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ON m-ISOMETRIC TOEPLITZ OPERATORS

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Abstract. In this paper, we study m-isometric Toeplitz operators T_φ with rational symbols. We characterize m-isometric Toeplitz operators T_φ by properties of the rational symbols φ. In addition, we give a necessary and sufficient condition for Toeplitz operators T_φ with analytic symbols φ to be m-expansive or m-contractive. Finally, we give some results for m-expansive and m-contractive Toeplitz operators T_φ with trigonometric polynomial symbols φ.

1. Introduction

Let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. In 1990's, Agler and Stankus [2] intensively studied the following operators; for a fixed positive integer m, we denote

$$B^m(T) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^{*j} T^j \tag{1.1}$$

for an operator $T \in \mathcal{L}(\mathcal{H})$. We say that $T \in \mathcal{L}(\mathcal{H})$ is *m-expansive* if $B^m(T) \leq 0$ for some positive integer m. In particular, if $B^m(T) = 0$, then T is said to be *m-isometric.* When $B^m(T) \geq 0$, we say that T is *m-contractive.*

The class of m-isometric operators has been widely investigated in latest years. In [1], J. Agler characterized subnormality with the positivity of $B^m(T)$ and also extended his results to the concept of m-isometric operators. The theory of these operators was investigated especially by Agler and Stankus [2–4]. In these papers, they developed a theory for the m-isometric operators with rich connections to Toeplitz operators and function theory. Recently, there has been worked on products of m-isometries [6] and m-isometric composition operators [17]. Many researchers have extensively studied the isometric Toeplitz operators in various ways; see [11–13] and the references therein. Based on these papers, we are studying the m-isometric Toeplitz operators.

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A function $\theta \in H^\infty$ satisfies $|\theta| = 1$ a.e. on $\mathbb{T}$ is an inner function. If θ is an inner function, the degree of θ, denoted by $\deg \theta$, is defined as $n + s$ if θ is a finite Blaschke product of the form

$$\theta(z) = e^{i\xi} z^s \prod_{j=1}^{n} \frac{z - \alpha_j}{1 - \overline{\alpha_j} z} \quad (|\alpha_j| < 1 \text{ for } j = 1, 2, \ldots, n),$$

otherwise the degree of θ is infinite. For an inner function θ, write

$$\mathcal{H}(\theta) := H^2 \ominus \theta H^2.$$

In [14], it was shown that if $f \in H^\infty$ is a rational function, then we can write

$$f = \theta \overline{a},$$

where θ is a finite Blaschke product and $a \in H^\infty$ satisfies that the inner parts of a and θ are coprime. For φ in $L^\infty(\mathbb{T})$ of the unit circle $\mathbb{T} = \partial\mathbb{D}$, the *Toeplitz operator* T_φ *with symbol* φ on the Hardy space $H^2(\mathbb{T})$ is given by

$$T_\varphi f := P(\varphi f) \quad (f \in H^2(\mathbb{T})),$$

where P denotes the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$.

If φ, ψ are in $L^\infty(\mathbb{T})$, then it is well-known that

1) $T_{\varphi+\psi} = T_\varphi + T_\psi$,
2) ${T_\varphi}^* = T_{\overline{\varphi}}$,
3) $T_{\overline{\varphi}} T_\psi = T_{\overline{\varphi}\psi}$ if φ or ψ is analytic.

These properties enable us to establish several consequences of m-isometric operators.

This paper is organized as follows. In Section 2, we study some properties of m-isometric Toeplitz operators. In particular, we give several results for the m-isometric Toeplitz operators with rational symbols. In Section 3, we establish some results for the m-expansive and m-contractive Toeplitz operators.

2. m-isometric operators

First, we briefly recall the definitions and some elementary properties of Toeplitz operators and m-isometric operators. We refer the reader to [2–4, 8] for further references.

