

## A NOTE ON PROOF OF GORDON'S CONJECTURE

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ABSTRACT. In this paper, we show a proof of Gordon's Conjecture by using Qiu's labels and two new labels.

### 1. Introduction

All 3-manifolds in this paper are assumed to be compact and orientable. All surfaces in 3-manifolds are assumed to be orientable.

Let  $M$  be a 3-manifold. If there is a closed surface  $S$  which cuts  $M$  into two compression bodies  $V$  and  $W$  with  $S = \partial_+ W = \partial_+ V$ , then we say  $M$  has a Heegaard splitting, denoted by  $M = V \cup_S W$ ; and  $S$  is called a Heegaard surface of  $M$ . If there are essential disks  $B \subset V$  and  $D \subset W$  such that  $\partial B = \partial D$  (resp.  $\partial B \cap \partial D = \emptyset$ ), then  $M = V \cup_S W$  is said to be reducible (resp. weakly reducible); otherwise,  $M = V \cup_S W$  is said to be irreducible (resp. strongly irreducible). If there are essential disks  $B \subset V$  and  $D \subset W$  such that  $|B \cap D| = 1$ , then  $M = V \cup_S W$  is said to be stabilized; otherwise,  $M = V \cup_S W$  is said to be unstabilized.

Let  $M$  be a 3-manifold,  $F$  be a connected closed surface in  $M$ , which cuts  $M$  into two 3-manifolds  $M_1$  and  $M_2$ . Suppose that  $M_i = V_i \cup_{S_i} W_i$  is a Heegaard splitting of  $M_i$  ( $i = 1, 2$ ). Then,  $M$  has a natural Heegaard splitting  $M = V \cup_S W$  called the amalgamation of  $M_1 = V_1 \cup_{S_1} W_1$  and  $M_2 = V_2 \cup_{S_2} W_2$  along  $F$ , see [8]. From this construction, we have  $g(M) \leq g(M_1) + g(M_2) - g(F)$ . So, there is an interesting question as follows:

**Question 1.1.** *When  $M = V \cup_S W$  is unstabilized?*

If  $g(F) = 0$ , then it is the Gordon's Conjecture ([2]). Bachman ([1]), Qiu ([6]), Qiu and Scharlemann ([7]) give an affirmative answer about this question. But for generally case, it is not true. There are two counterexamples, such that  $g(M) < g(M_1) + g(M_2) - g(F)$ , see [4] and [9]. In [3], Kabayashi and Qiu

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proved the uniqueness of minimal Heegaard splitting  $M = V \cup_S W$  by using sufficiently complicated manifolds, i.e., the amalgamation of  $M_1 = V_1 \cup_{S_1} W_1$  and  $M_2 = V_2 \cup_{S_2} W_2$  along  $F$ . In [5], Lackenby proved the uniqueness of minimal Heegaard splitting  $M = V \cup_S W$  by using sufficiently complicated map, i.e., the amalgamation of  $M_1 = V_1 \cup_{S_1} W_1$  and  $M_2 = V_2 \cup_{S_2} W_2$  along  $F$ .

If  $g(F) = 0$ , then  $S$  can be isotoped, such that  $F \cap S$  is an essential simple closed curve on  $S$ . Hence,  $M = V \cup_S W$  is the reducible Heegaard splitting and  $F$  is the reducing 2-sphere. So,  $F$  cuts  $V$  into  $V^1$  and  $V^2$  and cuts  $W$  into  $W'_1$  and  $W'_2$  such that  $M_1 = V^1 \cup W'_1$  and  $M_2 = V^2 \cup W'_2$ . Let  $W^i = W'_i \cup_{\partial F = \partial B_i^3} B_i^3$  ( $i = 1, 2$ ), where  $B_i^3$  is a 3-ball. Then,  $W^i$  is a compression body and  $M^i = V^i \cup_{S^i} W^i$  is a Heegaard splitting of  $M^i$  with  $S^i = \partial_+ V^i = \partial_+ W^i$ . So,  $M = V \cup_S W$  is said to be the connected sum of  $M^1 = V^1 \cup_{S^1} W^1$  and  $M^2 = V^2 \cup_{S^2} W^2$ . In this paper, we show a proof of Gordon's Conjecture by using Qiu's labels in [6] and two new labels as follows:

**Theorem 1.2.** *The connected sum  $M = V \cup_S W$  of  $M^1 = V^1 \cup_{S^1} W^1$  and  $M^2 = V^2 \cup_{S^2} W^2$  is stabilized if and only if one of  $M^1 = V^1 \cup_{S^1} W^1$  and  $M^2 = V^2 \cup_{S^2} W^2$  is stabilized.*

## 2. The proof of Theorem 1.2

*Proof.* If one of  $M^1 = V^1 \cup_{S^1} W^1$  and  $M^2 = V^2 \cup_{S^2} W^2$  is stabilized, then by the construction of Heegaard splitting of connected sum,  $M = V \cup_S W$  is stabilized. So, we only prove that if  $M = V \cup_S W$  is stabilized, then one of  $M^1 = V^1 \cup_{S^1} W^1$  and  $M^2 = V^2 \cup_{S^2} W^2$  is stabilized.

Since  $M = V \cup_S W$  is stabilized, there are two disks  $D_V \subset V$  and  $D_W \subset W$  such that  $|D_V \cap D_W| = 1$ . Let  $x = D_V \cap D_W$ ,  $F_V = F \cap V$  and  $F_W = F \cap W$ , where  $F$  is the reducing 2-sphere of  $M = V \cup_S W$ . Then  $F_V$  is an essential disk in  $V$  and  $F_W$  is an essential disk in  $W$ .

**Proposition 2.1.** *If either  $D_V \cap F_V = \emptyset$  or  $D_W \cap F_W = \emptyset$ , then one of  $M^1 = V^1 \cup_{S^1} W^1$  and  $M^2 = V^2 \cup_{S^2} W^2$  is stabilized.*

*Proof.* If  $D_V \cap F_V = \emptyset$ , then  $D_V$  is a properly embedded disk in  $V^1$  or  $V^2$ . We may assume that  $D_V$  lies in  $V^1$ . If  $D_W \cap F_W = \emptyset$ , since  $|D_V \cap D_W| = 1$ ,  $D_W$  is a properly embedded disk in  $W^1$ . Hence,  $M^1 = V^1 \cup_{S^1} W^1$  is stabilized and Proposition 2.1 holds. So, we may assume that  $D_W \cap F_W \neq \emptyset$  and  $|D_W \cap F_W|$  is minimal. Hence, each component of  $D_W \cap F_W$  is an arc on both  $D_W$  and  $F_W$ . Let  $S'_i = S^i \cap S$  ( $i = 1, 2$ ). Since  $|D_W \cap F_W|$  is minimal, each component of  $\partial D_W \cap S'_i$  is an essential arc on  $S'_i$ . Let  $D_1^W$  be a subdisk of  $D_W$ , which is cut by  $F_W$ , such that  $|D_V \cap D_1^W| = 1$ . Since  $D_V \cap F_V = \emptyset$ , we can push all components of  $\partial D_1^W \cap F_W$  into  $S'_1$ , after isotopy, still denote it by  $D_1^W$ . Then,  $D_1^W$  is a properly embedded disk in  $W^1$  and  $|D_V \cap D_1^W| = 1$ . So,  $M^1 = V^1 \cup_{S^1} W^1$  is stabilized and Proposition 2.1 holds.

