Bull. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/BKMS.b170053
pISSN: 1015-8634 / eISSN: 2234-3016

A SUFFICIENT CONDITION FOR ACYCLIC 5-CHOOSABILITY OF PLANAR GRAPHS WITHOUT 5-CYCLES

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Abstract. A proper vertex coloring of a graph G is acyclic if G contains no bicolored cycle. A graph G is acyclically L-list colorable if for a given list assignment $L = \{L(v) : v \in V(G)\}$, there exists an acyclic coloring ϕ of G such that $\phi(v) \in L(v)$ for all $v \in V(G)$. A graph G is acyclically k-choosable if G is acyclically L-list colorable for any list assignment with $L(v) \geq k$ for all $v \in V(G)$. Let G be a planar graph without 5-cycles and adjacent 4-cycles. In this article, we prove that G is acyclically 5-choosable if every vertex v in G is incident with at most one i-cycle, $i \in \{6, 7\}$.

1. Introduction

For a given graph $G = (V(G), E(G))$, a *proper k-coloring* of G is an assignment of k colors to the vertices of G such that no two adjacent vertices are assigned the same colour. A proper vertex coloring of G is *acyclic* if every cycle in G uses at least three colors [12]. For a given list assignment $L = \{L(v) : v \in V(G)\}$, a graph G is *acyclically L-list colorable* if G has an acyclic coloring ϕ such that $\phi(v) \in L(v)$ for each vertex $v \in V(G)$, and we say that ϕ is an *acyclic L-coloring* of G. A graph G is *acyclically k-choosable* if G is acyclically L-list colorable for any list assignment L with $L(v) \geq k$ for each $v \in V(G)$. The acyclic list chromatic number of G, denoted by $\chi_a^l(G)$, is the smallest integer k such that G is acyclically k-choosable. In this paper, we only consider finite, simple and undirected graphs.

Borodin et al. [2] proved that every planar graph is acyclically 7-choosable and put forward to a challenging conjecture as follows:

Conjecture 1. *Every planar graph is acyclically 5-choosable.*

So far, Conjecture 1 remains open, and it has been verified only for some restricted planar graphs. It is proved that Conjecture 1 holds for planar graphs

Received January 19, 2017; Revised June 1, 2017; Accepted July 21, 2017.
2010 *Mathematics Subject Classification.* 05C15.
Key words and phrases. planar graph, acyclic coloring, choosable, adjacent cycles, minimal counterexample.

with girth at least 5 by Montassier and Ochemin [15], without 4-cycles by Borodin and Ivanova [3]. Particularly, Borodin and Inanova [4] verified that a planar graph is acyclically 5-choosable if it does not contain an i-cycle adjacent to a j-cycle, where $3 \leq j \leq 5$ if $i = 3$ and $4 \leq j \leq 6$ if $i = 4$. This result absorbs most of the previous work in this direction. For the acyclic 6-choosability, Wang and Chen [17] proved that every planar graph without 4-cycles is acyclically 6-choosable. Later, Wang et al. [18] proved that every planar graph G is acyclic 6-choosable if G does not contain 4-cycles adjacent to i-cycles for each $i \in \{3, 4, 5, 6\}$. For the acyclic 4-choosability of a planar graph G, it is proved that G is acyclic 4-choosable if $g(G) \geq 5$ [14], or G does not contain 4-cycles and triangular 6-cycles [7], or G does not contain an i-cycle adjacent to a j-cycle, where $3 \leq j \leq 6$ if $i = 3$ and $4 \leq j \leq 7$ if $i = 4$ [5], or G does not contain 4-, 7- and 8-cycles [10], or G does not contain 4- and 5-cycles [6,9]. For the graphs embedded on a special surface, Hou and Liu [13] proved that every toroidal graph is acyclically 8-choosable.

The purpose of this paper is to give a sufficient conditions for the acyclic 5-choosable planar graphs in which the number of adjacent 4^--cycles is increased to a certain extent.

Theorem 2. *Let G be a planar graph without 5-cycles and adjacent 4-cycles. Then G is acyclically 5-choosable if every vertex in G is incident with at most one i-cycle, $i \in \{6, 7\}$.*

A plane graph is a planar embedding of a planar graph in the Euclidean plane. For a plane graph G, let $F(G)$ denote its face set. For a vertex $v \in V(G)$, $N_G(v)$ denotes the set of vertices adjacent to v, $d_G(v) = |N_G(v)|$ denotes the degree of v. The degree of a face f of G, denoted by $d_G(f)$, is the number of edges incident with f where each cut edge is counted twice. A k-, k^+- and k^--vertex (face) is a vertex (face) of degree k, at least k and at most k, respectively. A vertex u is called a k-neighbor (resp. k^--neighbor, k^+-neighbor) of a vertex v if $uv \in E(G)$ and $d_G(u) = k$ (resp. $d_G(u) \leq k$, $d_G(u) \geq k$). A k-cycle is a cycle of length k. Two cycles (or faces) are said to be *intersecting* if they have at least one common vertex, and *adjacent* if they have at least one common edge. An edge uv is a (b_1, b_2)-edge if $d_G(u) = b_1$ and $d_G(v) = b_2$. A face $f \in F(G)$ is usually denoted by $f = [u_1 u_2 \cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are the boundary vertices of f in a cyclic order. For convenience, a face $f = [u_1 u_2 \cdots u_n]$ is called an $(a_1, a_2, \ldots, a_n)$-face if the degree of the vertex u_i is a_i for $i = 1, 2, \ldots, n$. For $x \in V(G) \cup F(G)$, let $n_i(x)$ denote the number of i-vertices adjacent to or incident with x and $f_k(v)$ denote the number of k-faces incident with a vertex $v \in V(G)$. For a vertex $v \in V(G)$, if v is incident with two adjacent 3-faces with a common edge vu, then the two adjacent 3-faces are called *bad* faces of v. Let $f_b(v)$ denote the number of bad 3-faces of v. For a 3-face $[uvw]$, if each of the two edges vu and vw is not incident with other 3-face except the face $[uvw]$, then $[uvw]$ is called an *independent* face of v, and let $\iota(v)$ denote the number of independent faces of v. If there is no confusion in the context, we write $d_G(v)$, $N_G(v)$ as $d(v)$ and $N(v)$ for short. For other undefined notations, we refer the readers to [1].

2. Structural properties of the minimum counterexample to Theorem 2

Suppose that G is a counterexample to Theorem 2 with the minimum number of vertices embedded in the plane. Then G is connected. Let L be a list assignment of G with $|L(v)| = 5$ for all $v \in V(G)$, and G is not acyclically L-list colorable, but any proper subgraph G' with $|V(G')| < |V(G)|$ is acyclically L-list colorable.

Firstly, we give the following known lemmas, the proofs of which were provided in [8, 11, 16].

Lemma 3.

($C1$) [16] *There are no* 1*-vertices.*
($C2$) [16] *No* 2*-vertex is adjacent to a* 4^-*-vertex.*
($C3$) [16] *Let* v *be a* 3*-vertex. If* v *is adjacent to a* 3*-vertex, then* v *is not adjacent to other* 4^-*-vertex;*
($C4$) *Let* v *be a* 5*-vertex. Then*
  ($C4.1$) [16] $n_2(v) \leq 1$;
  ($C4.2$) [11] *If* $n_2(v) = 1$ *and* v *is incident with a* 3*-face* f*, then* $n_3(f) = 0$.
($C5$) *Let* v *be a* 6*-vertex. Then*
  ($C5.1$) [16] $n_2(v) \leq 4$;
  ($C5.2$) [16] *If* $n_2(v) = 4$*, then* v *is not adjacent to any* 3*-vertex;*
  ($C5.3$) [11] *If* $n_2(v) = 2$ *and* v *is incident with a* $(3, 3, 6)$*-face, then* $n_3(v) \leq 2$;
  ($C5.4$) [11] *If* $n_2(v) = 3$*, then* $n_3(v) \leq 1$;
  ($C5.5$) [11] *If* $n_2(v) = 4$*, then* $f_3(v) = 0$;
  ($C5.6$) [8] *If* v *is incident to a* $(3, 4, 6)$*-face, then* $n_2(v) \leq 2$.
($C6$) *Let* v *be a* 7*-vertex. Then*
  ($C6.1$) [16] $n_2(v) \leq 5$;
  ($C6.2$) [11] *If* $n_2(v) = 4$*, then* $n_3(v) \leq 2$;
  ($C6.3$) [11] *If* $n_2(v) = 5$*, then* $n_3(v) = 0$ *and* $f_3(v) = 0$.
($C7$) [16] *No* 3*-face* $[xyz]$ *with* $d(x) \leq d(y) \leq d(z)$ *satisfies one of the following:*
  ($C7.1$) $d(x) = 2$;
  ($C7.2$) $d(x) = d(y) = 3$ *and* $d(z) \leq 5$;
  ($C7.3$) $d(x) = 3$ *and* $d(y) = d(z) = 4$.
($C8$) [8] *Let* v *be a* 8*-vertex. Then*
  ($C8.1$) $n_2(v) \leq 6$;
  ($C8.2$) *If* $f_3(v) = 1$*, then* $n_2(v) \leq 5$.

