Bull. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/BKMS.b170051
pISSN: 1015-8634 / eISSN: 2234-3016

COEFFICIENT ESTIMATES FOR A SUBCLASS OF ANALYTIC BI-UNIVALENT FUNCTIONS

Ebrahim Analouei Adegani, Serap Bulut, and Ahmad Zireh

Abstract. In this work, we use the Faber polynomial expansions to find upper bounds for the coefficients of analytic bi-univalent functions in subclass $\Sigma(\tau,\gamma,\varphi)$ which is defined by subordination conditions in the open unit disk $\mathbb{U}$. In certain cases, our estimates improve some of those existing coefficient bounds.

1. Introduction

Let $\mathbb{C}$ be the set of complex numbers and $\mathbb{N} = \{1, 2, \dots\}$. Let $\mathcal{A}$ be a class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

We also denote by $\mathcal{S}$ the class of all functions in class $\mathcal{A}$ which are univalent in $\mathbb{U}$.

An analytic function f is said to be subordinate to another analytic function g, written as

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z) = \sum_{n=1}^{\infty} c_n z^n$, which is analytic in $\mathbb{U}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \qquad (z \in \mathbb{U}),$$

such that $f(z) = g(w(z))$. In particular, if the function g is univalent in $\mathbb{U}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$. For the Schwarz function $w(z)$ we note that $|c_n| \le 1$, (see [6]).

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

Received January 18, 2017; Revised August 4, 2017; Accepted November 24, 2017.
2010 *Mathematics Subject Classification.* Primary 30C45, 30C80.
Key words and phrases. analytic functions, univalent functions, bi-univalent functions, coefficient estimates, Faber polynomials, subordination.

and

$$f\left(f^{-1}(w)\right)=w \qquad \left(|w|<r_0(f)\,;\ r_0(f)\geq \frac{1}{4}\right).$$

where

$$g(w)=f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots$$

$$=: w+\sum_{n=2}^{\infty}A_nw^n. \tag{1.2}$$

A function $f\in\mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. The class consisting of bi-univalent functions is denoted by Σ. Finding bounds for the coefficients of classes of analytic bi-univalent functions was first introduced and studied by Lewin in [10] where it was proved that $|a_2|<1.51$. Not much is known about the bounds on the higher coefficients $|a_n|$ for $n>3$. Ali et al. [4] remarked that finding the bounds for $|a_n|$ $(n>3)$ for the bi-univalent functions is an open problem.

The Faber polynomials introduced by Faber [7] play an important role in various areas of mathematical sciences, especially in geometric function theory. By using the Faber polynomial expansion of functions $f\in\mathcal{S}$ of the form (1.1), the coefficients of its inverse map $g=f^{-1}$ may be expressed, (see for details [2] and [3]),

$$g(w)=f^{-1}(w)=w+\sum_{n=2}^{\infty}\frac{1}{n}K_{n-1}^{-n}(a_2,a_3,\ldots,a_n)w^n, \tag{1.3}$$

where

$$\begin{aligned}K_{n-1}^{-n}=&\frac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1}+\frac{(-n)!}{(2(-n+1))!(n-3)!}a_2^{n-3}a_3\\&+\frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4+\frac{(-n)!}{(2(-n+2))!(n-5)!}a_2^{n-5}\\&\times[a_5+(-n+2)a_3^2]+\frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}[a_6+(-2n+5)a_3a_4]\\&+\sum_{j\geq 7}a_2^{n-j}V_j,\end{aligned}$$

such that V_j with $7\leq j\leq n$ is a homogeneous polynomial in the variables $a_2,a_3,\ldots,a_n$, (see for details [3]). In particular, the first three terms of K_{n-1}^{-n} are

$$K_1^{-2}=-2a_2,\quad K_2^{-3}=3\left(2a_2^2-a_3\right),\quad K_3^{-4}=-4(5a_2^3-5a_2a_3+a_4).$$