Given a positive integer m, it follows from definition that an operator $T \in \mathcal{L}(\mathcal{H})$ is an m-isometry if and only if

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \|T^j x\|^2 = 0 \text{ for all } x \in \mathcal{H}. \tag{2.1}$$

The above formulation was used to define m-isometries on a Banach space by Sid Ahmed [5] and by Botelho [7] on l_p spaces and general function spaces.

Using the identity (2.1) and the Toeplitz operator with symbol φ, we consider the following equation

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\|T_\varphi^j k\|^2 = 0 \tag{2.2}$$

for all $k \in H^2(\mathbb{T})$.

In [8], A. Brown and P.R. Halmos characterize isometric Toeplitz operators T_φ by properties of the symbols φ.

Lemma 2.1 ([8])**.** *A Toeplitz operator T_φ is an isometric operator if and only if φ is inner.*

We recapture the following lemma for the convenience of the readers.

Lemma 2.2 ([2])**.** *If T is an m-isometry, then it is an $m+1$-isometry.*

Proof. If T is an m-isometry, then $B^m(T) = 0$ from (1.1). Since $B^{m+1}(T) = T^*B^m(T)T - B^m(T)$, $B^{m+1}(T) = 0$. Hence T is an $m+1$- isometry. This completes the proof. □

Next, we give several results of m-isometric Toeplitz operators. The following results are the consequences of m-isometric Toeplitz operators with rational symbols.

Lemma 2.3. *If a Toeplitz operator T_φ with rational symbols φ is an m-isometry, then φ is analytic.*

Proof. Suppose that $\varphi(z) = f + \overline{g}$ is a rational function. Then we can write

$$f(z) = \theta_1\overline{a} \quad \text{and} \quad g(z) = \theta_2\overline{b}$$

for some finite Blaschke products θ_1 and θ_2, where $a \in \mathcal{H}(\theta_1)$ and $b \in \mathcal{H}(\theta_2)$. Since T_φ is an m-isometry,

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T_\varphi^{*j}T_\varphi^j k = 0$$

holds for all $k \in H^2(\mathbb{T})$. Put $k(z) = c$ for some nonzero constant c. Then

$$\begin{aligned}
&\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T_\varphi^{*j}T_\varphi^j c \\
&= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T_{\overline{f}+g}^j T_{f+\overline{g}}^j c \\
&= T_{\overline{f}+g}^m T_{f+\overline{g}}^m c - mT_{\overline{f}+g}^{m-1}T_{f+\overline{g}}^{m-1}c + \cdots + (-1)^{m-1}T_{\overline{f}+g}T_{f+\overline{g}}c + (-1)^m c.
\end{aligned}$$

Since the maximal degree term of the above relation is included only in $T_{\overline{f}+g}^m T_{f+\overline{g}}^m c$ term and the maximal degree term $c\theta_1^m\theta_2^m\overline{a}^m\overline{b}^m$ must be a zero, we have either f or g is zero. If $f = 0$, i.e., $\varphi = \overline{g}$, then for some nonzero constant $c \in H^2(\mathbb{T})$,

$$\begin{aligned} 0 &= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T_\varphi^{*j} T_\varphi^j c \\ &= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T_g^j T_{\overline{g}}^j c \\ &= (-1)^m c, \end{aligned}$$

we have a contradiction. Therefore $g = 0$ and hence φ is analytic. This completes the proof. □

Recall that $T \in \mathcal{L}(\mathcal{H})$ is said to be subnormal if T has a normal extension, i.e., $T = N|_{\mathcal{H}}$, where N is a normal operator on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that $\mathcal{H}$ is invariant for N. From Lemma 2.3, we get the following corollary immediately.

Corollary 2.4. *Every m-isometric Toeplitz operators T_φ with rational symbols φ is subnormal.*

We next show that every m-isometric Toeplitz operators with rational symbol is an isometry.