In all figures of this paper, a vertex v is drawn in black if v has no other neighbors besides the ones already depicted and in white otherwise.

Lemma 4. *If a 3-vertex $v \in V(G)$ is incident to two adjacent 3-faces $[vv_1v_2]$ and $[vv_2v_3]$, then $\min\{d(v_1), d(v_3)\} \geq 5$.*

Proof. Suppose to the contrary that one of v_1 and v_3 is a 4^--vertex. Without loss of generality, assume that $d(v_1) \leq 4$. Let $G' = G - v$, then by the minimality of G, G' admits an acyclic L-coloring ϕ. Note that $\phi(v_2) \neq \phi(v_3)$ and $\phi(v_2) \neq \phi(v_1)$. If $\phi(v_1) \neq \phi(v_3)$, then we color v with a color in $L(v) \setminus \{\phi(v_1), \phi(v_2), \phi(v_3)\}$ to get an acyclic L-coloring of G, a contradiction. Otherwise we color v with a color in $L(v) \setminus (\{\phi(v_2), \phi(v_3)\} \cup \{\phi(x) : (x \in N(v_1)) \wedge (x \notin \{v, v_2\})\})$. So we extend ϕ to an acyclic L-coloring of G, a contradiction. □

Lemma 5. *Let v be 5-vertex. Then G can not contain the configurations shown in Figure 1.*

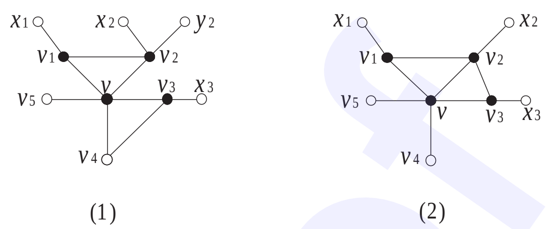

FIGURE 1. A 5-vertex v.

Proof. (1) Suppose to the contrary that G contains the configuration in Fig 1(1). Let $G' = G - v_1$. By the minimality of G, G' admits an acyclic L-coloring ϕ. Note that $\phi(v) \neq \phi(v_2)$. If $\phi(x_1) \notin \{\phi(v), \phi(v_2)\}$, then we color v_1 with a color in $L(v_1) \setminus \{\phi(x_1), \phi(v), \phi(v_2)\}$ to get an acyclic L-coloring of G, a contradiction. So we have $\phi(x_1) \in \{\phi(v), \phi(v_2)\}$. If $\phi(x_1) = \phi(v_2)$, then we color v_1 with a color in $L(v_1) \setminus \{\phi(x_1), \phi(v), \phi(x_2), \phi(y_2)\}$. Otherwise $\phi(x_1) = \phi(v)$. Thus, each color c in $L(v_1)$ appears in $\{\phi(v), \phi(v_2), \phi(v_3), \phi(v_4), \phi(v_5)\}$ for otherwise we can properly color v_1 with $d \in L(v_1) \setminus \{\phi(v), \phi(v_2), \phi(v_3), \phi(v_4), \phi(v_5)\}$ to get an acyclic L-coloring of G, a contradiction. Without loss of generality, we assume that $L(v_1) = \{1, 2, 3, 4, 5\}$, $\phi(x_1) = \phi(v) = 1$ and $\phi(v_i) = i$ for $i = 2, 3, 4, 5$.

Firstly, we have $\phi(x_3) = 1$ for otherwise we can properly color v_1 with 3. Secondly, we have $L(v) = \{1, 2, 3, 4, 5\}$ for otherwise we can recolor v with a color $\alpha \in L(v) \setminus L(v_1)$ and color v_1 with a color in $L(v_1) \setminus \{1, 2\}$ to get an acyclic L-coloring of G, a contradiction. Hence, we recolor v with 3, recolor v_3 with a color in $L(v_3) \setminus \{1, 3, 4\}$, and color v_1 with a color in $L(v_1) \setminus \{1, 2, 3\}$. Hence we extend ϕ to an acyclic L-coloring of G, a contradiction.

(2) The proof of the forbidden configuration as shown in Fig 1(2) is similar to the above. □

Lemma 6. *G can not contain the configurations as shown in Figure* 2.

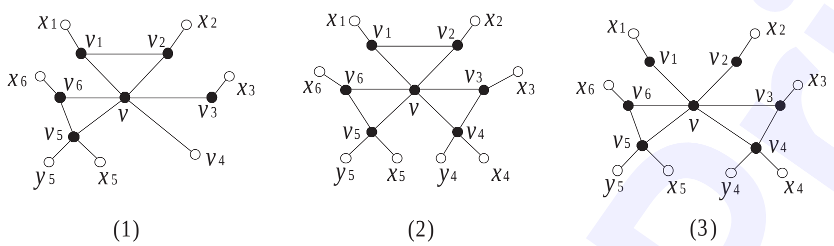

FIGURE 2. A 6-vertex v.

Proof. (1) Suppose that G contains the configuration as shown in Figure 2(1). By the minimality of G, $G' = G - \{v, v_1, v_2, v_3\}$ has an acyclic L-coloring ϕ. Obviously, $\phi(v_5) \neq \phi(v_6)$. We divide this problem into three cases.

Case 1. Suppose that $\phi(v_4), \phi(v_5)$ and $\phi(v_6)$ are pairwise different.

Then there exists a color $c \in L(v) \setminus \{\phi(v_4), \phi(v_5), \phi(v_6)\}$ appearing at most once in $\{x_1, x_2, x_3\}$. We color v with c. Assume that $\phi(x_i) = c$, $i \in \{1, 2, 3\}$, then we color v_i with a color in $L(v_i) \setminus \{c, \phi(v_4), \phi(v_5), \phi(v_6)\}$. Now, we color the vertices in $U = \{v_1, v_2, v_3\} \backslash \{v_i\}$. Let $v_k \in U$. If $d(v_k) = 2$, then we color v_k with a color in $L(v_k) \backslash \{c, \phi(x_k)\}$. If $d(v_k) = 3$, then let $v_t \in N(v) \cap N(v_k)$. If v_t is not assigned any color, then we color v_k with a color in $L(v_k) \setminus \{c, \phi(x_k)\}$. Otherwise, assume that v_t is colored with α. If $\phi(x_k) = \alpha$, then we color v_k with a color in $L(v_k) \setminus \{c, \alpha, \phi(x_t)\}$. Otherwise, we color v_k with a color in $L(v_k) \setminus (\{c, \phi(x_k), \alpha\}$. Hence, we extend ϕ to an acyclic L-coloring of G, a contradiction.

Case 2. Suppose that $\phi(v_4) = \phi(v_6)$.

If $\phi(x_6) = \phi(v_5)$, we recolor v_6 with a color in

$$L(v_6) \setminus \{\phi(v_6), \phi(v_5), \phi(x_5), \phi(y_5)\}$$

and then go back to the previous case. Otherwise, we recolor v_6 with a color in $L(v_6) \setminus \{\phi(v_6), \phi(v_5), \phi(x_6)\}$ and then go back to Case 1.

Case 3. Suppose that $\phi(v_4) = \phi(v_5)$.

If $\phi(x_5) = \phi(y_5)$, then there exists a color $c' \in L(v) \setminus \{\phi(v_5), \phi(v_6), \phi(x_5)\}$ appearing at most once in $\{x_1, x_2, x_3\}$. Then we color v with c and the left is the same to Case 1. Otherwise, $\phi(x_5) \neq \phi(y_5)$. If $\phi(v_6) \in \{\phi(x_5), \phi(y_5)\}$, then we recolor v_5 with a color in $L(v_5) \setminus \{\phi(v_5), \phi(x_5), \phi(y_5), \phi(x_6)\}$ and then go back to Case 1. Otherwise, we recolor v_5 with a color in $L(v_5) \setminus$

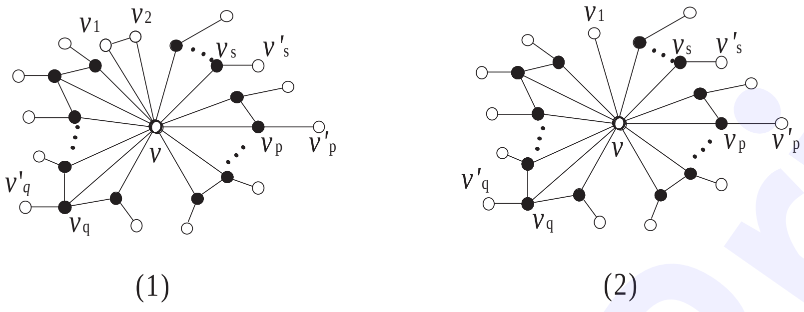

FIGURE 3. A configuration in Lemma 7.

$\{\phi(v_5), \phi(x_5), \phi(y_5), \phi(v_6)\}$ and then go back to Case 1. Hence, in any case, we can extend ϕ to an acyclic L-coloring of G, a contradiction.