If  $D_W \cap F_W = \emptyset$ , then  $D_W$  is a properly embedded disk in  $W^1$  or  $W^2$ . We may assume that  $D_W$  lies in  $W^1$ . If  $D_V \cap F_V = \emptyset$ , then by the same argument

as above,  $M^1 = V^1 \cup_{S^1} W^1$  is stabilized and Proposition 2.1 holds. So, we may assume that  $D_V \cap F_V \neq \emptyset$  and  $|D_V \cap F_V|$  is minimal. Hence, each component of  $D_V \cap F_V$  is an arc on both  $D_V$  and  $F_V$ , and each component of  $\partial D_V \cap S'_i$  is an essential arc on  $S'_i$  ( $i = 1, 2$ ). Let  $D_1^V$  be a subdisk of  $D_V$ , which is cut by  $F_V$ , such that  $|D_1^V \cap D_W| = 1$ . Then,  $D_1^V$  is a properly embedded disk in  $V^1$ . Hence,  $M^1 = V^1 \cup_{S^1} W^1$  is stabilized and Proposition 2.1 holds.  $\square$

By Proposition 2.1, we may assume that  $D_V \cap F_V \neq \emptyset$ ,  $D_W \cap F_W \neq \emptyset$ , both  $|D_V \cap F_V|$  and  $|D_W \cap F_W|$  are minimal. Hence, each component of  $D_V \cap F_V$  is an arc on both  $D_V$  and  $F_V$ , each component of  $D_W \cap F_W$  is an arc on both  $D_W$  and  $F_W$ , each component of  $\partial D_V \cap S'_i$  is essential on  $S'_i$ , and each component of  $\partial D_W \cap S'_i$  is essential on  $S'_i$  ( $i = 1, 2$ ). After isotopy, we may assume that  $x = D_V \cap D_W$  lies in  $S'_1$ . Let  $|D_V \cap F_V| = p$  and  $|D_W \cap F_W| = n$ . Now we show Qiu's labels (see [6]) and two new labels for each arc of  $D_V \cap F_V$  on  $F_V$  and  $D_W \cap F_W$  on  $F_W$  as follows:

For each component  $e$  of  $D_V \cap F_V$  on  $F_V$ ,  $e$  cuts  $D_V$  into two disks  $V_e'$  and  $V_e''$ , such that  $x$  lies in  $\partial V_e'$ . Let  $V_e$  be a subdisk of  $D_V$ , which is cut by  $F_V$ , such that  $\partial V_e$  contains  $e$  and  $V_e \subset V_e''$ , see Figure 3 in [6]. Then,  $V_e$  is a properly embedded disk in  $V^1$  or  $V^2$ . If  $V_e$  lies in  $V^1$ , then we label  $e$  with “+”; if  $V_e$  lies in  $V^2$ , then we label  $e$  with “-”. Similarly, for each component  $e$  of  $D_W \cap F_W$  on  $F_W$ ,  $e$  cuts  $D_W$  into two disks  $W_e'$  and  $W_e''$ , such that  $x$  lies in  $\partial W_e'$ . Let  $W_e^1$  be a subdisk of  $D_W$ , which is cut by  $F_W$ , such that  $\partial W_e^1$  contains  $e$  and  $W_e^1 \subset W_e''$ . Then,  $W_e^1$  is a properly embedded disk in  $W_1^1$  or  $W_2^1$ . If  $W_e^1$  lies in  $W_1^1$ , then we label  $e$  with “+”; if  $W_e^1$  lies in  $W_2^1$ , then we label  $e$  with “-”.

Since  $|D_V \cap F_V| = p$  and  $|D_W \cap F_W| = n$ , we label the arcs of  $D_V \cap F_V$  on  $F_V$  with  $\{v_1, \dots, v_p\}$  and label the arcs of  $D_W \cap F_W$  on  $F_W$  with  $\{w'_1, \dots, w'_n\}$ , such that if  $V_{v_i}'' \subsetneq V_{v_k}''$  and  $W_{w'_j}'' \subsetneq W_{w'_l}''$ , then  $i < k$  and  $j < l$ . So, each subdisk of  $D_V$  which is cut by  $F_V$  and does not contain  $x$  is denoted by  $V_{v_i}$  ( $1 \leq i \leq p$ ) and each subdisk of  $D_W$  which is cut by  $F_W$  and does not contain  $x$  is denoted by  $W_{w'_j}^1$  ( $1 \leq j \leq n$ ). For convenience, we denote  $V_{v_i}$  by  $V_i$  and denote  $W_{w'_j}^1$  by  $W_j^1$ . Let  $V_x$  be the subdisk of  $D_V$  which is cut by  $F_V$ , such that  $\partial V_x$  contains  $x$ ,  $W_x^1$  be the subdisk of  $D_W$  which is cut by  $F_W$ , such that  $\partial W_x^1$  contains  $x$ .

*Remark 2.2.* Since  $x$  lies in  $S'_1$ , each subdisk of  $D_V$  which is cut by  $F_V$  and lies in  $V^1$ , is either  $V_i$ , where the label  $v_i$  is “+”, or  $V_x$ ; each subdisk of  $D_V$  which is cut by  $F_V$  and lies in  $V^2$ , is  $V_i$ , where the label  $v_i$  is “-”; each subdisk of  $D_W$  which is cut by  $F_W$  and lies in  $W_1^1$ , is either  $W_j^1$ , where the label  $w'_j$  is “+”, or  $W_x^1$ ; and each subdisk of  $D_W$  which is cut by  $F_W$  and lies in  $W_2^1$ , is  $W_j^1$ , where the label  $w'_j$  is “-”.

For each component  $w'_j$  ( $1 \leq j \leq n$ ) of  $D_W \cap F_W$  on  $F_W$ ,  $w_j$  is said to be the dual arc of  $w'_j$  on  $F_V$ , if  $\partial w_j = \partial w'_j$ . After isotopy, we may assume that

for each component  $v_i$  ( $1 \leq i \leq p$ ) of  $D_V \cap F_V$  on  $F_V$ ,  $|w_j \cap v_i| \leq 1$ . We may assume that  $w_j$  and  $w'_j$  have the same labels. For each subdisk  $W_j^1$  ( $1 \leq j \leq n$  or  $j = x$ ) of  $D_W$  which is cut by  $F_W$ , we can push each arc  $w'_k$  of  $\partial W_j^1 \cap F_W$  on  $F_W$  into  $F_V$ , such that  $w'_k$  is replaced by  $w_k$  on  $F_V$ . After isotopy, we denote it by  $W_j$ . Then,  $W_j$  is a properly embedded disk in  $W^1$  or  $W^2$ .

So, for each arc  $v_i$  ( $1 \leq i \leq p$ ) of  $D_V \cap F_V$  on  $F_V$  and each dual arc  $w_j$  ( $1 \leq j \leq n$ ) of  $D_W \cap F_W$  on  $F_V$ ,  $|v_i \cap w_j| \leq 1$ . Let  $I(v_i) = \{r \mid v_r \subset \partial V_i \text{ and } v_r \neq v_i\}$ ,  $I(w_j) = \{r \mid w_r \subset \partial W_j \text{ and } w_r \neq w_j\}$ ,  $I(v) = \{r \mid v_r \subset \partial V_x\}$  and  $I(w) = \{r \mid w_r \subset \partial W_x\}$ . Then, there are some properties for  $I(v_i)$ ,  $I(w_j)$ ,  $I(v)$ ,  $I(w)$ ,  $V_i$ ,  $V_x$ ,  $W_j$  and  $W_x$  as follows:

**Proposition 2.3** ([6]). (1) *If  $r \in I(v_i)$ , then  $r < i$ ;*

(2) *if  $r \in I(w_j)$ , then  $r < j$ ;*

(3) *the label  $v_i$  is “+” if and only if the label  $v_r$  is “-” for each  $r \in I(v_i)$ ;*

(4) *the label  $w_j$  is “+” if and only if the label  $w_r$  is “-” for each  $r \in I(w_j)$ ;*