Theorem 2.5. *Let φ be a rational function. A Toeplitz operator T_φ is an m-isometry if and only if T_φ is an isometry.*

Proof. If T_φ is an m-isometry, Lemma 2.3 ensures that φ is analytic. Put $\varphi = f$ where $f \in H^\infty$. Then

$$T_\varphi^{*m-j} T_\varphi^{m-j} = T_{\overline{f}^{m-j}} T_{f^{m-j}} = T_{\overline{f}^{m-j} f^{m-j}}.$$

Hence

$$\begin{aligned} 0 &= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T_\varphi^{*j} T_\varphi^j \\ &= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T_{\overline{f}^j f^j} \\ &= T_{\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \overline{f}^j f^j} \\ &= T_{(\overline{f} f - 1)^m}. \end{aligned}$$

Thus $\overline{f} f = 1$, or equivalently, $\varphi \overline{\varphi} = 1$ and by Lemma 2.1, T_φ is an isometry. The converse implication is trivial by Lemma 2.2. □

From Theorem 2.5, we get the following results.

Corollary 2.6. *Suppose that φ is a rational function. If T_φ and T_φ^* are m-isometric operators, then T_φ is unitary and $\sigma(T_\varphi) \subset \partial\mathbb{D}$.*

Corollary 2.7. *Suppose that φ is a rational function. Then T_φ is an m-isometry if and only if φ is a finite Blaschke product.*

Proof. If T_φ is an m-isometry, then it is an isometry from Theorem 2.5. Hence it follows from Lemma 2.1 that φ is inner. Since φ is rational, φ is a finite Blaschke product. □

As some applications of m-isometric Toeplitz operators, we talk about the hyponormal Toeplitz operators. Hyponormal operators are closely connected to m-isometric operators; see [9, 18]. Recall that $T \in \mathcal{L}(\mathcal{H})$ is said to be hyponormal if its self-commutator $[T^*, T] := T^*T - TT^* \geq 0$. As considering Toeplitz operators with symbol $\varphi \in L^\infty(\mathbb{T})$, the relationship between the positivity of the self-commutator $[T_\varphi^*, T_\varphi]$ and the symbol φ was solved by C. Cowen [10] in 1988.

Lemma 2.8 (**Cowen's Theorem [10]**). *For $\varphi \in L^\infty(\mathbb{T})$, write*

$$\mathcal{E}(\varphi) := \{k \in H^\infty(\mathbb{T}) : \|k\|_\infty \leq 1 \text{ and } \varphi - k\overline{\varphi} \in H^\infty(\mathbb{T})\}.$$

Then T_φ is hyponormal if and only if $\mathcal{E}(\varphi)$ is nonempty.

The following lemma is a result on hyponormal Toeplitz operators with a finite rank self-commutator.

Lemma 2.9 (**Nakazi-Takahashi Theorem [16]**). *A Toeplitz operator T_φ is hypornormal and $[T_\varphi^*, T_\varphi]$ is a finite rank operator if and only if there exists a finite Blaschke product k in $\mathcal{E}(\varphi)$. In this case, we can choose k such that* $\deg(k) = \operatorname{rank}[T_\varphi^*, T_\varphi]$.

Using Lemma 2.9, authors in [14] characterized the rank of self-commutator as follows.

Lemma 2.10 ([14]). *Let $\varphi = \overline{g} + f \in L^\infty$, where f and g are in H^2. If φ is of bounded type and T_φ is hyponormal then*

$$\operatorname{rank}[T_\varphi^*, T_\varphi] = \min\{\deg(k) : k \text{ is an inner function in } \mathcal{E}(\varphi)\}.$$

Next, we deduced the rank of self-commutator of m-isometric Toeplitz operators.

Theorem 2.11. *Suppose that T_φ is an m-isometric Toeplitz operator with rational symbols. Then* $\deg(\varphi) = \operatorname{rank}[T_\varphi^*, T_\varphi]$.