In general, for any $p\in\mathbb{Z}=\{0,\pm1,\pm2,\ldots\}$, an expansion of K_n^p is (see for details [2, 15] or [3, page 349])

$$K_n^p=pa_{n+1}+\frac{p(p-1)}{2}D_n^2+\frac{p!}{(p-3)!3!}D_n^3+\cdots+\frac{p!}{(p-n)!n!}D_n^n,$$

where $D_n^p = D_n^p(a_2, a_3, \ldots)$ and by [9] (see for details [1, 13, 15])

$$D_n^m(a_2, a_3, \ldots, a_{n+1}) = \sum \frac{m!(a_2)^{\mu_1} \cdots (a_{n+1})^{\mu_n}}{\mu_1! \cdots \mu_n!}, \tag{1.4}$$

where the sum is taken over all nonnegative integers $\mu_1, \ldots, \mu_n$ satisfying

$$\begin{cases} \mu_1 + \mu_2 + \cdots + \mu_n = m, \\ \mu_1 + 2\mu_2 + \cdots + n\mu_n = n. \end{cases}$$

It is clear that $D_n^n(a_2, a_3, \ldots, a_{n+1}) = a_2^n$.

Throughout this paper, we assume that φ is an analytic function with positive real part in the unit disk $\mathbb{U}$, satisfying $\varphi(0) = 1,\ \varphi'(0) > 0$ and $\varphi(\mathbb{U})$ is symmetric with respect to the real axis. Such a function has series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \qquad (B_1 > 0).$$

Let that $u(z)$ and $v(z)$ are analytic in the unit disk $\mathbb{U}$ with $u(0) = v(0) = 0,\ |u(z)| < 1,\ |v(z)| < 1$, and suppose that

$$u(z) = z(p_1 + \sum_{n=2}^{\infty} p_n z^{n-1}) \text{ and } v(z) = z(q_1 + \sum_{n=2}^{\infty} q_n z^{n-1}) \ (z \in \mathbb{U}). \tag{1.5}$$

Then

$$|p_1| \le 1,\ |p_n| \le 1 - |p_1|^2,\ |q_1| \le 1,\ |q_n| \le 1 - |q_1|^2,\ (n \in \mathbb{N}\setminus\{1\}), \tag{1.6}$$

for example, see [11, Page 171].

Recently, Srivastava and Bansal [12] (see also [5, Page 57]) introduced the subclass of Σ as follow.

Definition 1. Let $0 \le \gamma \le 1$ and $\tau \in \mathbb{C}\setminus\{0\}$. A function $f \in \Sigma$ is said to be in the subclass $\Sigma(\tau, \gamma, \varphi)$ if each of the following subordination conditions holds true:

$$1 + \frac{1}{\tau}[f'(z) + \gamma z f''(z) - 1] \prec \varphi(z) \qquad (z \in \mathbb{U})$$

and

$$1 + \frac{1}{\tau}[g'(w) + \gamma w g''(w) - 1] \prec \varphi(w) \qquad (w \in \mathbb{U})$$

where $g = f^{-1}$ is given by (1.2).

The purpose of our study is to obtain bounds for the general coefficients $|a_n|$ $(n \ge 3)$ by using Faber polynomial expansion under certain conditions for analytic bi-univalent functions in subclass $\Sigma(\tau, \gamma, \varphi)$ and also we obtain improvements on the bounds for the first two coefficients $|a_2|$ and $|a_3|$ of functions in this subclass. In certain cases, our estimates improve some of those existing coefficient bounds.

2. Coefficient estimates

Now, we obtain an upper bound for the coefficients $|a_n|$ of functions in the subclass $\Sigma(\tau,\gamma,\varphi)$.