(5) *if  $r \in I(v)$ , then the label  $v_r$  is “-”;*

(6) *if  $r \in I(w)$ , then the label  $w_r$  is “-”;*

(7)  *$p \in I(v)$ ,  $n \in I(w)$ ;*

(8) *there are four sets of pairwise disjoint properly embedded disks  $\{V_i \mid 1 \leq i \leq p \text{ and the label } v_i \text{ is “+”}\} \cup \{V_x\}$  in  $V^1$ ,  $\{V_i \mid 1 \leq i \leq p \text{ and the label } v_i \text{ is “-”}\}$  in  $V^2$ ,  $\{W_j \mid 1 \leq j \leq n \text{ and the label } w_j \text{ is “+”}\} \cup \{W_x\}$  in  $W^1$ , and  $\{W_j \mid 1 \leq j \leq n \text{ and the label } w_j \text{ is “-”}\}$  in  $W^2$ , satisfying the following conditions:*

(i)  $V_i \cap F_V = v_i \cup_{r \in I(v_i)} v_r$ ,  $W_j \cap F_V = w_j \cup_{r \in I(w_j)} w_r$ ,  $V_x \cap F_V = \cup_{r \in I(v)} v_r$ ,  $W_x \cap F_V = \cup_{r \in I(w)} w_r$ ;

(ii) *if  $V_i$  lies in  $V^1$  and  $W_j$  lies in  $W^1$ , then  $V_i \cap W_j = V_i \cap W_j \cap F_V$ ,  $V_i \cap W_x = V_i \cap W_x \cap F_V$ ,  $V_x \cap W_j = V_x \cap W_j \cap F_V$ , and  $V_x \cap W_x = \{x\} \cup (V_x \cap W_x \cap F_V)$ ;*

(iii) *if  $V_i$  lies in  $V^2$  and  $W_j$  lies in  $W^2$ , then  $V_i \cap W_j = V_i \cap W_j \cap F_V$ .*

Since  $F_V$  cuts  $V$  into  $V^1$  and  $V^2$ , let  $F_V^k$  ( $k = 1, 2$ ) be a copy of  $F_V$ , such that  $F_V^k$  lies in  $S^k$ ,  $v_i^k$  be a copy of  $v_i$  on  $F_V^k$  and  $w_j^k$  be a copy of  $w_j$  on  $F_V^k$  ( $1 \leq i \leq p; 1 \leq j \leq n$ ). We may assume that  $v_i^k$  and  $v_i$  have the same label, and  $w_j^k$  and  $w_j$  have the same label. For convenience,  $v_i^1 = v_i^2$  means that both  $v_i^1$  and  $v_i^2$  are the copies of  $v_i$ , and  $w_j^1 = w_j^2$  means that both  $w_j^1$  and  $w_j^2$  are the copies of  $w_j$ .

**Outline of the proof of Theorem 2.** By using Qiu’s labels and two new labels, we band sum disks of  $\{V_i \mid 1 \leq i \leq p \text{ and the label } v_i \text{ is “+”}\} \cup \{V_x\}$  in  $V^1$  along some arcs obtained from  $\{w_1^1, w_2^1, \dots, w_n^1\}$  on  $S^1$ , band sum disks of  $\{W_j \mid 1 \leq j \leq n \text{ and the label } w_j \text{ is “+”}\} \cup \{W_x\}$  in  $W^1$  along some arcs obtained from  $\{v_1^1, v_2^1, \dots, v_p^1\}$  on  $S^1$ , band sum disks of  $\{V_i \mid 1 \leq i \leq p \text{ and the label } v_i \text{ is “-”}\}$  in  $V^2$  along some arcs obtained from  $\{w_1^2, w_2^2, \dots, w_n^2\}$  on  $S^2$ , and band sum disks of  $\{W_j \mid 1 \leq j \leq n \text{ and the label } w_j \text{ is “-”}\}$  in  $W^2$  along some arcs obtained from  $\{v_1^2, v_2^2, \dots, v_p^2\}$  on  $S^2$ . Finally, either there are two disks  $D_{V^1} \subset V^1$  and  $D_{W^1} \subset W^1$  with  $|D_{V^1} \cap D_{W^1}| = 1$  or there are two disks

$D_{V^2} \subset V^2$  and  $D_{W^2} \subset W^2$  with  $|D_{V^2} \cap D_{W^2}| = 1$ . So, one of  $M^1 = V^1 \cup_{S^1} W^1$  and  $M^2 = V^2 \cup_{S^2} W^2$  is stabilized.

**Proposition 2.4.** *Either there are two disks  $D_{V^1} \subset V^1$  and  $D_{W^1} \subset W^1$  with  $|D_{V^1} \cap D_{W^1}| = 1$ , where  $D_{V^1}$  is obtained by banding sum disks of  $\{V_i \mid 1 \leq i \leq p\}$  and the label  $v_i$  is “+”  $\cup \{V_x\}$  in  $V^1$  along some arcs obtained from  $\{w_1^1, w_2^1, \dots, w_n^1\}$  on  $S^1$ , and  $D_{W^1}$  is obtained by banding sum disks of  $\{W_j \mid 1 \leq j \leq n\}$  and the label  $w_j$  is “+”  $\cup \{W_x\}$  in  $W^1$  along some arcs obtained from  $\{v_1^1, v_2^1, \dots, v_p^1\}$  on  $S^1$ , or there are two disks  $D_{V^2} \subset V^2$  and  $D_{W^2} \subset W^2$  with  $|D_{V^2} \cap D_{W^2}| = 1$ , where  $D_{V^2}$  is obtained by banding sum disks of  $\{V_i \mid 1 \leq i \leq p\}$  and the label  $v_i$  is “-” in  $V^2$  along some arcs obtained from  $\{w_1^2, w_2^2, \dots, w_n^2\}$  on  $S^2$ , and  $D_{W^2}$  is obtained by banding sum disks of  $\{W_j \mid 1 \leq j \leq n\}$  and the label  $w_j$  is “-” in  $W^2$  along some arcs obtained from  $\{v_1^2, v_2^2, \dots, v_p^2\}$  on  $S^2$ .*

*Proof.* We consider all arcs  $\{v_1, v_2, \dots, v_p\}$  of  $D_V \cap F_V$  on  $F_V$  in sequence. If we consider all dual arcs  $\{w_1, w_2, \dots, w_n\}$  of  $D_W \cap F_W$  on  $F_V$  in sequence, then the argument is the same. So, we may assume that  $p \leq n$ . First, we consider  $v_1^1$  on  $F_V^1$ . Let  $m^1$  be the minimal label among all arcs of  $\{w_j^1 \mid 1 \leq j \leq n\}$  on  $F_V^1$  with  $|w_{m^1}^1 \cap v_1^1| = 1$ . If  $m^1 = \emptyset$ , then for each arc  $w_j^1$  ( $1 \leq j \leq n$ ),  $w_j^1 \cap v_1^1 = \emptyset$ . If  $m^1 \neq \emptyset$ , then  $|v_1^1 \cap w_{m^1}^1| = 1$  ( $1 \leq m^1 \leq n$ ). Since  $v_i^1 = v_i^2$  ( $1 \leq i \leq p$ ) and  $w_j^1 = w_j^2$  ( $1 \leq j \leq n$ ),  $|v_i^1 \cap w_j^1| = |v_i^2 \cap w_j^2|$ . So,  $m^1$  is the minimal label among all arcs of  $\{w_j^2 \mid 1 \leq j \leq n\}$  on  $F_V^2$  with  $|w_{m^1}^2 \cap v_1^2| = 1$ . We may assume that the label  $v_1$  on  $F_V$  is “+”. If the label  $v_1$  on  $F_V$  is “-”, then the argument is the same.