Proof. By Corollary 2.4, T_φ is subnormal and hence hyponormal. Since T_φ is m-isometric, from Corollary 2.7, φ is a finite Blaschke product. Put $\varphi = \theta$ where θ is a finite Blaschke product. Thus if $k \in \mathcal{E}(\varphi)$ is inner then $k = \theta h$ for some $h \in H^\infty$ $(0 \in \mathcal{E}(\varphi))$. By Lemma 2.10, $\operatorname{rank}[T_\varphi^*, T_\varphi] = \deg(\theta) = \deg(\varphi)$. This completes the proof. □

Example 2.12. Suppose that φ is a finite Blaschke product of the form $\varphi(z) = \prod_{j=1}^{n} \frac{z-\alpha_j}{1-\overline{\alpha_j}z}$ ($|\alpha_j| < 1$ for $j = 1, 2, \ldots, n$). Then $[T_\varphi^*, T_\varphi] = T_\varphi^* T_\varphi - T_\varphi T_\varphi^* = I - T_\varphi T_\varphi^*$. By Theorem 2.11, $\text{rank}(I - T_\varphi T_\varphi^*) = \deg(\varphi) = n$.

3. Expansive and contractive operators

In this section, we study the m-expansive and m-contractive Toeplitz operators with trigonometric polynomial symbols.

It follows from definition that for $\varphi \in L^\infty(\mathbb{T})$, a Toeplitz operator T_φ is m-expansive if and only if

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \|T_\varphi^j k\|^2 \le 0 \quad \text{for all} \quad k \in H^2(\mathbb{T}) \tag{3.1}$$

and m-contractive if and only if

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \|T_\varphi^j k\|^2 \ge 0 \quad \text{for all} \quad k \in H^2(\mathbb{T}). \tag{3.2}$$

By definition of expansive operators and properties of Toeplitz operators, the following lemma is easily checked.

Lemma 3.1. *For $\varphi \in L^\infty(\mathbb{T})$,*

(i) *T_φ is expansive if and only if $\|\varphi\|_\infty \le 1$;*
(ii) *T_φ is contractive if and only if $\|\varphi\|_\infty \ge 1$.*

Proof. (i) Suppose that $\|\varphi\|_\infty \le 1$. Let $\varphi = f + \overline{g}$ with $f, g \in H^\infty$. Then

$$\|T_\varphi k\|^2 = \|P(\varphi k)\|^2 \le \|\varphi k\|^2 \le \|\varphi\|_\infty^2 \|k\|^2 \le \|k\|^2.$$

Hence T_φ is expansive. Conversely, suppose that T_φ is expansive, i.e., $\|T_\varphi k\|^2 \le \|k\|^2$ for all $k \in H^2$. Then $\|T_\varphi\| \le 1$. Since $\|T_\varphi\| = \|\varphi\|_\infty$, $\|\varphi\|_\infty \le 1$.

(ii) Since T_φ is contractive if and only if $B(T) \ge 0$, we get the result with the same method. This complete the proof. □

Theorem 3.2. *Suppose that T_φ is a Toeplitz operator with trigonometric polynomial symbol $\varphi = f + \overline{g}$ where $f, g \in H^\infty(\mathbb{T})$. If T_φ is 2-expansive, then $\|f\| = 1$ and $P(\overline{g}f) = 0$.*

Proof. Suppose that $\varphi(z) = f + \overline{g}$ where $f, g \in H^\infty(\mathbb{T})$. Put $k(z) = \sum_{i=0}^{\infty} c_i z^i$. Then we have

$$\|T_\varphi k\|^2 = \|P(fk + \overline{g}k)\|^2 = \|fk + P(\overline{g}k)\|^2$$

and

$$\|T_\varphi^2 k\|^2 = \|f^2 k + fP(\overline{g}k) + P(\overline{g}fk) + P(\overline{g}P(\overline{g}k))\|^2.$$