Theorem 1. *For* $0\le\gamma\le 1$ *and* $\tau\in\mathbb{C}\backslash\{0\}$*, let the function* $f\in\Sigma(\tau,\gamma,\varphi)$ *be given by* (1.1)*. If* $a_k=0$ *for* $2\le k\le n-1$*, then*

$$|a_n|\le\frac{|\tau|\,B_1}{n\,[1+\gamma(n-1)]}\qquad(n\ge 3)\,. \tag{2.1}$$

Proof. For analytic functions f, given by (1.1), we have

$$1+\frac{1}{\tau}[f'(z)+\gamma zf''(z)-1]=1+\sum_{n=2}^{\infty}\frac{1}{\tau}[1+\gamma(n-1)]na_nz^{n-1} \tag{2.2}$$

and for its inverse map $g=f^{-1}$, given by (1.2), we have

$$1+\frac{1}{\tau}[g'(w)+\gamma wg''(w)-1]=1+\sum_{n=2}^{\infty}\frac{1}{\tau}[1+\gamma(n-1)]nA_nw^{n-1}. \tag{2.3}$$

Considering the equality (1.3), the above equality yields

$$\begin{aligned}&1+\frac{1}{\tau}[g'(w)+\gamma wg''(w)-1]\\&=1+\sum_{n=2}^{\infty}\frac{1}{\tau}[1+\gamma(n-1)]K_{n-1}^{-n}(a_2,a_3,\ldots,a_n)w^{n-1}.\end{aligned} \tag{2.4}$$

On the other hand, since $f\in\Sigma(\tau,\gamma,\varphi)$ and $g=f^{-1}\in\Sigma(\tau,\gamma,\varphi)$, there are two Schwarz functions $u,v:\mathbb{U}\to\mathbb{U}$ with $u(0)=v(0)=0$, which are given by (1.5), so that

$$1+\frac{1}{\tau}[f'(z)+\gamma zf''(z)-1]=\varphi(u(z)), \tag{2.5}$$

and

$$1+\frac{1}{\tau}[g'(w)+\gamma wg''(w)-1]=\varphi(v(w)). \tag{2.6}$$

Also, by (1.4) we get

$$\begin{aligned}\varphi(u(z))&=1+B_1p_1z+(B_1p_2+B_2p_1^2)z^2+\cdots\\&=1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}B_kD_n^k(p_1,p_2,\ldots,p_n)z^n,\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\varphi(v(w))&=1+B_1q_1w+(B_1q_2+B_2q_1^2)w^2+\cdots\\&=1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}B_kD_n^k(q_1,q_2,\ldots,q_n)w^n.\end{aligned} \tag{2.8}$$

Comparing the corresponding coefficients of (2.2) and (2.5) with (2.7) we obtain

$$\frac{1}{\tau}[1+\gamma(n-1)]na_n = \sum_{k=1}^{n-1} B_k D_{n-1}^k(p_1, p_2, \ldots, p_{n-1}) \qquad (n \geq 2)\,. \tag{2.9}$$

Similarly, from (2.3) and (2.6) with (2.8) we get

$$\begin{aligned} &\frac{1}{\tau}[1+\gamma(n-1)]K_{n-1}^{-n}(a_2, a_3, \ldots, a_n) \\ &= \sum_{k=1}^{n-1} B_k D_{n-1}^k(q_1, q_2, \ldots, q_{n-1}) \qquad (n \geq 2)\,. \end{aligned} \tag{2.10}$$

Now, from $a_k = 0$ for $2 \leq k \leq n-1$, we have $A_n = -a_n$ and the equalities (2.9) and (2.10) yield

$$\begin{aligned} [1+\gamma(n-1)]na_n &= \tau B_1 p_{n-1}, \\ -[1+\gamma(n-1)]na_n &= \tau B_1 q_{n-1}. \end{aligned} \tag{2.11}$$

Now taking the absolute values of either of the above two equations in (2.11) and using (1.6), we obtain (2.1) and this completes the proof of the theorem. □

Corollary 1. *For $0 \leq \gamma \leq 1,\ \tau \in \mathbb{C}\setminus\{0\}$ and $\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}$ $(0 < \alpha \leq 1)$, let the function $f \in \Sigma(\tau,\gamma,\varphi)$ be given by (1.1). If $a_k = 0$ for $2 \leq k \leq n-1$, then*

$$|a_n| \leq \frac{2\,|\tau|\,\alpha}{n\,[1+\gamma(n-1)]} \qquad (n \geq 3)\,.$$

Corollary 2. *For $0 \leq \gamma \leq 1,\ \tau \in \mathbb{C}\setminus\{0\}$ and $\varphi(z) = \frac{1+(1-2\beta)z}{1-z}$ $(0 \leq \beta < 1)$, let the function $f \in \Sigma(\tau,\gamma,\varphi)$ be given by (1.1). If $a_k = 0$ for $2 \leq k \leq n-1$, then*

$$|a_n| \leq \frac{2\,|\tau|\,(1-\beta)}{n\,[1+\gamma(n-1)]} \qquad (n \geq 3)\,.$$