If  $m^1 = \emptyset$ , then for each arc  $w_j^k$  ( $1 \leq j \leq n; k = 1, 2$ ),  $w_j^k \cap v_1^k = \emptyset$ . Since the label  $v_1$  on  $F_V$  is “+”, the label  $v_1^k$  ( $k = 1, 2$ ) on  $F_V^k$  is “+”. We label  $v_1^1$  on  $F_V^1$  with “ $\times$ ” and label  $v_1^2$  on  $F_V^2$  with “ $\circ$ ”. The label “ $\times$ ” on  $v_1^1$  means that we delete the arc  $v_1^1$  on  $F_V^1$ , and the label “ $\circ$ ” on  $v_1^2$  means that we retain the arc  $v_1^2$  on  $F_V^2$ . For each arc  $v_i^2$  ( $2 \leq i \leq p$ ) and  $w_j^2$  ( $1 \leq j \leq n$ ), since  $v_i^2 \cap v_1^2 = \emptyset$  and  $w_j^2 \cap v_1^2 = \emptyset$ , there is no influence on  $v_1^2$  when we consider  $v_i^2$  and  $w_j^2$ . Hence, the label “ $\circ$ ” on  $v_1^2$  means that we retain the arc  $v_1^2$  on  $F_V^2$ . For convenience, for each arc  $v_i^k$  ( $2 \leq i \leq p$ ) and  $w_j^k$  ( $1 \leq j \leq n$ ) on  $S^k$  ( $k = 1, 2$ ), we denote them by  $v_{i_1}^k$  and  $w_{j_1}^k$ . We may assume that  $v_{i_1}^k$  and  $v_i^k$  have the same label, and  $w_{j_1}^k$  and  $w_j^k$  have the same label. For each disk  $V_i$  ( $2 \leq i \leq p$  or  $i = x$ ) and  $W_j$  ( $1 \leq j \leq n$  or  $j = x$ ), we denote them by  $V_{i_1}$  and  $W_{j_1}$ . Since  $v_1^2$  is retained, we also denote it by  $v_{1_1}^2$ . But in the future banding sum process, we do not consider  $v_{1_1}^2$ .

If  $m^1 \neq \emptyset$ , then  $|v_1^1 \cap w_{m^1}^1| = |v_1^2 \cap w_{m^1}^2| = 1$ .

**Lemma 2.5.** *If the label  $w_{m^1}$  on  $F_V$  is “+”, then  $M^1 = V^1 \cup_{S^1} W^1$  is stabilized.*

*Proof.* Since the label  $v_1$  on  $F_V$  is “+”,  $V_1$  is a properly embedded disk in  $V^1$ . For each  $r \in I(v_1)$ , by (1) in Proposition 2.3,  $r < 1$ . So,  $I(v_1) = \emptyset$ . By (i) of (8) in Proposition 2.3,  $V_1 \cap F_V^1 = v_1^1 \cup_{r \in I(v_1)} v_r^1 = v_1^1$ . Since the label  $w_{m^1}$  on

$F_V$  is “+”,  $W_{m^1}$  is a properly embedded disk in  $W^1$ . For each  $r \in I(w_{m^1})$ , by (2) in Proposition 2.3,  $r < m^1$ . By the minimality of  $m^1$ ,  $w_r^1 \cap v_1^1 = \emptyset$ . By (8) in Proposition 2.3,  $|V_1 \cap W_{m^1}| = |V_1 \cap W_{m^1} \cap F_V^1| = |v_1^1 \cap (w_{m^1}^1 \cup_{r \in I(w_{m^1})} w_r^1)| = |v_1^1 \cap w_{m^1}^1| = 1$ . So,  $M^1 = V^1 \cup_{S^1} W^1$  is stabilized.  $\square$

By Lemma 2.5, Proposition 2.4 holds. So, we may assume that the label  $w_{m^1}$  on  $F_V$  is “-”. Then,  $W_{m^1}$  is a properly embedded disk in  $W^2$ . Now we label  $v_1^1$  and  $w_{m^1}^1$  on  $F_V^1$ , and label  $v_1^2$  and  $w_{m^1}^2$  on  $F_V^2$ , respectively:

**(I<sub>1</sub>) Label  $v_1^1$  and  $w_{m^1}^1$  on  $F_V^1$ .**

By (8) in Proposition 2.3,  $|V_1 \cap w_{m^1}^1| = |(V_1 \cap F_V^1) \cap w_{m^1}^1| = |v_1^1 \cap w_{m^1}^1| = 1$ . If there is a disk  $V_i$  of  $\{V_i \mid \text{the label } v_i \text{ is “+” and } 2 \leq i \leq p\} \cup \{V_x\}$  in  $V^1$  with  $\partial V_i \cap w_{m^1}^1 \neq \emptyset$ , then we band sum  $V_i$  and  $k$  copies of  $V_1$  along  $w_{m^1}^1$  in some order, where  $|\partial V_i \cap w_{m^1}^1| = k$ . After banding sum and isotopy, we obtain a properly embedded disk in  $V^1$  and denote it by  $V_{i_1}$ . So,  $V_{i_1} \cap V_1 = \emptyset$  and  $\partial V_{i_1} \cap (w_{m^1}^1 \cup v_1^1) = \emptyset$ . If there is a disk  $V_i$  of  $\{V_i \mid \text{the label } v_i \text{ is “+” and } 2 \leq i \leq p\} \cup \{V_x\}$  in  $V^1$  with  $\partial V_i \cap w_{m^1}^1 = \emptyset$ , then we do nothing and denote it by  $V_{i_1}$ . After isotopy, we obtain a collection of mutually disjoint disks  $\{V_{i_1} \mid \text{the label } v_i \text{ is “+” and } 2 \leq i \leq p\} \cup \{V_{x_1}\}$  in  $V^1$ . For each disk  $W_l$  of  $\{W_j \mid \text{the label } w_j \text{ is “+”, } 1 \leq j \leq n \text{ and } j \neq m^1\} \cup \{W_x\}$  in  $W^1$ , we do nothing and denote it by  $W_{l_1}$ . So, we obtain a collection of mutually disjoint disks  $\{W_{j_1} \mid \text{the label } w_j \text{ is “+”, } 1 \leq j \leq n \text{ and } j \neq m^1\} \cup \{W_{x_1}\}$  in  $W^1$ .

This procedure can be viewed as for each arc  $v_i^1$  ( $2 \leq i \leq p$ ), if  $|v_i^1 \cap w_{m^1}^1| = 1$ , then we band sum  $v_i^1$  and a copy  $\partial V_1^i$  of  $\partial V_1$  along  $w_{m^1}^1$ , where  $V_1^i$  is a copy of  $V_1$  and  $\partial V_1^i \cap F_V^1$  lies between  $v_1^1$  and  $v_i^1$ . After banding sum and isotopy, we obtain a new arc and denote it by  $v_{i_1}^1$ . Before banding sum, if there is an arc  $v_k^1$  ( $k \neq 1, i$ ) with  $|v_k^1 \cap w_{m^1}^1| = 1$ , such that  $v_k^1$  lies between  $v_1^1$  and  $v_i^1$ , then  $v_k^1$  lies between  $\partial V_1^i \cap F_V^1$  and  $v_i^1$ . Let  $\partial V_1^k$  be a copy of  $\partial V_1$ , where  $V_1^k$  is a copy of  $V_1$ , such that  $\partial V_1^k \cap F_V^1$  lies between  $\partial V_1^i \cap F_V^1$  and  $v_k^1$ . Then, we band sum  $v_k^1$  and  $\partial V_1^k$  along  $w_{m^1}^1$ . After banding sum and isotopy, we obtain a new arc and denote it by  $v_{k_1}^1$ , such that  $v_{k_1}^1 \cap v_{i_1}^1 = \emptyset$ , see Figure 1. If  $v_i^1 \cap w_{m^1}^1 = \emptyset$ , then we do nothing and denote it by  $v_{i_1}^1$ .