(2) Suppose that G contains the configuration as shown in Figure 2(2). By the minimality of G, $G' = G - \{v, v_1, v_2, v_3\}$ has an acyclic L-coloring ϕ. By the same argument as in (1), there always exists a color $c \in L(v) \setminus \{\phi(v_4), \phi(v_5), \phi(v_6)\}$ which appears at most once in $\{x_1, x_2, x_3\}$. Then we color v with c and the colorings of v_1 and v_2 are in the same way as in (1). In the following, we mainly discuss the coloring of v_3.

If $\phi(x_3) = c$, then $\phi(x_3) \neq \phi(v_4)$. Then we color v_3 with a color in $L(v_3) \setminus \{c, \phi(v_4), \phi(v_5), \phi(v_6)\}$. If $\phi(x_3) \neq c$, then we color v_3 with a color in $L(v_3) \setminus \{c, \phi(x_3), \phi(x_4), \phi(y_4)\}$ if $\phi(x_3) = \phi(v_4)$ and color v_3 with a color in $L(v_3) \setminus \{c, \phi(x_3), \phi(v_4)\}$ otherwise. Hence we extend ϕ to an acyclic L-coloring of G, a contradiction.

(3) The proof of (3) is similar to those of (1) and (2). □

Lemma 7. *G does not contain a d-vertex v ($6 \leq d \leq 10$) such that except at most one neighbor or two adjacent neighbors of v, the other neighbors of v are 4^--vertices (these vertices are called to be small) and these 4^--vertices satisfy the following conditions (see Figure 3):*

(1) *For every 3-neighbor u of v, uv is incident with exact one 3-face. Furthermore, if u is adjacent to another 3-neighbor w of v, then u, w, v forms a $(3,3,d)$-face;*

(2) *For every 4-neighbor w of v, wv is incident with two $(3,4,d)$-faces.*

Proof. Suppose to the contrary that G contains such a d-vertex with $6 \leq d \leq 10$. We just settle the case as shown in Figure 3(1) (the case as shown in Figure 3(2) can be settled similarly). We denote all small neighbors of v to be $v_3, v_4, \ldots, v_d$. For any v_i ($3 \leq i \leq d$), we have $|N(v_i) \setminus (N(v) \cup \{v\})| = 1$, so we always denote v_i' to be the neighbor of v_i which is not adjacent to v. Let $S = \{v_i' : 3 \leq i \leq d\}$.

By the minimality of G, $G' = G - \{v, v_3, \ldots, v_d\}$ has an acyclic L-coloring ϕ. Let $L' = L(v) \setminus \{\phi(v_1), \phi(v_2)\}$. Then $|L'| \geq 3$. Since $d \leq 10$, $|S| \leq 8$, and it follows that there exists a color $c \in L'$ which appears at most twice in S. Firstly, we color v with c. Without loss of generality, assume that $\phi(v'_i) = \phi(v'_j) = c$ for $3 \leq i < j \leq d$, and we color v_i with a color in $L(v_i) \setminus \{c, \phi(v_1), \phi(v_2)\}$ and color v_j with a color in $L(v_j) \setminus \{c, \phi(v_i), \phi(v_1), \phi(v_2)\}$.

Then we color the vertices in $U = \{v_3, \ldots, v_d\} \setminus \{v_i, v_j\}$. Let $v_k \in U$. Suppose that $d(v_k) = 2$. Then we color v_k with a color in $L(v_k) \setminus \{c, \phi(v'_k)\}$. Suppose that $d(v_k) = 3$. Let $v_t \in N(v) \cap N(v_k)$. Note that $d(v_t) \leq 4$. If v_t is not colored, then we color v_k with a color in $L(v_k) \setminus \{c, \phi(v'_k)\}$. If v_t is colored with α, then we discuss this problems in two cases. The first case is that $\phi(v'_k) \neq \alpha$, then we color v_k with a color in $L(v_k) \setminus \{c, \alpha, \phi(v'_k)\}$. The second case is that $\phi(v'_k) = \alpha$, then we color v_k with a color in $L(v_k) \setminus (\{c, \phi(v'_k), \phi(v'_t), \xi\}$, where ξ is the color of the vertex u with $u \in N(v_t) \setminus \{v, v_k, v'_t\}$.

Suppose that $d(v_k) = 4$. Let $\{v_s, v_t\} = N(v) \cap N(v_k)$. If at most one of v_s and v_t is colored, then we color v_k with a color in $L(v_k) \setminus \{c, \phi(v'_k), \gamma, \phi(v'_p) : p \in \{s, t\}$ and γ is the color of $v_p\}$. Otherwise, assume that v_s and v_t are colored with η and θ, respectively. If $|\{\eta, \theta, \phi(v'_k)\}| = 3$, then we color v_k with a color in $L(v_k) \setminus \{c, \eta, \theta, \phi(v'_k)\}$. If $|\{\eta, \theta, \phi(v'_k)\}| = 2$, then there exist one 3-vertex $v_p \in \{v_s, v_t\}$ such that the color of v_p appears twice in $\{\eta, \theta, \phi(v'_k)\}$. Then $|T| = |\{c, \eta, \theta, \phi(v'_k), \phi(v'_p)\}| \leq 4$ and we color v_k with a color in $L(v_k) \setminus T$. If $|\{\eta, \theta, \phi(v'_k)\}| = 1$, then $|T'| = |\{c, \eta, \theta, \phi(v'_k), \phi(v'_s), \phi(v'_t)\}| \leq 4$ and we color v_k with a color in $L(v_k) \setminus T'$.

It is easy to check that the coloring obtained as above is an acyclic L-coloring of G, a contradiction. □

Lemma 8. *G can not contain the configurations as shown in Figure 4, where v_4 is a 4^--vertex in each of Figure* 4(1), (2) and (3).

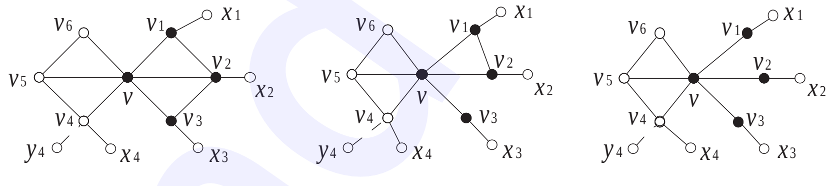

FIGURE 4. A 6-vertex v with $f_b(v) \geq 2$.

Proof. Suppose to the contrary that G contains the structure in Figure 4(1). We only prove the case that v_4 is a 4-vertex (the proof of the case that v_4 is a 3-vertex is much easier). Let $G' = G - \{v, v_1, v_2, v_3\}$, then G' admits an acyclic L-coloring ϕ. It is easy to see that $\phi(v_5) \neq \phi(v_6)$.

Case 1. Suppose that $\phi(v_4) \neq \phi(v_6)$.

Then $\phi(v_4), \phi(v_5)$ and $\phi(v_6)$ are pairwise different. Hence, there exists a color $c \in L(v) \setminus \{\phi(v_4), \phi(v_5), \phi(v_6)\}$ appearing at most once in $\{x_1, x_2, x_3\}$. Then we color v with c. For the coloring process of the vertices v_1, v_2, v_3 is the same as that of Lemma 7.

Case 2. Suppose that $\phi(v_4) = \phi(v_6)$.

Let $T = \{\phi(v_4), \phi(v_5), \phi(v_6), \phi(x_4), \phi(y_4)\}$. Firstly, we assume that $\phi(x_4) \neq \phi(y_4)$. If $\phi(v_5) \notin \{\phi(x_4), \phi(y_4)\}$, then we color v_4 with a color in $L(v_4) \setminus \{\phi(v_4), \phi(v_5), \phi(x_4), \phi(y_4)\}$ and go back to Case 1. If $\phi(v_5) \in \{\phi(x_4), \phi(y_4)\}$, then $|T| = 3$. Hence, there exists a color $c \in L(v) \setminus T$ which appears at most once in $\{x_1, x_2, x_3\}$. Then we color v with c. For the coloring process of the vertices v_1, v_2, v_3 is the same as that of Lemma 7. Secondly, assume that $\phi(x_4) = \phi(y_4)$. Then we have $|T| \leq 3$. It follows that there exists a color in $c \in L(v) \setminus T$ which appears at most once in $\{x_1, x_2, x_3\}$ and go back to the previous case.

The proofs of the other two forbidden configuration as shown in Figure 4(2), (3) are the same as the above. □

By Lemma 7, we have the following two results:

Corollary 9. *Let v be 9-vertex. Then*
(C9.1) $n_2(v) \leq 7$.
(C9.2) If $n_2(v) = 7$, then $f_3(v) = 0$.

3. The proof of Theorem 2

Let G be a minimal counterexample to Theorem 2 (i.e., with the least number of vertices). From Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ and the relation that $\sum_{v\in V(G)} d_G(v) = \sum_{f\in F(G)} d_G(f) = 2|E(G)|$, we have

$$\sum_{v\in V(G)} (2d(v) - 6) + \sum_{f\in F(G)} (d(f) - 6) = -12. \tag{1}$$

Then we assign an initial charge $c(v) = 2d(v) - 6$ to $v \in V(G)$ and $c(f) = d(f) - 6$ to $f \in F(G)$. By applying a set of discharging rules and the structural properties of G, we can obtain the final charge $c'(x)$ such that $c'(x) \geq 0$ for every element $x \in V(G) \cup F(G)$, which is a contradiction to equation (1) in final and completes our proof.

