity of meromorphic functions. In present paper, we also show the following as the applications of Theorem 1.5 and Theorem 1.6.

Theorem 5.1. *Suppose that f and g are two nonconstant meromorphic functions with $\Theta(\infty, f) = \Theta(\infty, g) = 1$, where f has a nonzero period $c \in \mathbb{C}$ with hyper-order $\rho_2(f) < 1$. Let k_1, k_2 be two positive integers, $a_1, a_2, a_3 \in \mathcal{S}(f)$ be three distinct periodic functions with period c such that f and g share a_3 CM and*

$$\overline{E}_{k_1}(a_j, f) = \overline{E}_{k_2}(a_j, g), \quad j = 1, 2.$$

Then g is a periodic function with period T , where $T \in \{c, 2c\}$, that is, $g(z) = g(z + T)$ for all $z \in \mathbb{C}$.

Proof. We just have to check that g satisfies the conditions of Theorem 1.5 or Theorem 1.6. By the Nevanlinna's second fundamental theorem, we have

$$\begin{aligned} T(r, f) &\leq \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f - a_1}\right) + \bar{N}\left(r, \frac{1}{f - a_3}\right) + S(r, f) \\ &\leq \bar{N}_{k_1}\left(r, \frac{1}{f - a_1}\right) + \frac{1}{k_1 + 1}N\left(r, \frac{1}{f - a_1}\right) + \bar{N}\left(r, \frac{1}{f - a_3}\right) + S(r, f) \\ &\leq \bar{N}_{k_1}\left(r, \frac{1}{g - a_1}\right) + \frac{1}{k_1 + 1}T(r, f) + \bar{N}\left(r, \frac{1}{g - a_3}\right) + S(r, f) \\ &\leq 2T(r, g) + \frac{1}{2}T(r, f) + S(r, f), \end{aligned}$$

that is

$$T(r, f) \leq 4T(r, g) + S(r, f).$$

Similarly, we also have

$$T(r, g) \leq 4T(r, f) + S(r, g).$$

Thus, it follows that

$$(15) \quad S(r, f) = S(r, g) \doteq S(r)$$

and

$$(16) \quad \rho_2(g) = \rho_2(f) < 1.$$

On the other hand, since $f(z) \equiv f(z + c)$ and f, g share a_3 CM, we know that $g(z), g(z + c)$ share a_3 CM. Furthermore, it is obvious that

$$\bar{E}_{k_1}(a_1, g) = \bar{E}_{k_1}(a_1, f) = \bar{E}_{k_1}(a_1, f(z + c)) = \bar{E}_{k_1}(a_1, g(z + c))$$

and

$$\bar{E}_{k_2}(a_2, g) = \bar{E}_{k_2}(a_2, f) = \bar{E}_{k_2}(a_2, f(z + c)) = \bar{E}_{k_2}(a_2, g(z + c)).$$

Therefore, Theorem 5.1 follows from Theorem 1.5 and Theorem 1.6 immediately. \square

References

- [1] G. Brosch, *Eindeutigkeitsätze für meromorphe Funktionen*, Thesis, Technical University of Aachen, 1989.
- [2] Y. M. Chiang and S. J. Feng, *On the Nevanlinna characteristic of $f(z + \eta)$ and difference equations in the complex plane*, Ramanujan J. **16** (2008), no. 1, 105–129.
- [3] K. S. Charak, R. J. Korhonen, and G. Kumar, *A note on partial sharing of values of meromorphic functions with their shifts*, J. Math. Anal. Appl. **435** (2016), no. 2, 1241–1248.
- [4] S. J. Chen and A. Z. Xu, *Periodicity and unicity of meromorphic functions with three shared values*, J. Math. Anal. Appl. **385** (2012), no. 1, 485–490.
- [5] S. J. Chen and W. C. Lin, *Periodicity and uniqueness of meromorphic functions concerning Three sharing values*, Houston J. Math. 2016, Preprint.

- [6] J. Heittokangas, R. Korhonen, I. Laine and J. Rieppo, *Value sharing results for shifts of meromorphic functions, and sufficient conditions for periodicity*, J. Math. Anal. Appl. **355** (2009), no. 1, 352–363.
- [7] ———, *Uniqueness of meromorphic functions sharing values with their shifts*, Complex Var. Elliptic Equ. **56** (2011), no. 1-4, 81–92.
- [8] R. G. Halburd and R. J. Korhonen, *Difference analogue of the lemma on the logarithmic derivative with applications to difference equations*, J. Math. Anal. Appl. **314** (2006), no. 2, 477–487.
- [9] R. Halburd, R. Korhonen, and K. Tohge, *Holomorphic curves with shift-invariant hyperplane preimages*, Trans. Amer. Math. Soc. **366** (2014), no. 8, 4267–4298.
- [10] R. G. Halburd and R. J. Korhonen, *Nevanlinna theory for the difference operator*, Ann. Acad. Sci. Fenn. Math. **31** (2006), no. 2, 463–478.
- [11] X. M. Li and H. X. Yi, *Meromorphic functions sharing four values with their difference operators or shifts*, Bull. Korean Math. Soc. **53** (2016), no. 4, 1213–1235.
- [12] K. Liu, *Meromorphic functions sharing a set with applications to difference equations*, J. Math. Anal. Appl. **359** (2009), no. 1, 384–393.
- [13] K. Liu and L.-Z. Yang, *Value distribution of the difference operator*, Arch. Math. (Basel) **92** (2009), no. 3, 270–278.
- [14] M. Ozawa, *On the existence of prime periodic entire functions*, Kōdai Math. Sem. Rep. **29** (1977/78), no. 3, 308–321.
- [15] C. C. Yang and H. X. Yi, *Uniqueness theory of meromorphic functions*, Mathematics and its Applications, **557**, Kluwer Academic Publishers Group, Dordrecht, 2003.
- [16] K. Yamanoi, *The second main theorem for small functions and related problems*, Acta Math. **192** (2004), no. 2, 225–294.
- [17] J. H. Zheng, *Unicity theorem for period meromorphic functions that share three values*, Chi. Sci. Bull. **37** (1992), no. 1, 12–15.

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