After banding sum and isotopy, we obtain a collection of mutually disjoint arcs  $\{v_{i_1}^1 \mid 2 \leq i \leq p\}$  on  $S^1$ . Also, for each arc  $w_j^1$  ( $1 \leq j \leq n$  and  $j \neq m^1$ ) before banding sum, we do nothing and denote it by  $w_{j_1}^1$  after banding sum. So, there is a collection of mutually disjoint arcs  $\{w_{j_1}^1 \mid 1 \leq j \leq n \text{ and } j \neq m^1\}$  on  $S^1$ . Hence,  $v_i^1$  and  $w_j^1$  ( $2 \leq i \leq p; 1 \leq j \leq n$  and  $j \neq m^1$ ) represent the arcs before banding sum,  $v_{i_1}^1$  and  $w_{j_1}^1$  represent the arcs after banding sum. We may assume that  $v_i^1$  and  $v_{i_1}^1$  ( $2 \leq i \leq p$ ) have the same label, and  $w_j^1$  and  $w_{j_1}^1$  ( $1 \leq j \leq n$  and  $j \neq m^1$ ) have the same label.

For each arc  $v_{i_1}^1$  ( $2 \leq i \leq p$ ) and  $w_{j_1}^1$  ( $1 \leq j \leq n$  and  $j \neq m^1$ ),  $v_{i_1}^1 \cap (v_1^1 \cup w_{m^1}^1) = \emptyset$ ,  $|v_{i_1}^1 \cap w_{j_1}^1| \leq 1$  and  $|v_{i_1}^1 \cap F_V^1| \leq 2$ . Since the label  $v_1^1$  on  $F_V^1$  is “+” and the label  $w_{m^1}^1$  on  $F_V^1$  is “-”, we label  $v_1^1$  on  $F_V^1$  with “ $\times$ ” and label  $w_{m^1}^1$  on

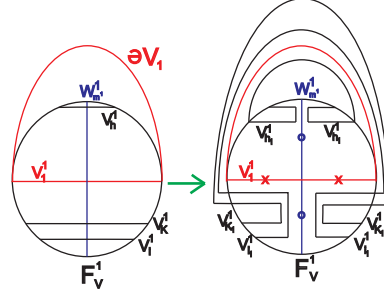


FIGURE 1. Band sum subdisk  $V_i$  and  $V_1$  in  $V^1$  along  $w_{m^1}^1$  ( $2 \leq i \leq p$ )

$F_V^1$  with “o”, see Figure 1. The label “x” on  $v_1^1$  means that we delete the arc  $v_1^1$  on  $F_V^1$ , the label “o” on  $w_{m^1}^1$  means that we retain the arc  $w_{m^1}^1$  on  $F_V^1$ . For each arc  $v_{i_1}^1$  ( $2 \leq i \leq p$ ) and  $w_{j_1}^1$  ( $1 \leq j \leq n$  and  $j \neq m^1$ ), since  $v_{i_1}^1 \cap w_{m^1}^1 = \emptyset$  and  $w_{j_1}^1 \cap w_{m^1}^1 = \emptyset$ , there is no influence on  $w_{m^1}^1$  when we consider  $v_{i_1}^1$  and  $w_{j_1}^1$ . Hence, the label “o” on  $w_{m^1}^1$  means that we retain the arc  $w_{m^1}^1$ . So, we also denote it by  $w_{m^1}^1$ , but in the future banding sum process, we do not need to consider it.

By (8) in Proposition 2.3 and the argument as above, we have:

**Lemma 2.6.** *There are two sets of pairwise disjoint properly embedded disks  $\{V_{i_1} \mid \text{the label } v_{i_1} \text{ is “+” and } 2 \leq i \leq p\} \cup \{V_{x_1}\}$  in  $V^1$ , and  $\{W_{j_1} \mid \text{the label } w_{j_1} \text{ is “+”}, 1 \leq j \leq n \text{ and } j \neq m^1\} \cup \{W_{x_1}\}$  in  $W^1$ , satisfying the following conditions:*

- (1)  $V_{i_1} \cap F_V^1 = (v_{i_1}^1 \cap F_V^1) \cup_{r \in I(v_{i_1})} (v_{r_1}^1 \cap F_V^1)$ ,  $W_{j_1} \cap F_V^1 = w_{j_1}^1 \cup_{r \in I(w_{j_1})} w_{r_1}^1$ ,  $V_{x_1} \cap F_V^1 = \cup_{r \in I(v)} (v_{r_1}^1 \cap F_V^1)$ ,  $W_{x_1} \cap F_V^1 = \cup_{r \in I(w)} w_{r_1}^1$ ;
- (2)  $V_{i_1} \cap W_{j_1} = V_{i_1} \cap W_{j_1} \cap F_V^1$ ,  $V_{i_1} \cap W_{x_1} = V_{i_1} \cap W_{x_1} \cap F_V^1$ ,  $V_{x_1} \cap W_{j_1} = V_{x_1} \cap W_{j_1} \cap F_V^1$ ,  $V_{x_1} \cap W_{x_1} = \{x\} \cup (V_{x_1} \cap W_{x_1} \cap F_V^1)$ .

*Remark 2.7.* For each  $2 \leq i \leq p$ ,  $1 \leq j \leq n$  and  $j \neq m^1$ , if  $|v_i^1 \cap w_{m^1}^1| = 1$  and  $|w_j^1 \cap (v_1^1 \cup v_i^1)| = 1$  before banding sum, then  $|v_{i_1}^1 \cap w_{j_1}^1| = 1$  after banding sum; if  $|v_i^1 \cap w_{m^1}^1| = 1$  and  $|w_j^1 \cap (v_1^1 \cup v_i^1)| = 0$  or  $2$  before banding sum, then  $v_{i_1}^1 \cap w_{j_1}^1 = \emptyset$  after banding sum and isotopy; if  $v_i^1 \cap w_{m^1}^1 = \emptyset$  before banding sum, then  $|v_{i_1}^1 \cap w_{j_1}^1| = |v_i^1 \cap w_j^1|$ , see Figure 2. After banding sum and isotopy,  $|v_{i_1}^1 \cap w_{j_1}^1| \leq 1$ .

**(I<sub>2</sub>) Label  $v_1^2$  and  $w_{m^1}^2$  on  $F_V^2$ .**

Since the label  $w_{m^1}$  on  $F_V$  is “-”,  $W_{m^1}$  is a properly embedded disk in  $W^2$ . For each  $r \in I(w_{m^1})$ , by (2) in Proposition 2.3,  $r < m^1$ . By the minimality of  $m^1$ ,  $v_1^2 \cap w_r^2 = \emptyset$ . By (8) in Proposition 2.3,  $|W_{m^1} \cap v_1^2| = |(W_{m^1} \cap F_V^2) \cap v_1^2| = |(w_{m^1}^2 \cup_{r \in I(w_{m^1})} w_r^2) \cap v_1^2| = |w_{m^1}^2 \cap v_1^2| = 1$ . If there is a disk  $W_l$  of  $\{W_j \mid \text{the label } w_j \text{ is “-”}; 1 \leq j \leq n \text{ and } j \neq m^1\}$  in  $W^2$  with  $\partial W_l \cap v_1^2 \neq \emptyset$ , then we band