Suppose that $f = [v_1 v_2 \cdots v_k]$ is a k-face. We use $(d_G(v_1), \ldots, d_G(v_k)) \to (c_1, \ldots, c_k)$ to denote that the vertex v_i gives f the charge c_i for $i = 1, \ldots, k$.

The discharging rules are given as follows.

R1. Let $f = [v_1 v_2 v_3]$ be a 3-face with $d_G(v_1) \leq d_G(v_2) \leq d_G(v_3)$. Then
$(3, 3, 6^+) \to (0, 0, 3)$;
$(3, 4, 5^+) \to (0, 1, 2)$;
$(3, 5^+, 5^+) \to (0, \frac{3}{2}, \frac{3}{2})$;
$(4^+, 4^+, 4^+) \to (1, 1, 1)$.

R2. Let $f = [v_1v_2v_3v_4]$ be a 4-face. Then
$(2, 5^+, 3^-, 5^+) \to (0, 1, 0, 1)$;
$(3, 3, 5^+, 5^+) \to (0, 0, 1, 1)$;
$(3, 4^+, 3, 4^+) \to (0, 1, 0, 1)$;
$(3^-, 4^+, 4^+, 4^+) \to (0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3})$;
$(4^+, 4^+, 4^+, 4^+) \to (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

R3. Let v be a 2-vertex. Then v receives 1 from each of its neighbors.

R4. Let f be a 7-face. Then f sends $\frac{1}{7}$ to each vertex incident with f.

R5. Let f be a 8^+-face. Then f sends $\frac{1}{4}$ to each vertex incident with f.

R6. Let v be a 4^--vertex and $\sigma(v)$ denote the charges obtained from its incident 7^+-faces. Then v sends $\sigma(v)$ uniformly among its 5^+-neighbors.

By the assumption of Theorem 2, we have the following result.

Fact 1. For any vertex $v \in V(G)$, we have

(1) $f_3(v) \le \lfloor \frac{2}{3} d(v) \rfloor$ if $n_2(v) = 0$ and $f_3(v) \le \lceil \frac{2}{3}(d(v) - n_2(v) - 1) \rceil$ if $n_2(v) > 0$.
(2) There is no 4-cycle adjacent to a 4-cycle.
(3) Let f be a 3-face in G. Then f is not adjacent to 4-, 5- and 6-faces.
(4) $f_6(v) + f_7(v) \le 1$.
(5) If v is a 2-vertex with $f_4(v) \ge 1$ in G, then $f_4(v) = 1$ and $f_5(v) = f_6(v) = f_7(v) = 0$.

Let $f \in F(G)$. Suppose that f is a 3-face $f = [v_1v_2v_3]$ with $d(v_1) \le d(v_2) \le d(v_3)$. By Lemma 3 and $(C7.1)$, $d(v_1) \ge 3$. If $d(v_1) = d(v_2) = 3$, then by Lemma 3 $(C7.2)$, $d(v_3) \ge 6$ and it follows that $c'(f) \ge 3 - 6 + 3 = 0$ by R1. If $d(v_1) = 3$ and $d(v_2) = 4$, then by Lemma 3 $(C7.3)$, $d(v_3) \ge 5$ and it follows that $c'(f) \ge 3 - 6 + 1 + 2 = 0$ by R1. If $d(v_1) = 3$ and $d(v_3) \ge d(v_2) \ge 5$, then $c'(f) \ge 3 - 6 + 2 \times \frac{3}{2} = 0$ by R1. If $d(v_1) \ge 4$, then $c'(f) \ge 3 - 6 + 3 \times 1 = 0$ by R1.

Suppose that f is a 4-face. By Lemma 3$(C1)$-$(C3)$ and R2, we can easily check that $c'(f) \ge 0$. There are no 5-face in G. Suppose that f is a 6-face. Then by discharging rules, $c'(v) = c(v) = 6 - 6 = 0$. Suppose that f is a 7-face. Then $c'(f) \ge d(f) - 6 - 7 \times \frac{1}{7} \ge 0$ by R4. Suppose that f is a 8^+-face. Then $c'(f) \ge d(f) - 6 - \frac{1}{4} d(f) \ge 0$ by R5.

Let $v \in V(G)$. By Lemma 3$(C1)$, $d(v) \ge 2$. Suppose that $d(v) = 2$. By Lemma 3$(C7.1)$, $f_3(v) = 0$. Hence, by R2-R6, $c'(v) \ge 2 \times 2 - 6 + 2 \times 1 = 0$. Suppose that $d(v) = 3$. By R1-R2 and R4-R6, $c'(v) \ge c(v) \ge 2 \times 3 - 6 = 0$. Suppose that $d(v) = 4$. By Fact 1(2), $f_3(v) + f_4(v) \le 2$. By Lemma 3$(C2)$, $n_2(v) = 0$. It follows that $c'(v) \ge 2 \times 4 - 6 - 1 \times 2 = 0$ by R1-R2 and R4-R6.

For a vertex $v \in V(G)$, let $v_1, \ldots, v_{d(v)}$ denote the neighbors of v in a clockwise order and let f_i be a face incident with vv_i and vv_{i+1}, where the sum of the subscription is taken modular $d(v)$.

For a 5^+-vertex v, let $[vuw]$ and $[vwx]$ be two bad 3-faces of v. The subgraph induces by the two 3-faces is called to be a *diamond* of v. By Lemma 4 and R1, we divide all diamonds of v into the following four classes.

1-diamond: $d(u) = d(x) = 3$ *and* $d(w) = 4$;
2-diamond: $d(u) = 3$, $d(w) = 4$ *and* $d(x) \geq 4$, *or* $d(w) = 3$, $d(u) \geq 5$ *and* $d(x) \geq 5$, *or* $d(u) = d(x) = 3$ *and* $d(w) \geq 5$;
3-diamond: $d(u) = 3$, $d(w) \geq 5$ *and* $d(x) \geq 4$;
4-diamond: $d(u) \geq 4$, $d(w) \geq 4$ *and* $d(x) \geq 4$.

We denote by $\rho_i(v)$ the charge which v sends to a i-diamond of v, and $f^i_{dia}(v)$ the number of i-diamonds of v, where $i = 1, 2, 3, 4$. By R1, we have $\rho_1 = 4$, $\rho_2 = 3$, $\rho_3 = \frac{5}{2}$, $\rho_4 = 2$, and $f_b(v) = 2\sum_{i=1}^{4} f^i_{dia}(v)$.

Suppose that $d(v) = 5$. Then $n_2(v) \leq 1$ by Lemma 3($C4.1$). Let $n_2(v) = 0$, then $f_3(v)+f_4(v) \leq 3$ by Fact 1(1)-(3). If $f_3(v) \leq 2$ and v is not incident with a $(3,4,5)$-face, then $c'(v) \geq 2\times 5-6-2\times\frac{3}{2}-1 = 0$ by R1-R2. If $f_3(v) \leq 2$ and v is incident with a $(3,4,5)$-face, then by Lemma 5, any other 3-face incident with v is not incident with 3-vertices. If follows that $c'(v) \geq 2\times 5-6-(2+1)-1 = 0$ by R1-R2. If $f_3(v) = 3$, then $f^1_{dia}(v) = 0$ by Lemma 5 (Figure 1(2)), $f_4(v) = 0$ and $\sum_{i=2}^{4} f^i_{dia}(v) = 1$ by Fact 1(2). Let f_1, f_2 and f_4 be 3-faces. If one of the three 3-faces is a $(3,4,5)$-face, then $n_3(v) = 1$ by Lemma 5 (Figure 1). It follows that $c'(v) \geq 2\times 5-6-2-1\times 2 = 0$ by R1. Otherwise, we firstly assume that $f^2_{dia}(v) = 1$. If the two bad 3-faces of v are adjacent $(3,5,5^+)$-faces f_1 and f_2 with a common $(5,5^+)$-edge, then each of the two 3-neighbors v_1 and v_3 receives at least $\frac{1}{7}+\frac{1}{4}$ from its incident 7^+-faces by R4, R5 and Fact 1. Hence, $c'(v) \geq 2\times 5-6-3\times\frac{3}{2}+\frac{1}{7}+\frac{1}{4}+2\times\frac{1}{3}\times(\frac{1}{7}+\frac{1}{4}) = \frac{13}{84}$ by R1 and R4-R7. If the two bad 3-faces of v are adjacent $(3,5,5^+)$-faces f_1 and f_2 with a common $(3,5)$-edge, then v_2 receives at least $\frac{1}{4}$ from its incident 7^+-faces if one of f_3 and f_5 is a 7-face and $\frac{1}{7}$ otherwise by R4, R5 and Fact 1(4). Hence, $c'(v) \geq 2\times 5-6-3\times\frac{3}{2}+\min\{\frac{1}{4}+\frac{1}{7}+\frac{1}{3}\times\frac{1}{4}+\frac{1}{3}\times\frac{1}{7}, 2\times\frac{1}{4}+\frac{1}{3}\times\frac{1}{4}+\frac{1}{3}\times\frac{1}{7}\} = \frac{1}{42}$ if one of v_4 and v_5 is a 3-vertex and $c'(v) \geq 2\times 5-6-2\times\frac{3}{2}-1 = 0$ otherwise by R1 and R4-R7. Secondly, assume that $f^2_{dia}(v) = 0$, then $c'(v) \geq 2\times 5-6-(\frac{5}{2}+\frac{3}{2}) = 0$ by Lemma 4 and R1-R2. Let $n_2(v) = 1$. Then $f_3(v)+f_4(v) \leq 3$ by Fact 1(2), (3). By Lemma 3($C4.2$) and R1-R3, $c'(v) \geq 2\times 5-6-2\times 1-1-1 = 0$.