From the relation (3.1), T_φ is 2-expansive if and only if

$$\|f^2 k + fP(\overline{g}k) + P(\overline{g}fk) + P(\overline{g}P(\overline{g}k))\|^2 - 2\|fk + P(\overline{g}k)\|^2 + \|k\|^2 \le 0 \tag{3.3}$$

for all $k \in H^2(\mathbb{T})$. Put $k(z) = c$ for some nonzero constant c. Then from (3.3) we have

$$\|cf^2 + cP(\overline{g}f)\|^2 - 2\|cf\|^2 + |c|^2 \leq 0$$

or equivalently,

$$\|cf^2\|^2 + \|cP(\overline{g}f)\|^2 + 2\mathrm{Re}\langle cf^2, cP(\overline{g}f)\rangle - 2\|cf\|^2 + |c|^2 \leq 0.$$

Since $\mathrm{Re}\langle cf^2, cP(\overline{g}f)\rangle \leq |c|^2 \, \|f^2\|\|P(\overline{g}f)\|$, we have

$$|c|^2 \left\{(\|f\|^2 - 1)^2 + \|P(\overline{g}f)\|^2 + 2\|f^2\|\|P(\overline{g}f)\|\right\} \leq 0.$$

Hence if T_φ is 2-expansive, then $\|f\| = 1$ and $P(\overline{g}f) = 0$. □

In the following example, we show that the converse of Theorem 3.2 does not hold.

Example 3.3. Suppose that $\varphi(z) = f + \overline{g} = 1 + \overline{z}$. Then $\|f\| = 1$ and $P(\overline{g}f) = 0$. But for $k(z) = 1 + z$, a straightforward calculation shows that

$$\|T_\varphi^2 k\|^2 - 2\|T_\varphi k\|^2 + \|k\|^2 = 2 > 0.$$

Therefore T_φ is not 2-expansive. Hence the converse of Theorem 3.2 does not hold.

Corollary 3.4. *Suppose that $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$ with $a_N \neq 0$ where $\|\varphi\|_\infty \leq 1$. Then T_φ is expansive but not 2-expansive.*

Proof. From Lemma 3.1, it is obvious that T_φ is expansive. Set $f(z) = \sum_{n=0}^{N} a_n z^n$ and $g(z) = \sum_{n=1}^{N} a_{-n} z^n$. Since $P(\overline{g}f) \neq 0$, T_φ is not 2-expansive from Theorem 3.2. □

Corollary 3.5. *Suppose that T_φ is hyponormal with polynomial symbols and 2-expansive if and only if $\varphi \in H^\infty(\mathbb{T})$ with $\|\varphi\| = 1$.*

Proof. For $\varphi = f + \overline{g}$, if T_φ is hyponormal, then deg $f \geq$ deg g. From Theorem 3.2, if T_φ 2-expansive, then $\|f\| = 1$ and $P(\overline{g}f) = 0$. Set $f(z) = \sum_{n=0}^{N} a_n z^n$ and $g(z) = \sum_{n=1}^{m} a_{-n} z^n$ with $N \geq m$ and $a_N a_{-m} \neq 0$. Since

$$P(\overline{g}f) = P\left(\sum_{n=0}^{N} a_n z^n \cdot \sum_{n=1}^{m} \overline{a_{-n} z^n}\right) = P\left(\sum_{i=0}^{N}\sum_{j=0}^{m} a_i \overline{a_j} z^i \overline{z}^j\right)$$

$P(\overline{g}f) = 0$ if and only if $a_N \overline{a_j} = 0$ for some nonzero a_j, which is a contradiction. Hence we conclude that $g = 0$, and so $\varphi = f$ with $\|\varphi\| = 1$. Conversely, suppose that $\varphi \in H^\infty$ with $\|\varphi\| = 1$. Then T_φ is hyponormal. And from Lemmas 2.1 and 2.2, T_φ is an 2-isometry and hence 2-expansive. This completes the proof. □

Next, we consider the m-expansive Toeplitz operators with analytic symbols.