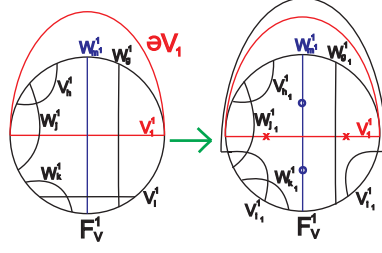


FIGURE 2.  $v_i^1 \cap w_j^1$  and  $v_{i_1}^1 \cap w_{j_1}^1$  ( $2 \leq i \leq p$ ;  $1 \leq j \leq n$  and  $j \neq m^1$ )

sum  $W_l$  and  $k$  copies of  $W_{m^1}$  along  $v_1^2$  in some order, where  $|\partial W_l \cap v_1^2| = k$ . After banding sum and isotopy, we obtain a properly embedded disk in  $W^2$  and denote it by  $W_{l_1}$ . So,  $W_{l_1} \cap W_{m^1} = \emptyset$  and  $\partial W_{l_1} \cap (w_{m^1}^2 \cup v_1^2) = \emptyset$ . If there is a disk  $W_l$  of  $\{W_j \mid \text{the label } w_j \text{ is “-”}; 1 \leq j \leq n \text{ and } j \neq m^1\}$  in  $W^2$  with  $\partial W_l \cap v_1^2 = \emptyset$ , then we do nothing and denote it by  $W_{l_1}$ . After isotopy, we obtain a collection of mutually disjoint disks  $\{W_{j_1} \mid \text{the label } w_j \text{ is “-”}; 1 \leq j \leq n \text{ and } j \neq m^1\}$  in  $W^2$ . For each disk  $V_i$  of  $\{V_i \mid \text{the label } v_i \text{ is “-” and } 2 \leq i \leq p\}$  in  $V^2$ , we do nothing and denote it by  $V_{i_1}$ . So, we obtain a collection of mutually disjoint disks  $\{V_{i_1} \mid \text{the label } v_i \text{ is “-” and } 2 \leq i \leq p\}$  in  $V^2$ .

This procedure can be viewed as for each arc  $w_j^2$  ( $1 \leq j \leq n$  and  $j \neq m^1$ ), if  $|w_j^2 \cap v_1^2| = 1$ , then we band sum  $w_j^2$  and a copy  $\partial W_{m^1}^j$  of  $\partial W_{m^1}$  along  $v_1^2$ , where  $W_{m^1}^j$  is a copy of  $W_{m^1}$  and one component of  $\partial W_{m^1}^j \cap F_V^2$  which is a copy of  $w_{m^1}^2$  lies between  $w_{m^1}^2$  and  $w_j^2$ . After banding sum and isotopy, we obtain a new arc and denote it by  $w_{j_1}^2$ . Before banding sum, if there is an arc  $w_k^2$  ( $k \neq m^1, j$ ) with  $|w_k^2 \cap v_1^2| = 1$ , such that  $w_k^2$  lies between  $w_{m^1}^2$  and  $w_j^2$ , then,  $w_k^2$  lies between one component of  $\partial W_{m^1}^j \cap F_V^2$  which is a copy of  $w_{m^1}^2$  and  $w_j^2$ . Let  $\partial W_{m^1}^k$  be a copy of  $\partial W_{m^1}$ , where  $W_{m^1}^k$  is a copy of  $W_{m^1}$ , such that one component of  $\partial W_{m^1}^k \cap F_V^2$  which is a copy of  $w_{m^1}^2$  lies between one component of  $\partial W_{m^1}^j \cap F_V^2$  which is a copy of  $w_{m^1}^2$  and  $w_k^2$ . Then, we band sum  $w_k^2$  and  $\partial W_{m^1}^k$  along  $v_1^2$ . After banding sum and isotopy, we obtain a new arc and denote it by  $w_{k_1}^2$ , such that  $w_{k_1}^2 \cap w_{j_1}^2 = \emptyset$ , see Figure 3. If  $w_j^2 \cap v_1^2 = \emptyset$ , then we do nothing and denote it by  $w_{j_1}^2$ .

After banding sum and isotopy, we obtain a collection of mutually disjoint arcs  $\{w_{j_1}^2 \mid 1 \leq j \leq n \text{ and } j \neq m^1\}$  on  $S^2$ . For each arc  $v_i^2$  ( $2 \leq i \leq p$ ) before banding sum, we do nothing and denote it by  $v_{i_1}^2$  after banding sum. Then, there is a collection of mutually disjoint arcs  $\{v_{i_1}^2 \mid 2 \leq i \leq p\}$  on  $S^2$ . Hence,  $v_i^2$  and  $w_j^2$  ( $2 \leq i \leq p$ ;  $1 \leq j \leq n$  and  $j \neq m^1$ ) represent the arcs before banding sum,  $v_{i_1}^2$  and  $w_{j_1}^2$  represent the arcs after banding sum. So, we may assume



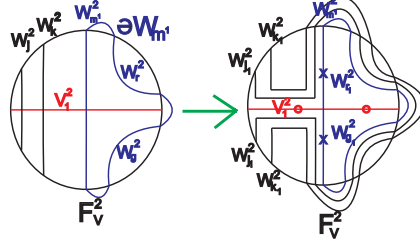


FIGURE 3. Band sum subdisk  $W_j$  and  $W_{m^1}$  in  $W^2$  along  $v_1^2$  ( $1 \leq j \leq n$  and  $j \neq m^1$ )

that  $v_i^2$  and  $v_{i_1}^2$  ( $2 \leq i \leq p$ ) have the same label, and  $w_j^2$  and  $w_{j_1}^2$  ( $1 \leq j \leq n$  and  $j \neq m^1$ ) have the same label.

For each arc  $w_{j_1}^2$  ( $1 \leq j \leq n$  and  $j \neq m^1$ ) on  $S^2$ ,  $w_{j_1}^2 \cap (v_1^2 \cup w_{m^1}^2) = \emptyset$ . Since the label  $w_{m^1}^2$  on  $F_V^2$  is “-” and the label  $v_1^2$  on  $F_V^2$  is “+”, we label  $w_{m^1}^2$  on  $F_V^2$  with “ $\times$ ” and label  $v_1^2$  on  $F_V^2$  with “ $\circ$ ”, see Figure 3. The label “ $\times$ ” on  $w_{m^1}^2$  means that we delete the arc  $w_{m^1}^2$  on  $F_V^2$ , the label “ $\circ$ ” on  $v_1^2$  means that we retain the arc  $v_1^2$  on  $F_V^2$ . For each arc  $v_{i_1}^2$  ( $2 \leq i \leq p$ ) and  $w_{j_1}^2$  ( $1 \leq j \leq n$  and  $j \neq m^1$ ), since  $v_{i_1}^2 \cap v_1^2 = \emptyset$  and  $w_{j_1}^2 \cap v_1^2 = \emptyset$ , there is no influence on  $v_1^2$  when we consider  $v_{i_1}^2$  and  $w_{j_1}^2$ . Hence, the label “ $\circ$ ” on  $v_1^2$  means that we retain the arc  $v_1^2$ . So, we also denote it by  $v_{1_1}^2$ , but in the future banding sum process, we do not need to consider it.

By (8) in Proposition 2.3 and the argument as above, we have:

**Lemma 2.8.** *There are two sets of pairwise disjoint properly embedded disks  $\{V_{i_1} \mid \text{the label } v_i \text{ is “-” and } 2 \leq i \leq p\}$  in  $V^2$ , and  $\{W_{j_1} \mid \text{the label } w_j \text{ is “-”}; 1 \leq j \leq n \text{ and } j \neq m^1\}$  in  $W^2$ , satisfying the following conditions:*

- (1)  $V_{i_1} \cap F_V^2 = v_{i_1}^2 \cup_{r \in I(v_i)} v_{r_1}^2$ ,  $W_{j_1} \cap F_V^2 = (w_{j_1}^2 \cap F_V^2) \cup_{r \in I(w_j)} (w_{r_1}^2 \cap F_V^2)$ ;
- (2)  $V_{i_1} \cap W_{j_1} = V_{i_1} \cap W_{j_1} \cap F_V^2$ .