Suppose that $d(v) = 6$. Then $c(v) = 2\times 6-6 = 6$ and $n_2(v) \leq 4$ by Lemma 3($C5.1$). Let $f_{3,4}(v) = (a, b)$ denote that $f_3(v) = a$ and $f_4(v) = b$ and let $f_{3,4}(v) = (a^-, b)$ denote that $f_3(v) \leq a$ and $f_4(v) = b$. Similarly, we can define $f_{3,4}(v) = (a, b^-)$. We will discuss this problem in terms of $n_2(v)$.

If $n_2(v) = 0$, then $f_3(v) \leq 4$ by Fact 1(1), and it follows that $c'(v) \geq 6-3 = 3$ for $f_{3,4}(v) = (0, 3^-)$ by R2; $c'(v) \geq 6-3-2 = 1$ for $f_{3,4}(v) = (1, 2^-)$ by R1-R2. Assume that $f_{3,4}(v) = (2, 1^-)$. If $\sum_{i=1}^{4} f^i_{dia}(v) = 1$, then $c'(v) \geq 6-4-1 = 1$ by Lemma 4 and R1-R2. Otherwise, $\iota(v) = 2$. If v is incident with at most one $(3,3,6)$-face, then $c'(v) \geq 6-3-2-1 = 0$ by R1-R2. Otherwise, v is incident with two $(3,3,6)$-faces. Let f_1 and f_3 be two 3-faces. Since every vertex is incident with at most one j-cycle, $j \in \{6, 7\}$, v_1, v_2, v_3

and v_4 receive at least $2\times\frac{1}{7}+2\times\frac{1}{4}=\frac{11}{14}$ from their incident 7^+-faces. Then $c'(v)\geq 6-3-3-1+\frac{1}{7}+2\times\frac{1}{4}+\frac{1}{2}\times(2\times\frac{1}{7}+2\times\frac{1}{4})=\frac{1}{28}$ by R1-R2 and R4-R6. For the case $f_{3,4}(v)=(3,0)$, we should consider the following subcases. The first subcase is that $\sum_{i=1}^{4} f_{dia}^{i}=1$, then $c'(v)\geq 6-\max\{4+2,3+3\}=0$ by Lemma 4, Lemma 7 and R1-R2. The second subcase is that $\sum_{i=1}^{4} f_{dia}^{i}=0$, then by Lemma 7, v is incident with at most one $(3,3,6)$-face. Note that $f_4(v)=0$ by Fact 1(3). If v is not incident with a $(3,3,6)$-face, then $c'(v)\geq 6-2\times 3=0$ by R1. Otherwise, let f_1 be a $(3,3,6)$-face. Then by Lemma 6 (Figure 2(2)), v is incident with at most one $(3,4,6)$-face. Hence, by Fact 1(4), R1 and R4-R5, we have $c'(v)\geq 6-3-2-\frac{3}{2}+\frac{1}{7}+2\times\frac{1}{4}=\frac{1}{7}$. For the case $f_{3,4}(v)=(4,0)$, $\sum_{i=1}^{4} f_{dia}^{i}(v)=2$ by Fact 1(2). If $f_{dia}^{1}(v)=0$, then $c'(v)\geq 6-2\times 3=0$ by Lemma 4 and R1. Otherwise, by Lemma 7, $f_{dia}^{1}(v)=1$. Assume that f_1 and f_2 construct a 1- diamond of v, then f_5 and f_6 are two bad 3-faces of v by Fact 1(2). Furthermore, by Lemma 4, $d(v_1)=d(v_3)=3$ and $d(v_2)=4$. If $f_{dia}^{2}(v)=0$, then by Lemma 8 (Figure 4(1)), $f_{dia}^{3}(v)=0$ and $f_{dia}^{4}(v)=1$. Hence, we have $c'(v)\geq 6-4-2=0$ by Lemma 4 and R1. If $f_{dia}^{2}(v)=1$, then by Lemma 8 (Figure 4(1)), we have $d(v_5)=3$ and $\min\{d(v_4),d(v_6)\}\geq 5$. Hence, by Fact 1, R1 and R4-R6, $c'(v)\geq 6-4-3+\frac{1}{7}+\frac{1}{4}+(\frac{1}{7}+\frac{1}{4})\times\frac{1}{2}\times 3+\frac{1}{7}\times\frac{1}{3}=\frac{5}{168}$.

If $n_2(v)=1$, then $f_3(v)\leq 3$ by Fact 1(1). Hence, we have $c'(v)\geq 6-1-3\times 1=2$ for $f_{3,4}(v)=(0,3^-)$ by R2-R3; $c'(v)\geq 6-3-1-2=0$ for $f_{3,4}(v)=(1,2^-)$ by R1-R3. If $f_{3,4}(v)=(2,1^-)$, then by Fact 1(2), $\sum_{i=1}^{4} f_{dia}^{i}(v)\leq 1$. If $\sum_{i=1}^{4} f_{dia}^{i}(v)=1$, then by Lemma 4 and R1-R3, $c'(v)\geq 6-4-1-1=0$. Otherwise, by Lemma 6, v is incident with at most one $(3,3,6)$-face. If v is incident with a $(3,3,6)$-face, then by Lemma 6 (Figure 2 (1)), v is not incident with a $(3,4,6)$-face. So by Fact 1(2), (3) and R1-R5, $c'(v)\geq 6-3-\frac{3}{2}-1-1+\frac{1}{7}+2\times\frac{1}{4}=\frac{1}{7}$. Otherwise, by Fact 1(2), (3) and R1-R3, $c'(v)\geq 6-2\times 2-1-1=0$. For the case $f_{3,4}(v)=(3,0)$, $\sum_{i=1}^{4} f_{dia}^{i}(v)=1$, $\iota(v)=1$ by Fact 1. By Lemma 7, $f_{dia}^{1}(v)=0$. If the independent 3-face of v is not a $(3,3,6)$-face, then $c'(v)\geq 6-3-2-1=0$ by R1-R3. Otherwise, assume that $f_1=[vv_1v_2]$ is an independent $(3,3,6)$-face, f_3 and f_4 are bad 3-faces of v and v_6 is a 2-vertex. If $\sum_{i=3}^{4} f_{dia}^{i}(v)=1$, then by Lemma 8 (Figure 4(2)), $f_{dia}^{3}(v)=0$ and $f_{dia}^{4}(v)=1$. Hence, by Lemma 4 and R1, $c'(v)\geq 6-3-2-1=0$. If $f_{dia}^{2}(v)=1$, then by Lemma 8 (Figure 4(2)), $d(v_4)=3$ and $\min\{d(v_3),d(v_5)\}\geq 5$. Hence, by Fact 1, R1 and R3-R6, $c'(v)\geq 6-4-3+\frac{1}{7}+\frac{1}{4}\times 2+(\frac{1}{7}+\frac{1}{4})\times\frac{1}{2}+\frac{1}{7}\times\frac{1}{3}+\frac{1}{4}\times\frac{1}{2}=\frac{1}{84}$.

If $n_2(v)=2$, then $f_3(v)\leq 2$ by Fact 1(1). If $f_{3,4}(v)=(0,3^-)$, then $c'(v)\geq 6-2-3=1$ by R2-R3. For the case $f_{3,4}(v)=(1,2^-)$, if v is not incident with a $(3,3,6)$-face, then $c'(v)\geq 6-2-2-2=0$ by R1-R3. Otherwise, $c'(v)\geq 6-3-2-2+\min\{\frac{1}{7}+\frac{1}{4}\times 2+\frac{1}{7}\times\frac{1}{2}\times 2+\frac{1}{4}\times\frac{1}{2}\times 2,\frac{1}{4}\times 2+\frac{1}{4}\times\frac{1}{2}\times 2+\frac{1}{4}\times\frac{1}{2}\times 2\}=0$ by Fact 1(5) and R1-R6. For the case $f_{3,4}(v)=(2,1^-)$, if $\sum_{i=1}^{4} f_{dia}^{i}(v)=1$, then by Lemma 7, $f_{dia}^{1}(v)=0$. Hence, we have $c'(v)\geq 6-3-2-1=0$ by R1-R3. Otherwise, $\iota(v)=2$. By Lemma 7, v is not incident with a $(3,3,6)$-face and it follows that $n_3(v)\leq 2$. If $n_3(v)\leq 1$, then $c'(v)\geq 6-2-1-2-1=0$ by

R1-R3. If $n_3(v)=2$, by Lemma 6, v is incident with at most one $(3,4,6)$-face. It follows that $c'(v)\geq 6-\max\{2\times\frac{3}{2}+2+1, 2+\frac{3}{2}+2+1-\frac{1}{7}-\frac{1}{4}\times 2\}=0$ by Fact 1 and R1-R5.