Theorem 3.6. *Suppose that φ is analytic. Then*

(i) *If m is even, then T_φ is m-expansive if and only if T_φ is an isometry*;

(ii) *If m is odd, then T_φ is m-expansive if and only if $|\varphi| \le 1$. In particular, if $|\varphi| = 1$, then T_φ is an isometry.*

Proof. (i) Clearly T_φ is an m-isometry implies T_φ is m-expansive. If $\varphi(z)$ is an analytic, then from the relation (1.1), we have

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T_\varphi^{*j}T_\varphi^{j} = \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T_{\overline{\varphi}^j\varphi^j} = T_{\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\overline{\varphi}^j\varphi^j} = T_{(\overline{\varphi}\varphi-1)^m}.$$

Since T_φ is m-expansive, we have

$$\big\langle P((\overline{\varphi}\varphi-1)^m k), k\big\rangle \le 0 \tag{3.4}$$

for all $k \in H^2(\mathbb{T})$. Hence

$$\big\langle P((\overline{\varphi}\varphi-1)^m k), k\big\rangle = \big\langle (\overline{\varphi}\varphi-1)^m k, k\big\rangle = \|(|\varphi|^2-1)^{\frac{m}{2}}k\|^2 \le 0.$$

Hence we have that $\|(|\varphi|^2-1)^{\frac{m}{2}}k\|^2 \le 0$ if and only if $|\varphi| = 1$. Moreover, by Lemma 2.1, T_φ is an isometry.

(ii) From the inequality (3.4), T_φ is m-expansive if and only if

$$\big\langle P((\overline{\varphi}\varphi-1)^m k), k\big\rangle = \big\langle (|\varphi|^2-1)^m k, k\big\rangle \le 0.$$

Applying the Hölder-McCarthy inequality introduced in [15], we have

$$\big\langle (|\varphi|^2-1)k, k\big\rangle^m \le \|k\|^{2(m-1)}\big\langle (|\varphi|^2-1)^m k, k\big\rangle.$$

Hence T_φ is m-expansive if and only if $|\varphi| \le 1$. This complete the proof. □

Corollary 3.7. *Every subnormal and m-expansive Toeplitz operator is an isometry where m is a positive even number.*

Corollary 3.8. *Suppose that φ is analytic. If T_φ is m-expansive, then $\|T_\varphi\| \le 1$.*

Proof. Since $\|T_\varphi\| = \|\varphi\|_\infty$, the proof follows from Theorem 3.6. □

Example 3.9. Suppose that $\varphi(z) = \frac{c(z-\alpha)}{1-\overline{\alpha}z}$ where $\alpha, c \in \mathbb{C}$. By Theorem 3.6, for every positive even number m, T_φ is m-expansive, and T_φ is an m-isometry if and only if $|c| = 1$. But from Lemma 3.1 and Theorem 3.6, T_φ is expansive (contractive) if and only if $|c| \le 1$ ($|c| \ge 1$), respectively.

Now, we study m-contractive Toeplitz operators with trigonometric polynomial symbols. The following result is necessary and sufficient conditions for the 2-contractive Toeplitz operators with coanalytic inner symbol.

Proposition 3.10. *Suppose that $\varphi(z) = a_{-k}\overline{z}^k$. Then T_φ is 2-contractive if and only if $|a_{-k}| \le \frac{1}{\sqrt{2}}$.*

Proof. From the relation (3.2) for $m = 2$, put $k(z) = \sum_{i=0}^{\infty} c_i z^i$ ($c_i \in \mathbb{C}$ ($i = 0, 1, 2, \ldots$)). Then

$$\|T_\varphi k\|^2 = \Big\| a_{-k} \sum_{i=k}^{\infty} c_i z^{i+k} \Big\|^2 = |a_{-k}|^2 \sum_{i=k}^{\infty} |c_i|^2$$

and

$$\|T_\varphi^2 k\|^2 = \Big\| P\Big(a_{-k}^2 \sum_{i=2k}^{\infty} c_i z^{i+2k} \Big) \Big\|^2 = |a_{-k}|^4 \sum_{i=2k}^{\infty} |c_i|^2.$$