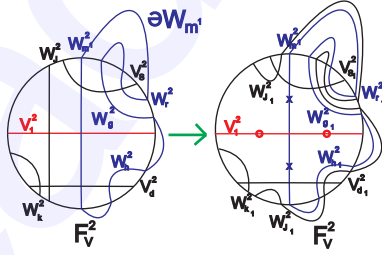


FIGURE 4.  $v_i^2 \cap w_j^2$  and  $v_{i_1}^2 \cap w_{j_1}^2$  ( $2 \leq i \leq p; 1 \leq j \leq n$  and  $j \neq m^1$ )

*Remark 2.9.* By the argument as above, for each arc  $v_{i_1}^2$  ( $2 \leq i \leq p$ ) and  $w_{j_1}^2$  ( $1 \leq j \leq n$  and  $j \neq m^1$ ) on  $S^2$ , if  $w_j^2 \cap v_1^2 = \emptyset$ , then  $w_{j_1}^2$  lies in  $F_V^2$  and  $|w_{j_1}^2 \cap v_{i_1}^2| \leq 1$ ; if  $|w_j^2 \cap v_1^2| = 1$  and  $v_i^2 \cap (\cup_{r \in I(w_{m^1})} w_r^2) = \emptyset$ , then  $|w_{j_1}^2 \cap v_{i_1}^2| \leq 1$ ; if  $|w_j^2 \cap v_1^2| = 1$  and  $v_i^2 \cap (\cup_{r \in I(w_{m^1})} w_r^2) \neq \emptyset$ , then  $|w_{j_1}^2 \cap v_{i_1}^2| \geq 1$ . Particularly, if  $|w_{j_1}^2 \cap v_{i_1}^2| \geq 2$ , then  $j$  is not the minimal label among all arcs of  $\{w_l^2 \mid 1 \leq l \leq n \text{ and } l \neq m^1\}$  on  $S^2$  with  $w_{j_1}^2 \cap v_{i_1}^2 \neq \emptyset$ . Specifically, if  $w_j^2 \cap v_1^2 = \emptyset$ , then  $|w_{j_1}^2 \cap v_{i_1}^2| = |w_j^2 \cap v_i^2|$ ; if  $|w_j^2 \cap v_1^2| = 1$ ,  $|v_i^2 \cap (w_j^2 \cup w_{m^1}^2)| = 1$  and  $|v_i^2 \cap (\cup_{r \in I(w_{m^1})} w_r^2)| = k$ , then  $|w_{j_1}^2 \cap v_{i_1}^2| = k + 1$ ; if  $|w_j^2 \cap v_1^2| = 1$ ,  $|v_i^2 \cap (w_j^2 \cup w_{m^1}^2)| = 0$  or  $2$  and  $|v_i^2 \cap (\cup_{r \in I(w_{m^1})} w_r^2)| = k$ , then  $|w_{j_1}^2 \cap v_{i_1}^2| = k$  after isotopy, see Figure 4.

*Remark 2.10.* By  $(I_1)$ ,  $(I_2)$ , if  $m^1 = \emptyset$  and the label  $v_1$  on  $F_V$  is “+”, then we label  $v_1^1$  on  $F_V^1$  with “ $\times$ ” and label  $v_1^2$  on  $F_V^2$  with “ $\circ$ ”; if  $m^1 = \emptyset$  and the label  $v_1$  on  $F_V$  is “-”, then we label  $v_1^1$  on  $F_V^1$  with “ $\circ$ ” and label  $v_1^2$  on  $F_V^2$  with “ $\times$ ”; if  $m^1 \neq \emptyset$ , the label  $v_1$  on  $F_V$  is “+”, by Lemma 2.5, we may assume that the label  $w_{m^1}$  on  $F_V$  is “-”, then after banding sum, we label  $v_1^1$  on  $F_V^1$  with “ $\times$ ”, label  $w_{m^1}^1$  on  $F_V^1$  with “ $\circ$ ”, label  $v_1^2$  on  $F_V^2$  with “ $\circ$ ”, and label  $w_{m^1}^2$  on  $F_V^2$  with “ $\times$ ”; if  $m^1 \neq \emptyset$ , the label  $v_1$  on  $F_V$  is “-”, by Lemma 2.5, we may assume that the label  $w_{m^1}$  on  $F_V$  is “+”, then after banding sum, we label  $v_1^1$  on  $F_V^1$  with “ $\circ$ ”, label  $w_{m^1}^1$  on  $F_V^1$  with “ $\times$ ”, label  $v_1^2$  on  $F_V^2$  with “ $\times$ ”, and label  $w_{m^1}^2$  on  $F_V^2$  with “ $\circ$ ”.

Let  $\Delta v_{i_1}^k = \{j \mid v_{i_1}^k \cap w_{j_1}^k \neq \emptyset\}$  ( $2 \leq i \leq p; 1 \leq j \leq n$  and  $j \neq m^1; k = 1, 2$ ). Then, we have:

**Lemma 2.11.** *For  $2 \leq i \leq p$ , if  $m^1 = \emptyset$  and the label  $v_1$  on  $F_V$  is “+”, then  $\Delta v_{i_1}^1 = \Delta v_{i_1}^2$ ; if  $m^1 \neq \emptyset$ , the label  $v_1$  on  $F_V$  is “+” and the label  $w_{m^1}$  on  $F_V$  is “-”, then  $\Delta v_{i_1}^1 \subseteq \Delta v_{i_1}^2$ , and if  $j \in \Delta v_{i_1}^1 - \Delta v_{i_1}^2$ , then  $j$  is not the minimal label among all arcs of  $\{w_l^2 \mid 1 \leq l \leq n \text{ and } l \neq m^1\}$  on  $S^2$  with  $v_{i_1}^2 \cap w_{j_1}^2 \neq \emptyset$ .*

*Proof.* Before banding sum,  $v_i^1 = v_i^2$  ( $1 \leq i \leq p$ ) and  $w_j^1 = w_j^2$  ( $1 \leq j \leq n$ ). If  $m^1 = \emptyset$ , then we do not need to band sum. So,  $v_i^k = v_{i_1}^k$  and  $w_j^k = w_{j_1}^k$  ( $2 \leq i \leq p; 1 \leq j \leq n; k = 1, 2$ ). Since  $|v_i^1 \cap w_j^1| = |v_i^2 \cap w_j^2|$ ,  $|v_{i_1}^1 \cap w_{j_1}^1| = |v_{i_1}^2 \cap w_{j_1}^2|$  ( $2 \leq i \leq p; 1 \leq j \leq n$ ). Hence,  $\Delta v_{i_1}^1 = \Delta v_{i_1}^2$  ( $2 \leq i \leq p$ ). So, we may assume that  $m^1 \neq \emptyset$ . There are two cases:

**Case 1 in Lemma 2.11.**  $v_i^1 \cap w_{m^1}^1 = \emptyset$  for some  $2 \leq i \leq p$ .

Since  $v_i^1 = v_i^2$  and  $w_{m^1}^1 = w_{m^1}^2$ ,  $v_i^2 \cap w_{m^1}^2 = \emptyset$ . By  $(I_2)$ ,  $v_i^2 = v_{i_1}^2$ . Since  $v_i^1 \cap w_{m^1}^1 = \emptyset$ , by  $(I_1)$ ,  $v_i^1 = v_{i_1}^1$ . By Remark 2.7, for each  $1 \leq l \leq n$  and  $l \neq m^1$ ,  $|v_{i_1}^1 \cap w_l^1| \leq 1$ . Hence, for each  $j \in \Delta v_{i_1}^1$ ,  $|w_{j_1}^1 \cap v_{i_1}^1| = 1$ . By  $(I_1)$ , since  $w_j^1 = w_{j_1}^1$  and  $v_i^1 = v_{i_1}^1$ ,  $|v_i^1 \cap w_j^1| = 1$ . So,  $|v_i^2 \cap w_j^2| = 1$ . If  $w_j^2 \cap v_1^2 = \emptyset$ , then by  $(I_2)$ ,  $w_j^2 = w_{j_1}^2$ . Since  $v_i^2 = v_{i_1}^2$ ,  $|v_{i_1}^2 \cap w_{j_1}^2| = 1$ . Hence,  $j \in \Delta v_{i_1}^2$ . So,  $\Delta v_{i_1}^1 \subseteq \Delta v_{i_1}^2$ . If  $|w_j^2 \cap v_1^2| = 1$ , since  $|v_i^2 \cap (w_{m^1}^2 \cup w_j^2)| = 1$ , by Remark 2.9,  $|w_{j_1}^2 \cap v_{i_1}^2| = k + 1$ , where  $k = |v_i^2 \cap (\cup_{r \in I(w_{m^1})} w_r^2)|$ . Hence,  $j \in \Delta v_{i_1}^2$ . So,  $\Delta v_{i_1}^1 \subseteq \Delta v_{i_1}^2$ .