If $n_2(v)=3$, then $f_3(v)\leq 2$ by Lemma 3($C7.1$) and Fact 1(2). Then by R2 and R3, $c'(v)\geq 6-3-3=0$ for $f_{3,4}(v)=(0,3^-)$. If $f_3(v)=1$, then $f_4(v)\leq 2$ by Fact 1(2), (3). By Lemma 7 and Lemma 3($C5.6$), v is no incident with a $(3,4^-,6)$-face. Hence, by Fact 1 and R1-R6, $c'(v)\geq 6-\frac{3}{2}-3-2+\frac{1}{4}\times 2=0$ if v is incident with a $(3,5^+,6)$-face and $c'(v)\geq 6-1-3-2=0$ otherwise. If $f_3(v)=2$, then $f_4(v)\leq 1$ and $\sum_{i=1}^{4} f^i_{dia}(v)=1$ by Fact 1(2), (3) and Lemma 3($C7.1$). Furthermore, $f^1_{dia}(v)=0$ by Lemma 7. If $f^3_{dia}(v)+f^4_{dia}(v)=1$, then by Lemma 8 (Figure 4(3)), $f^3_{dia}(v)=0$ and $f^4_{dia}(v)=1$. Hence, by R1-R5, $c'(v)\geq 6-2-3-1=0$. If $f^2_{dia}(v)=1$, then by Lemma 8 (Figure 4(3)), the two bad 3-faces of the 2-diamond of v are two adjacent $(3,5^+,5^+)$-faces with a common $(3,6)$-edge. Hence, by Fact 1 and R1-R6, $c'(v)\geq 6-3-3-1+\frac{1}{7}+\frac{1}{4}\times 2+\frac{1}{4}\times\frac{1}{2}\times 3=\frac{1}{56}$.

If $n_2(v)=4$, then $f_3(v)=0$ and $f_4(v)\leq 3$ by Lemma 3($C5.5$) and Fact 1(2). Hence, we have $c'(v)\geq 6-4-3+\frac{1}{4}\times 2+\frac{1}{4}\times\frac{1}{2}\times 4=0$ by R2-R6.

Suppose that $d(v)=7$. Then $n_2(v)\leq 5$ by Lemma 3($C6.1$). At the same time, $c(v)=2\times 7-6=8$.

If $n_2(v)=0$, then $f_3(v)\leq 4$ by Fact 1(1). If $f_3(v)\leq 2$, then by Fact 1, Lemma 4 and R1-R2, we have $c'(v)\geq 8-\max\{3,3+2,4+2,3+3+1\}=1$. If $f_3(v)=3$, then $\sum_{i=1}^{4} f^i_{dia}(v)\leq 1$. Hence, if $\sum_{i=1}^{4} f^i_{dia}(v)=1$, then $f_4(v)\leq 1$ by Fact 1(2) and it follows that $c'(v)\geq 8-4-3-1=0$ by R1-R2; if $\sum_{i=1}^{4} f^i_{dia}(v)=0$, then $f_4(v)=0$ by Fact 1(2) and it follows that $c'(v)\geq 8-2\times 3-2=0$ by Lemma 7 and R1. If $f_3(v)=4$, then $f_4(v)=0$ and $1\leq\sum_{i=1}^{4} f^i_{dia}(v)\leq 2$ by Fact 1(2), (3). If $\sum_{i=1}^{4} f^i_{dia}(v)=2$, then by Lemma 7, Lemma 4 and R1, $c'(v)\geq 8-4-3=1$. If $\sum_{i=1}^{4} f^i_{dia}(v)=1$, we will discuss this problem into three subcases. The first subcase is that $f^1_{dia}(v)=1$, then by Lemma 7, Lemma 4 and R1, $c'(v)\geq 8-4-2\times 2=0$. The second subcase is that $f^2_{dia}(v)=1$, then by Lemma 4, R1 and R4-R6, we have $c'(v)\geq 8-3-3-2=0$ if v is incident with at most one independent $(3,3,7)$-face and $c'(v)\geq 8-3-3-3+(\frac{1}{7}+2\times\frac{1}{4})+(2\times\frac{1}{7}\times\frac{1}{2}+2\times\frac{1}{4}\times\frac{1}{2})=\frac{1}{28}$ if v is incident with two independent $(3,3,7)$-faces. The third subcase is that $\sum_{i=3}^{4} f^i_{dia}(v)=1$, then by Lemma 4, R1 and R4-R6, $c'(v)\geq 8-\max\{2+3+3,\frac{5}{2}+3+3-\frac{1}{7}-2\times\frac{1}{4}\}=0$.

If $n_2(v)=1$, then $f_3(v)\leq 4$ by Fact 1(1). If $f_3(v)\leq 2$, then by Fact 1(2), Lemma 4 and R1-R3, $c'(v)\geq 8-\max\{1+3,3+1+2,4+1+2,3+3+1+1\}=0$. If $f_3(v)=3$, then by Fact 1(2), (3) and Lemma 3($C7.1$), $\sum_{i=1}^{4} f^i_{dia}(v)\leq 1$ and $f_4(v)\leq 1$. By Lemma 7 and R1-R3, $c'(v)\geq 8-\max\{3+3+1+1,4+2+1+1\}=0$ if $\sum_{i=1}^{4} f^i_{dia}(v)=1$ and $c'(v)\geq 8-3-2\times 2-1=0$ if $\sum_{i=1}^{4} f^i_{dia}(v)=0$. If $f_3(v)=4$, then by Fact 1(2) and Lemma 3($C7.1$), $\sum_{i=1}^{4} f^i_{dia}(v)=2$ and $f_4(v)=0$. Hence, by Lemma 7 and R1-R3, $c'(v)\geq 8-4-3-1=0$.

If $n_2(v) = 2$, then $f_3(v) \leq 3$ by Fact 1(1). If $f_3(v) \leq 2$, then $c'(v) \geq 8 - \max\{2+3, 3+2+2, 4+2+2, 3+2+2+1\} = 0$ by Fact 1(2), (3), Lemma 4, Lemma 7 and R1-R3. If $f_3(v) = 3$, then by Fact 1(2), (3) and Lemma 3($C7.1$), $\sum_{i=1}^{4} f_{dia}^{i}(v) = 1$, $\iota(v) = 1$ and $f_4(v) \leq 1$. By Lemma 7, $f_{dia}^{1}(v) = 0$. Let $f_1 = [vv_1v_2]$ be an independent 3-face of v. If f_1 is not a $(3,3,6)$-face, then $c'(v) \geq 8 - 3 - 2 - 2 - 1 = 0$ by Lemma 4 and R1-R3. Otherwise, $c'(v) \geq 8 - 3 - 3 - 2 - 1 + \frac{1}{7} + \frac{1}{4} \times 2 + \frac{1}{7} \times \frac{1}{2} \times 2 + \frac{1}{4} \times \frac{1}{2} \times 2 = \frac{1}{28}$ by Lemma 4 and R1-R6.

If $n_2(v) = 3$, then $f_3(v) \leq 2$ by Fact 1(1). If $f_3(v) \leq 1$, then $c'(v) \geq 8 - \max\{3+3, 3+3+2\} = 0$ by Fact 1 and R1-R3. If $f_3(v) = 2$, then by Fact 1, Lemma 7 and R1-R3, $c'(v) \geq 8 - 3 - 3 - 2 = 0$ if $\sum_{i=1}^{4} f_{dia}^{i}(v) = 1$ and $c'(v) \geq 8 - 2 - 2 - 3 - 1 = 0$ otherwise.

If $n_2(v) = 4$, then $f_3(v) \leq 2$ by Fact 1(1). If $f_3(v) = 0$, then $f_4(v) \leq 3$ by Fact 1(2) and it follows that $c'(v) \geq 8 - (3+4) = 1$ by R2-R3. If $f_3(v) = 1$, then $f_4(v) \leq 2$ by Fact 1(2), (3) and it follows that $c'(v) \geq 8 - 2 - 4 - 2 = 0$ by Lemma 7 and R1-R3. If $f_3(v) = 2$, then by Fact 1(2), (3) and Lemma 3($C7.1$), $\sum_{i=1}^{4} f_{dia}^{i}(v) = 1$ and $f_4(v) \leq 2$. By Lemma 7, $f_{dia}^{1}(v) = 0$. Hence, we have $c'(v) \geq 8 - 3 - 4 - 2 + \frac{1}{4} \times 2 + \frac{1}{4} \times \frac{1}{2} \times 4 = 0$ by Lemma 4 and R1-R6.