Remark 1. By taking $\tau = 1$ in Corollary 2, we get results obtained by Srivastava et al. [13, Theorem 1], for all $0 \leq \gamma \leq 1$.

Theorem 2. *For $0 \leq \gamma \leq 1$ and $\tau \in \mathbb{C}\setminus\{0\}$, let the function $f \in \Sigma(\tau,\gamma,\varphi)$ be given by (1.1). Then one has the following*

$$|a_2| \leq \frac{|\tau|\,B_1\sqrt{B_1}}{\sqrt{4(1+\gamma)^2 B_1 + |3(1+2\gamma)\tau B_1^2 - 4(1+\gamma)^2 B_2|}} \tag{2.12}$$

and

$$|a_3| \leq \min\left\{k(\gamma), l(\gamma)\right\}, \tag{2.13}$$

where

$$l(\gamma) = \begin{cases} \frac{|\tau|B_1}{3(1+2\gamma)} \times \frac{3(1+2\gamma)|\tau|B_1^2 + \left|3(1+2\gamma)\tau B_1^2 - 4(1+\gamma)^2 B_2\right|}{4(1+\gamma)^2 B_1 + \left|3(1+2\gamma)\tau B_1^2 - 4(1+\gamma)^2 B_2\right|}, & B_1 \geq \frac{4(1+\gamma)^2}{3|\tau|(1+2\gamma)} \\ \frac{|\tau|B_1}{3(1+2\gamma)}, & 0 < B_1 \leq \frac{4(1+\gamma)^2}{3|\tau|(1+2\gamma)} \end{cases}$$

and

$$k(\gamma)=\begin{cases}\frac{\tau|B_2|}{3(1+2\gamma)}, & |B_2|>B_1\\ \frac{\tau B_1}{3(1+2\gamma)}, & |B_2|\leq B_1.\end{cases}$$

Proof. If we set $n=2$ and $n=3$ in (2.9) and (2.10) , respectively, we get

$$\frac{2(1+\gamma)}{\tau}a_2=B_1p_1, \tag{2.14}$$

$$\frac{3(1+2\gamma)}{\tau}a_3=B_1p_2+B_2p_1^2, \tag{2.15}$$

$$-\frac{2(1+\gamma)}{\tau}a_2=B_1q_1, \tag{2.16}$$

$$\frac{3(1+2\gamma)}{\tau}(2a_2^2-a_3)=B_1q_2+B_2q_1^2. \tag{2.17}$$

From (2.14) and (2.16), we get

$$p_1=-q_1. \tag{2.18}$$

Adding (2.15) and (2.17), and using (2.18), we have

$$6(1+2\gamma)a_2^2-2\tau B_2p_1^2=\tau B_1(p_2+q_2). \tag{2.19}$$

From (2.14), we get

$$[6\tau(1+2\gamma)B_1^2-8(1+\gamma)^2B_2]a_2^2=\tau^2B_1^3(p_2+q_2).$$

By (1.6), (2.14) and (2.18), we obtain

$$\begin{aligned}\left|6\tau(1+2\gamma)B_1^2-8(1+\gamma)^2B_2\right|\,|a_2|^2 &\leq |\tau|^2B_1^3\left(|p_2|+|q_2|\right)\\ &\leq 2|\tau|^2B_1^3\left(1-|p_1|^2\right)\\ &= 2|\tau|^2B_1^3-8(1+\gamma)^2B_1|a_2|^2.\end{aligned} \tag{2.20}$$

Consequently

$$|a_2|^2\leq\frac{|\tau|^2B_1^3}{4(1+\gamma)^2B_1+|3(1+2\gamma)\tau B_1^2-4(1+\gamma)^2B_2|}.$$

So we obtain the bound on $|a_2|$ in (2.12).