Hence, T_φ is 2-contractive if and only if

$$\begin{aligned} &\|T_\varphi^2 k\|^2 - 2\|T_\varphi k\|^2 + \|k\|^2 \\ &= |a_{-k}|^4 \sum_{i=2k}^{\infty} |c_i|^2 - 2|a_{-k}|^2 \sum_{i=k}^{\infty} |c_i|^2 + \sum_{i=0}^{\infty} |c_i|^2 \\ &= \sum_{i=2k}^{\infty} |c_i|^2 (|a_{-k}|^2 - 1)^2 + \sum_{i=k}^{2k} |c_i|^2 (-2|a_{-k}|^2 + 1) + \sum_{i=0}^{k} |c_i|^2 \geq 0 \end{aligned}$$

for all $c_i \in \mathbb{C}$ ($i = 0, 1, 2, \ldots$), or equivalently,

$$|a_{-k}| \leq \frac{1}{\sqrt{2}}.$$

This completes the proof. □

Corollary 3.11. *Suppose that $\varphi(z) = a_{-k}\overline{z}^k$. Then T_φ is never 2-expansive.*

Proof. We argue by contradiction. Suppose that T_φ is 2-expansive. Put $k(z) = c_0 (c_0 \neq 0)$. Then from the same arguments in proof of Proposition 3.10,

$$\|T_\varphi^2 k\|^2 - 2\|T_\varphi k\|^2 + \|k\|^2 = |c_0|^2 > 0.$$

Therefore we can conclude a contradiction. □

The following result is a consequence of 2-contractive Toeplitz operators with trigonometric polynomial symbols.

Proposition 3.12. *Suppose that $\varphi(z) = a_1 z + \overline{a_{-1} z}$ ($a_{-1} \neq 0$). If T_φ is 2-contractive, then*

$$(|a_1|^2 - 1)^2 + |a_{-1}|^2 (3|a_1|^2 - 2) \geq 0.$$

Proof. Put $k(z) = \sum_{i=0}^{\infty} c_i z^i$. Then from (3.2), we have

$$\begin{aligned} \|T_\varphi k\|^2 &= \Big\| a_1 \sum_{i=0}^{\infty} c_i z^{i+1} + \overline{a_{-1}} \sum_{i=1}^{\infty} c_i z^{i-1} \Big\|^2 \\ &= |a_1|^2 \sum_{i=0}^{\infty} |c_i|^2 + |a_{-1}|^2 \sum_{i=1}^{\infty} |c_i|^2 + 2\mathrm{Re}\Big\{ a_1 \overline{a_{-1}} \sum_{i=1}^{\infty} c_i \overline{c_{i+2}} \Big\}, \end{aligned}$$

and

$$\begin{aligned}\|T_\varphi^2 k\|^2 &= \Big\|P\Big((a_1 z + a_{-1}\overline{z})\Big(a_1\sum_{i=0}^{\infty} c_i z^{i+1} + \overline{a_{-1}}\sum_{i=1}^{\infty} c_i z^{i-1}\Big)\Big)\Big\|^2\\ &= \Big\|a_1^2\sum_{i=0}^{\infty} c_i z^{i+2} + a_1\overline{a_{-1}}\sum_{i=1}^{\infty} c_i z^{i} + a_1\overline{a_{-1}}\sum_{i=0}^{\infty} c_i z^{i} + \overline{a_{-1}}^2\sum_{i=2}^{\infty} c_i z^{i-2}\Big\|^2\\ &= |a_1|^4\sum_{i=0}^{\infty}|c_i|^2 + |a_1a_{-1}|^2\sum_{i=1}^{\infty}|c_i|^2 + |a_1a_{-1}|^2\sum_{i=0}^{\infty}|c_i|^2\\ &\quad + |a_{-1}|^4\sum_{i=2}^{\infty}|c_i|^2 + 4\mathrm{Re}\Big\{|a_1|^2 a_1 a_{-1}\sum_{i=0}^{\infty} c_i\overline{c_{i+2}}\Big\}\\ &\quad + 2\mathrm{Re}\Big\{a_1^2 a_{-1}^2\sum_{i=0}^{\infty} c_i\overline{c_{i+4}}\Big\} + |a_1a_{-1}|^2\sum_{i=1}^{\infty}|c_i|^2\\ &\quad + 2\mathrm{Re}\Big\{a_1|a_{-1}|^2 a_{-1}\sum_{i=1}^{\infty} c_i\overline{c_{i+2}}\Big\} + 2\mathrm{Re}\Big\{a_1|a_{-1}|^2 a_{-1}\sum_{i=0}^{\infty} c_i\overline{c_{i+2}}\Big\}.\end{aligned}$$