If $n_2(v) = 5$, then $f_3(v) = 0$ by Lemma 3($C6.3$). Hence, $c'(v) \geq 8 - 5 - 3 = 0$ by Fact 1(2) and R2-R3.

Suppose that $d(v) = 8$. Then $n_2(v) \leq 6$ by Lemma 3($C8.1$). At the same time, $c(v) = 2 \times 8 - 6 = 10$.

If $n_2(v) = 0$, then $f_3(v) \leq 5$ by Fact 1(1). If $f_3(v) \leq 3$, then $c'(v) \geq 10 - \max\{4, 3+3, 4+2, 2 \times 3 + 2, 4+3+1, 3 \times 3 + 1\} = 0$ by Lemma 4, Fact 1(2), (3), R1-R3. If $f_3(v) = 4$, then $f_4(v) \leq 1$ and $\sum_{i=1}^{4} f_{dia}^{i}(v) \leq 2$ by Fact 1(2), (3). If $\sum_{i=1}^{4} f_{dia}^{i}(v) \geq 1$, then $c'(v) \geq 10 - \max\{4+4+1, 4+3 \times 2\} = 0$ by Lemma 4, R1-R3. Otherwise, $f_4(v) = 0$ by Fact 1(3) and it follows that $c'(v) \geq 10 - 3 \times 2 - 2 \times 2 = 0$ by Lemma 7, R1 and R3. If $f_3(v) = 5$, then $\sum_{i=1}^{4} f_{dia}^{i}(v) = 2$ and $f_4(v) = 0$ by Fact 1(2), (3). If follows that $c'(v) \geq 10 - 4 - 3 - 3 = 0$ by Lemma 4, Lemma 7 and R1.

If $n_2(v) = 1$, then $f_3(v) \leq 4$ by Fact 1(1). If $f_3(v) \leq 2$, then $c'(v) \geq 10 - \max\{1+4, 3+1+3, 4+1+2, 3 \times 2 + 1 + 2\} = 1$ by Fact 1(2), (3), Lemma 4 and R1-R3. If $f_3(v) = 3$, then $f_4(v) \leq 1$ and $\sum_{i=1}^{4} f_{dia}^{i}(v) \leq 1$ by Fact 1(2), (3) and Lemma 3($C7.1$). If $\sum_{i=1}^{4} f_{dia}^{i}(v) = 1$, then $c'(v) \geq 10 - 4 - 3 - 1 - 1 = 1$ by Lemma 4 and R1-R3. Otherwise, $c'(v) \geq 10 - 3 \times 2 - 2 - 1 - 1 = 0$ by Lemma 7 and R1-R3. If $f_3(v) = 4$, then $f_4(v) \leq 1$ and $1 \leq \sum_{i=1}^{4} f_{dia}^{i}(v) \leq 2$ by Fact 1(2), (3) and Lemma 3($C7.1$). If $\sum_{i=1}^{4} f_{dia}^{i}(v) = 2$, then $c'(v) \geq 10 - (4 \times 2 + 1 + 1) = 0$ by R1-R3. If $\sum_{i=1}^{4} f_{dia}^{i}(v) = 1$, then $c'(v) \geq 10 - \max\{4+2+2+1, 3+3+3+1\} = 0$ by Lemma 7 and R1-R3.

If $n_2(v) = 2$, then $f_3(v) \leq 4$ by Fact 1(1). If $f_3(v) \leq 2$, then $c'(v) \geq 10 - \max\{2+4, 3+2+3, 4+2+2, 3 \times 2 + 2 + 2\} = 0$ by Fact 1 and R1-R3. If $f_3(v) = 3$, then $f_4(v) \leq 1$ by Fact 1(2), (3) and it follows that $c'(v) \geq$

$10 - \max\{4+2+2+1, 3+3+2+1\} = 1$ if $\sum_{i=1}^{4} f_{dia}^{i}(v) = 1$ and $c'(v) \geq 10-3-2-2-2-1 = 0$ if $\sum_{i=1}^{4} f_{dia}^{i}(v) = 0$ by Lemma 4, Lemma 7 and R1-R3. If $f_3(v) = 4$, $f_4(v) \leq 1$ and $\sum_{i=1}^{4} f_{dia}^{i}(v) = 2$ by Fact 1(2) and Lemma 3(C7.1). It follows that $c'(v) \geq 10-4-3-2-1 = 0$ by Lemma 4, Lemma 7 and R1-R3.

If $n_2(v) = 3$, then $f_3(v) \leq 3$ by Fact 1(1). If $f_3(v) \leq 1$, then $c'(v) \geq 10 - \max\{3+4, 3+3+3\} = 1$ by Fact 1(2), (3) and R1-R3. If $f_3(v) = 2$, then $f_4(v) \leq 2$ by Fact 1(2), (3). It follows that $c'(v) \geq 10-4-3-2 = 1$ by Lemma 4 and R1-R3 if $\sum_{i=1}^{4} f_{dia}^{i}(v) = 1$ and $c'(v) \geq 10-3-2-3-2 = 0$ by Lemma 7 and R1-R3 if $\sum_{i=1}^{4} f_{dia}^{i}(v) = 0$. If $f_3(v) = 3$, then $f_4(v) \leq 1$ and $\sum_{i=1}^{4} f_{dia}^{i}(v) = 1$ by Fact 1(2), (3) and Lemma 3(C7.1). Hence, $c'(v) \geq 10-(3+3)-3-1 = 0$ by Lemma 4, Lemma 7 and R1-R3.

If $n_2(v) = 4$, then $f_3(v) \leq 2$ by Fact 1(1). If $f_3(v) \leq 1$, then $c'(v) \geq 10 - \max\{4+4, 3+4+3\} = 0$ by Fact 1(2), (3) and R1-R3. If $f_3(v) = 2$, then $f_4(v) \leq 2$ by Fact 1(2), (3). It follows that $c'(v) \geq 10-4-4-2 = 0$ by Lemma 4 and R1-R3 if $\sum_{i=1}^{4} f_{dia}^{i}(v) = 1$ and $c'(v) \geq 10 - 2\times 2 - 4 - 2 = 0$ by Lemma 7 and R1-R3 if $\sum_{i=1}^{4} f_{dia}^{i}(v) = 0$.

If $n_2(v) = 5$, then $f_3(v) \leq 2$ by Fact 1(1). If $f_3(v) = 0$, then $f_4(v) \leq 4$ by Fact 1(2) and $c'(v) \geq 10-5-4 = 1$ by R2-R3. If $f_3(v) = 1$, then $f_4(v) \leq 3$ by Fact 1(2), (3) and $c'(v) \geq 10-2-5-3 = 0$ by Lemma 7 and R1-R3. If $f_3(v) = 2$, then $f_4(v) \leq 2$ and $\sum_{i=1}^{4} f_{dia}^{i}(v) = 1$ by Fact 1(2), (3) and Lemma 3(C7.1). Hence, $c'(v) \geq 10-3-5-2 = 0$ by Lemma 7 and R1-R3.

If $n_2(v) = 6$, then $f_3(v) = 0$ by Lemma 3(C8.2) and $f_4(v) \leq 4$ by Fact 1(2). Hence, $c'(v) \geq 10-6-4 = 0$ by R2-R3.

Suppose that $d(v) = 9$. Then $c(v) = 2\times 9 - 6 = 12$. By Lemma 9, $n_2(v) \leq 7$. We have some subcases below. If $n_2(v) = 0$, then $f_3(v) \leq 6$ by Fact 1(1). By Fact 1(2), (3), Lemma 4 and R2-R3, $c'(v) \geq 12 - \max\{4, 3+3, 4+3, 3\times 2+2, 4+3+2, 3\times 3+1, 4\times 2+1, 4+3\times 2+1, 3\times 4\} = 0$. If $n_2(v) = 1$, then $f_3(v) \leq 5$ by Fact 1(1). If follows that $c'(v) \geq 12 - \max\{1+4, 3+1+3, 4+1+3, 3\times 2+1+2, 4+3+1+2, 3\times 3+1+1, 4\times 2+1+1, 4+3\times 2+1+1, 4\times 2+3+1\} = 0$ by Fact 1(2), (3), Lemma 4 and R1-R3 if $\iota(v) \neq 4$. Otherwise, $c'(v) \geq 12 - 3\times 3 - 2 - 1 = 0$ by Lemma 7 and R1-R3. If $n_2(v) = 2$, then $f_3(v) \leq 4$ by Fact 1(1). If $f_3(v) \leq 3$, then $c'(v) \geq 12 - \max\{2+4, 3+2+3, 4+2+3, 3\times 2+2+2, 4+3+2+2, 3\times 3+2+1\} = 0$ by Fact 1(2), (3), Lemma 4 and R1-R3. If $f_3(v) = 4$, then $1 \leq \sum_{i=1}^{4} f_{dia}^{i}(v) \leq 2$ by Fact 1(2), (3) and Lemma 3(C7.1). Hence, $c'(v) \geq 12 - 4\times 2 - 2 - 1 = 1$ by Lemma 4 and R1-R3 if $\sum_{i=1}^{4} f_{dia}^{i}(v) = 2$ and $c'(v) \geq 12 - \max\{4+2\times 2, 3\times 3\} - 2 - 1 = 0$ Lemma 4, Lemma 7 and R1-R3 otherwise. If $n_2(v) = 3$, then $f_3(v) \leq 4$ by Fact 1(1). If $f_3(v) \leq 2$, then $c'(v) \geq 12 - \max\{3+4, 3+3+3, 4+3+3, 3\times 2+3+2\} = 1$ by Fact 1(2), (3), Lemma 4 and R1-R3. If $f_3(v) = 3$, then by Fact 1(2), (3) and R1-R3, we have $c'(v) \geq 12-4-3-3-2 = 0$ by Lemma 4 if $\sum_{i=1}^{4} f_{dia}^{i}(v) = 1$ and $c'(v) \geq 12 - 3 - 2\times 2 - 3 - 1 = 1$ by Lemma 7 if $\sum_{i=1}^{4} f_{dia}^{i}(v) = 0$.