Next, in order to find the bound on the coefficient $|a_3|$, by subtracting (2.17) from (2.15), and using (2.18), we get

$$6(1+2\gamma)a_3=6(1+2\gamma)a_2^2+\tau B_1(p_2-q_2).$$

Using (1.6), we have

$$\begin{aligned}6(1+2\gamma)|a_3| &\leq 6(1+2\gamma)|a_2|^2+|\tau|B_1\left(|p_2|+|q_2|\right)\\ &\leq 6(1+2\gamma)|a_2|^2+2|\tau|B_1(1-|p_1|^2).\end{aligned}$$

From (2.14), we get

$$3|\tau|(1+2\gamma)B_1|a_3|\leq|\tau|^2B_1^2+\left[3|\tau|(1+2\gamma)B_1-4(1+\gamma)^2\right]|a_2|^2. \tag{2.21}$$

On the other hand from (2.15), we have

$$3(1+2\gamma)|a_3| \leq \tau[B_1(1-|p_1|^2)+|B_2||p_1|^2].$$

Consequently,

$$|a_3| \leq \begin{cases} \frac{\tau|B_2|}{3(1+2\gamma)}, & |B_2| > B_1 \\ \frac{\tau B_1}{3(1+2\gamma)}, & |B_2| \leq B_1. \end{cases} \tag{2.22}$$

Hence, from (2.21) and (2.22), we obtain the desired estimate on $|a_3|$ given in (2.13). This completes the proof. □

For $\tau = 1$ and $\gamma = 0$ we have the following corollary.

Corollary 3. *Let the function $f \in \Sigma(1,0,\varphi)$ be given by* (1.1). *Then one has the following*

$$|a_3| \leq \min\{k(0), l(0)\},$$

where

$$l(0) = \begin{cases} \frac{B_1}{3} \times \frac{3B_1^2+|3B_1^2-4B_2|}{4B_1+|3B_1^2-4B_2|}, & B_1 \geq \frac{4}{3} \\ \frac{B_1}{3}, & 0 < B_1 \leq \frac{4}{3} \end{cases}$$

and

$$k(0) = \begin{cases} \frac{|B_2|}{3}, & |B_2| > B_1 \\ \frac{B_1}{3}, & |B_2| \leq B_1. \end{cases}$$

Remark 2. Theorem 2 is an improvement of the estimates obtained by Srivastava and Bansal [12, Theorem 1].

Remark 3. If we take $\gamma = 0$ and $\tau = 1$ in Theorem 2, then we get an improvement of the estimates obtained by Ali et al. [4, Theorem 2.1].

Remark 4. If we take $\tau = 1$ and $\varphi(z) = \left(\frac{1+z}{1-z}\right)^\alpha$ $(0 < \alpha \leq 1)$ in Theorem 2, then for $|a_3|$, we have an improvement of the estimates which were given by Frasin, and for $|a_2|$ is the same [8, Theorem 2.2] for all $0 < \gamma \leq 1$.

Remark 5. If we set $\varphi(z) = \left(\frac{1+z}{1-z}\right)^\alpha$ $(0 < \alpha \leq 1)$ in Corollary 3, then for $|a_3|$, we have the estimates which were given by Zaprawa [17, Corollary 3] and an improvement of the estimates which were given by Srivastava et al. [14, Theorem 1].

Remark 6. If we take $\tau = 1$ and $\varphi(z) = \frac{1+(1-2\beta)z}{1-z}$ $(0 \leq \beta < 1)$ in Theorem 2, then we get an improvement of the estimates obtained by Srivastava et al. [13, Theorem 1], for all $0 \leq \gamma \leq 1$ (see also [8, Theorem 3.2]).

Remark 7. If we set $\varphi(z) = \frac{1+(1-2\beta)z}{1-z}$ $(0 \leq \beta < 1)$ in Corollary 3, then we have the estimates which were given by Zaprawa [17, Corollary 4] and an improvement of the estimates which were given by Srivastava et al. [14, Theorem 2].