Put $c_1 = 1$ and $c_i = 0$ for all $i \geq 0,\ i \neq 1$, then

$$\begin{aligned}\|T_\varphi^2 k\|^2 - 2\|T_\varphi k\|^2 + \|k\|^2 &= |a_1|^4 - 2|a_1|^2 + 3|a_1a_{-1}|^2 - 2|a_{-1}|^2 + 1\\ &= (|a_1|^2-1)^2 + |a_{-1}|^2(3|a_1|^2 - 2) \geq 0.\end{aligned}$$

This completes the proof. □

Example 3.13. Suppose that $\varphi(z) = \frac{1}{2}z + \overline{z}$. Then from Proposition 3.12, T_φ is not 2-contractive but by Lemma 3.1, it is contractive.

Next, we consider the m-contractive Toeplitz operators with analytic symbols.

Theorem 3.14. *If φ is analytic, then T_φ is m-contractive where m is a positive even number.*

Proof. For a positive even number m, if $\varphi(z)$ is an analytic, then from (1.1), we have

$$\begin{aligned}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T_\varphi^{*j}T_\varphi^j &= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T_{\overline{\varphi}^j\varphi^j}\\ &= T_{\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\overline{\varphi}^j\varphi^j}\\ &= T_{(\overline{\varphi}\varphi-1)^m}.\end{aligned}$$

Hence, for a positive even number m, T_φ is m-contractive if and only if

$$\big\langle P((\overline{\varphi}\varphi - 1)^m k), k\big\rangle \geq 0$$

for all $k \in H^2(\mathbb{T})$. Hence

$$\begin{aligned}\left\langle P((\overline{\varphi}\varphi - 1)^m k), k\right\rangle &= \left\langle (\overline{\varphi}\varphi - 1)^m k, k\right\rangle \\ &= \left\langle (\overline{\varphi}\varphi - 1)^{\frac{m}{2}} k, (\overline{\varphi}\varphi - 1)^{\frac{m}{2}} k\right\rangle \\ &= \|(\overline{\varphi}\varphi - 1)^{\frac{m}{2}} k\|^2 \geq 0.\end{aligned}$$

This complete the proof. □

Remark 3.15. It is easy to confirm that Toeplitz operator T_φ with analytic symbols of the form $\varphi(z) = \sum_{n=0}^{N} a_n z^n$ with $\sum_{n=0}^{N} |a_n|^2 < 1$ is not contractive. Indeed, from Lemma 3.1, $a_{-k} = 0$ for all $k = 1, 2, \ldots, m$, T_φ is contractive if and only if $\sum_{n=0}^{N} |a_n|^2 \geq 1$. So we conclude that there exist a Toeplitz operator T_φ with analytic symbols that is not contractive.

Example 3.16. Consider the trigonometric polynomial

$$\varphi(z) = z + z^2.$$

From Lemma 3.1, T_φ is not contractive, but by Theorem 3.14, it is 2-contractive.

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