If $f_3(v) = 4$, then $f_4(v) \leq 1$ and $\sum_{i=1}^{4} f_{dia}^{i}(v) = 2$ by Fact 1(2), (3) and Lemma 3($C7.1$). It follows that $c'(v) \geq 12 - 4 \times 2 - 3 - 1 = 0$ by Lemma 4 and R1-R3. If $n_2(v) = 4$, then $f_3(v) \leq 3$ by Fact 1(1). If $f_3(v) \leq 2$, then $c'(v) \geq 12 - \max\{4+4, 3+4+3, 4+4+3, 3\times 2+4+2\} = 0$ by Fact 1(2), (3), Lemma 4 and R1-R3. If $f_3(v) = 3$, then $f_4(v) \leq 2$ and $\sum_{i=1}^{4} f_{dia}^{i}(v) = 1$ by Fact 1(2), (3) and Lemma 3($C7.1$) and it follows that $c'(v) \geq 12 - 3 - 3 - 4 - 2 = 0$ by Lemma 4, Lemma 7 and R1-R3. If $n_2(v) = 5$, then $f_3(v) \leq 2$ by Fact 1(1). If $f_3(v) \leq 1$, then $c'(v) \geq 12 - \max\{5+4, 3+5+3\} = 1$ by Fact 1(2), (3), Lemma 4 and R1-R3. If $f_3(v) = 2$, then $f_4(v) \leq 3$ by Fact 1(2), (3) and Lemma 3($C7.1$). Hence, we have $c'(v) \geq 12 - 4 - 5 - 3 = 0$ by Lemma 4 and R1-R3 if $\sum_{i=1}^{4} f_{dia}^{i}(v) = 1$ and $c'(v) \geq 12 - 3 - 2 - 5 - 2 = 0$ by Lemma 7 and R1-R3 otherwise. If $n_2(v) = 6$, then $f_3(v) \leq 2$ and $\sum_{i=1}^{4} f_{dia}^{i}(v) \leq 1$ by Fact 1(2), (3) and Lemma 3($C7.1$). If $f_3(v) \leq 1$, then $c'(v) \geq 12 - \max\{6+4, 3+6+3\} = 0$ by Fact 1(2), (3), R1 and R3. If $f_3(v) = 2$, then $c'(v) \geq 12 - 3 - 6 - 3 = 0$ by Fact 1(2), (3), Lemma 7 and R1-R3. If $n_2(v) = 7$, then $f_3(v) = 0$ by Lemma 9 ($C9.2$) and $f_4(v) \leq 4$ by Fact 1(2). It follows that $c'(v) \geq 12 - 7 - 4 = 1$ by R2-R3.

Suppose that $d(v) \geq 10$. By the assumption of Theorem 2, we have $\frac{3}{2}f_b(v) + 2\iota(v) + 2f_4(v) \leq d(v)$ and $\frac{3}{2}f_b(v) + 2\iota(v) + n_2(v) \leq d(v)$. Note that $f_3(v) = f_b(v) + \iota(v)$. Hence, by Fact 1, Lemma 4 and R1-R5, we can obtain that

$$\begin{aligned}
c'(v) &\geq 2d(v) - 6 - \big(4 \cdot \tfrac{f_b(v)}{2} + 3\iota(v) + f_4(v) + n_2(v)\big) \\
&\quad + \tfrac{1}{4}(d(v) - f_b(v) - \iota(v) - f_4(v) - 1) \\
&= 2d(v) - 6 - \big(\tfrac{9}{4}f_b(v) + \tfrac{13}{4}\iota(v) + \tfrac{5}{4}f_4(v) + n_2(v) - \tfrac{1}{4}d(v) + \tfrac{1}{4}\big) \\
&\geq 2d(v) - 6 - \big(\tfrac{9}{4}f_b(v) + \tfrac{13}{4}\iota(v) + \tfrac{5}{4}f_4(v) + (d(v) - \tfrac{3}{2}f_b(v) - 2\iota(v)) \\
&\quad - \tfrac{1}{4}d(v) + \tfrac{1}{4}\big) \\
&= 2d(v) - 6 - \big(\tfrac{3}{4}d(v) + \tfrac{3}{4}f_b(v) + \tfrac{5}{4}\iota(v) + \tfrac{5}{4}f_4(v) + \tfrac{1}{4}\big) \\
&\geq 2d(v) - 6 - \big(\tfrac{3}{4}d(v) + \tfrac{3}{4}f_b(v) + \tfrac{5}{4} \cdot \tfrac{1}{2} \cdot (d(v) - \tfrac{3}{2}f_b(v)) + \tfrac{1}{4}\big) \\
&\geq \tfrac{5}{8}d(v) - \tfrac{25}{4} \\
&\geq 0.
\end{aligned}$$

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, MacMillan, London, 1976.

[2] O. V. Borodin, D. G. Fon-Der Flass, A. V. Kostochka, A. Raspaud, and E. Sopena, *Acyclic list 7-coloring of planar graphs*, J. Graph Theory **40** (2002), no. 2, 83–90.

[3] O. V. Borodin and A. O. Ivanova, *Acyclic 5-choosability of planar graphs without 4-cycles*, Siberian Math. J. **52** (2011), no. 3, 411–425.

[4] ———, *Acyclic 5-choosability of planar graphs without adjacent short cycles*, J. Graph Theory **68** (2011), no. 2, 169–176.

[5] ———, *Acyclic 4-choosability of planar graphs without adjacent short cycles*, Discrete Math. **312** (2012), no. 22, 3335–3341.

[6] ———, *Acyclic 4-choosability of planar graphs with no 4- and 5-cycles*, J. Graph Theory **72** (2013), no. 4, 374–397.
[7] O. V. Borodin, A. O. Ivanova, and A. Raspaud, *Acyclic 4-choosability of planar graphs with neither 4-cycles nor triangular 6-cycles*, Discrete Math. **310** (2010), no. 21, 2946–2950.
[8] M. Chen and A. Raspaud, *A sufficient condition for planar graphs to be acyclically 5-choosable*, J. Graph Theory **70** (2012), no. 2, 135–151.
[9] ———, *Planar graphs without 4- and 5-cycles are acyclically 4-choosable*, Discrete Appl. Math. **161** (2013), no. 7-8, 921–931.
[10] M. Chen, A. Raspaud, N. Roussel, and X. D. Zhu, *Acyclic 4-choosability of planar graphs*, Discrete Math. **311** (2011), no. 1, 92–101.
[11] M. Chen and W. F. Wang, *Acyclic 5-choosability of planar graphs without 4-cycles*, Discrete Math. **308** (2008), no. 24, 6216–6225.
[12] B. Grünbaum, *Acyclic colorings of planar graphs*, Israel J. Math. **14** (1973), no. 3, 390–408.
[13] J. F. Hou and G. Z. Liu, *Every toroidal graph is acyclically 8-choosable*, Acta Math. Sin. (Engl. Ser.) **30** (2014), no. 2, 343–352.
[14] M. Montassier, *Acyclic 4-choosability of Planar Graphs with Girth at Least 5*, Graph theory in Paris, 299–310, Trends Math., Birkhäuser, Basel, 2007.
[15] M. Montassier, P. Ochem, and A. Raspaud, *On the acyclic choosability of graphs*, J. Graph Theory **51** (2006), no. 4, 281–300.
[16] M. Montassier, A. Raspaud, and W. Wang, *Acyclic 5-choosability of planar graphs without small cycles*, J. Graph Theory **54** (2007), no. 3, 245–260.
[17] W. F. Wang and M. Chen, *Planar graphs without 4-cycles are 6-choosable*, J. Graph Theory **61** (2009), no. 4, 307–323.
[18] W. F. Wang, G. Zhang, and M. Chen, *Acyclic 6-choosability of planar graphs without adjacent short cycles*, Sci. China Math. **57** (2014), no. 1, 197–209.

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