Remark 8. Recently, Yang and Liu [16] proved that if $f \in \mathcal{A}$ and

$$\Re\left(f'(z) + \gamma z f''(z)\right) > \beta \qquad (\gamma > 0,\ 0 \leq \beta < 1),$$

then $f \in \mathcal{S}$ if and only if

$$2(1-\beta)\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\gamma m + 1} \leq 1.$$

So the condition $2(1-\beta)\sum\limits_{m=1}^{\infty} \frac{(-1)^{m-1}}{\gamma m+1} \leq 1$ in [8, Definition 3.1] is not necessary.

Acknowledgements. The authors would like to thank the referees for their kind help.

References

[1] H. Airault, *Symmetric sums associated to the factorization of Grunsky coefficients*, in Groups and symmetries, CRM Proc. Lecture Notes Amer. Math. Soc. Providence, RI, **47** (2007), 3–16.
[2] H. Airault and A. Bouali, *Differential calculus on the Faber polynomials*, Bull. Sci. Math. **130** (2006), no. 3, 179–222.
[3] H. Airault and J. Ren, *An algebra of differential operators and generating functions on the set of univalent functions*, Bull. Sci. Math. **126** (2002), no. 5, 343–367.
[4] R. M. Ali and S. K. Lee, V. Ravichandran, and S. Subramaniam, *Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions*, Appl. Math. Lett. **25** (2012), no. 3, 344–351.
[5] E. Deniz, *Certain subclasses of bi-univalent functions satisfying subordinate conditions*, J. Class. Anal. **2** (2013), no. 1, 49–60.
[6] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, **259**, Springer-Verlag, New York, 1983.
[7] G. Faber, *Über polynomische Entwickelungen*, Math. Ann. **57** (1903), no. 3, 389–408.
[8] B. A. Frasin, *Coefficient bounds for certain classes of bi-univalent functions*, Hacet. J. Math. Stat. **43** (2014), no. 3, 383–389.
[9] J. M. Jahangiri and S. G. Hamidi, *Coefficient estimates for certain classes of bi-univalent functions*, Int. J. Math. Math. Sci. **2013**, Art. ID 190560, 4 pp.
[10] M. Lewin, *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc. **18** (1967), 63–68.
[11] Z. Nehari, *Conformal Mapping*, McGraw-Hill Book Co., Inc., New York, Toronto, London, 1952.
[12] H. M. Srivastava and D. Bansal, *Coefficient estimates for a subclass of analytic and bi-univalent functions*, J. Egyptian Math. Soc. **23** (2015), no. 2, 242–246.
[13] H. M. Srivastava, S. S. Eker, and R. M. Ali, *Coefficient bounds for a certain class of analytic and bi-univalent functions*, Filomat **29** (2015), no. 8, 1839–1845.
[14] H. M. Srivastava, A. K. Mishra, and P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett. **23** (2010), no. 10, 1188–1192.
[15] P. G. Todorov, *On the Faber polynomials of the univalent functions of class* Σ, J. Math. Anal. Appl. **162** (1991), no. 1, 268–276.
[16] D.-G. Yang and J.-L. Liu, *A class of analytic functions with missing coefficients*, Abstr. Appl. Anal. **2011**, Art. ID 456729, 16 pp.
[17] P. Zaprawa, *On the Fekete-Szegö problem for classes of bi-univalent functions*, Bull. Belg. Math. Soc. Simon Stevin **21** (2014), no. 1, 169–178.

Ebrahim Analouei Adegani
Department of Mathematics
Shahrood University of Technology
P.O.Box 316-36155, Shahrood, Iran
E-mail address: e_analoei@ymail.com

Serap Bulut
Kocaeli University
Faculty of Aviation and Space Sciences
Arslanbey Campus, 41285 Kartepe-Kocaeli, Turkey
E-mail address: serap.bulut@kocaeli.edu.tr

Ahmad Zireh
Department of Mathematics
Shahrood University of Technology
P.O.Box 316-36155, Shahrood, Iran
E-mail address: azireh@gmail.com