For each  $j \notin \Delta v_{i_1}^1$ ,  $v_{i_1}^1 \cap w_{j_1}^1 = \emptyset$ . Since  $v_{i_1}^1 = v_i^1$  and  $w_{j_1}^1 = w_j^1$ ,  $v_i^1 \cap w_j^1 = \emptyset$ . So,  $v_i^2 \cap w_j^2 = \emptyset$ . If  $w_j^2 \cap v_1^2 = \emptyset$ , then by  $(I_2)$ ,  $w_j^2 = w_{j_1}^2$ . Since  $v_i^2 = v_{i_1}^2$ ,  $w_{j_1}^2 \cap v_{i_1}^2 = \emptyset$ . Hence,  $j \notin \Delta v_{i_1}^2$ . If  $|w_j^2 \cap v_1^2| = 1$ , since  $v_i^2 \cap (w_j^2 \cup w_{m^1}^2) = \emptyset$ , by Remark 2.9,  $|v_{i_1}^2 \cap w_{j_1}^2| = k$ , where  $k = |v_i^2 \cap (\cup_{r \in I(w_{m^1})} w_r^2)|$ . If  $k = 0$ , then  $j \notin \Delta v_{i_1}^2$ . If  $k > 0$ , then  $j \in \Delta v_{i_1}^2 - \Delta v_{i_1}^1$ . Since  $|w_j^2 \cap v_1^2| = 1$ , by the minimality of  $m^1$  and (2) in Proposition 2.3,  $j > m^1 > r$ , where  $r \in I(w_{m^1})$ . Since  $k > 0$ , there is  $r \in I(w_{m^1})$  with  $|v_i^2 \cap w_r^2| = 1$ . By the minimality of  $m^1$ ,  $w_r^2 \cap v_1^2 = \emptyset$ . By  $(I_2)$ ,  $w_r^2 = w_{r_1}^2$ . Since  $v_i^2 = v_{i_1}^2$ ,  $|v_{i_1}^2 \cap w_{r_1}^2| = 1$ . Since  $j > r$ ,  $j$  is not the minimal label among all arcs of  $\{w_l^2 \mid 1 \leq l \leq n \text{ and } l \neq m^1\}$  on  $S^2$  with  $v_{i_1}^2 \cap w_{j_1}^2 \neq \emptyset$ .

**Case 2 in Lemma 2.11.**  $|v_i^1 \cap w_{m^1}^1| = 1$  for some  $2 \leq i \leq p$ .

Since  $v_i^1 = v_i^2$  and  $w_{m^1}^1 = w_{m^1}^2$ ,  $|v_i^2 \cap w_{m^1}^2| = 1$ . Since  $|v_i^1 \cap w_{m^1}^1| = 1$ ,  $v_i^1 \neq v_{i_1}^1$ . By Remark 2.7, for each  $1 \leq l \leq n$  and  $l \neq m^1$ ,  $|v_{i_1}^1 \cap w_l^1| \leq 1$ . Hence, for each  $j \in \Delta v_{i_1}^1$ ,  $|w_{j_1}^1 \cap v_{i_1}^1| = 1$ . By Remark 2.7,  $|w_j^1 \cap (v_1^1 \cup v_i^1)| = 1$ . So,  $|w_j^2 \cap (v_1^2 \cup v_i^2)| = 1$ . If  $w_j^2 \cap v_1^2 = \emptyset$  and  $|w_j^2 \cap v_i^2| = 1$ , then by  $(I_2)$ ,  $w_j^2 = w_{j_1}^2$ . Since  $v_i^2 = v_{i_1}^2$ ,  $|w_{j_1}^2 \cap v_{i_1}^2| = 1$ . Then,  $j \in \Delta v_{i_1}^2$ . So,  $\Delta v_{i_1}^1 \subseteq \Delta v_{i_1}^2$ . If  $|w_j^2 \cap v_1^2| = 1$  and  $w_j^2 \cap v_i^2 = \emptyset$ , then  $|v_i^2 \cap (w_{m^1}^2 \cup w_j^2)| = 1$ . By Remark 2.9,  $|w_{j_1}^2 \cap v_{i_1}^2| = k + 1$ , where  $k = |v_i^2 \cap (\cup_{r \in I(w_{m^1})} w_r^2)|$ . Hence,  $j \in \Delta v_{i_1}^2$ . So,  $\Delta v_{i_1}^1 \subseteq \Delta v_{i_1}^2$ .

For each  $j \notin \Delta v_{i_1}^1$ ,  $w_{j_1}^1 \cap v_{i_1}^1 = \emptyset$ . By Remark 2.7,  $|w_j^1 \cap (v_1^1 \cup v_i^1)| = 0$  or  $2$ . So,  $|w_j^2 \cap (v_1^2 \cup v_i^2)| = 0$  or  $2$ . If  $w_j^2 \cap (v_1^2 \cup v_i^2) = \emptyset$ , then by  $(I_2)$ ,  $w_j^2 = w_{j_1}^2$ . Since  $v_i^2 = v_{i_1}^2$ ,  $w_{j_1}^2 \cap v_{i_1}^2 = \emptyset$ . Hence,  $j \notin \Delta v_{i_1}^2$ . If  $|w_j^2 \cap (v_1^2 \cup v_i^2)| = 2$ , then  $|v_i^2 \cap (w_{m^1}^2 \cup w_j^2)| = 2$ . By Remark 2.9,  $|w_{j_1}^2 \cap v_{i_1}^2| = k$ , where  $k = |v_i^2 \cap (\cup_{r \in I(w_{m^1})} w_r^2)|$ . If  $k = 0$ , then  $j \notin \Delta v_{i_1}^2$ . If  $k > 0$ , then  $j \in \Delta v_{i_1}^2 - \Delta v_{i_1}^1$ . By the same argument as in Case 1 in Lemma 2.11,  $j$  is not minimal label among all arcs of  $\{w_l^2 \mid 1 \leq l \leq n \text{ and } l \neq m^1\}$  on  $S^2$  with  $v_{i_1}^2 \cap w_{j_1}^2 \neq \emptyset$ .  $\square$

*Remark 2.12.* By the same proof as in Lemma 2.11, for  $2 \leq i \leq p$ , if  $m^1 = \emptyset$  and the label  $v_1$  on  $F_V$  is “-”, then  $\Delta v_{i_1}^1 = \Delta v_{i_1}^2$ ; if  $m^1 \neq \emptyset$ , the label  $v_1$  on  $F_V$  is “-” and the label  $w_{m^1}$  on  $F_V$  is “+”, then  $\Delta v_{i_1}^2 \subseteq \Delta v_{i_1}^1$ , and if  $j \in \Delta v_{i_1}^1 - \Delta v_{i_1}^2$ , then  $j$  is not the minimal label among all arcs of  $\{w_l^1 \mid 1 \leq l \leq n \text{ and } l \neq m^1\}$  on  $S^1$  with  $v_{i_1}^1 \cap w_{j_1}^1 \neq \emptyset$ .

Second, we consider the arc  $v_{2_1}^1$  on  $S^1$ .

**Lemma 2.13.** *If  $m^2$  is the minimal label among all arcs of  $\{w_{j_1}^1 \mid 1 \leq j \leq n \text{ and } j \neq m^1\}$  on  $S^1$  with  $|w_{m^1}^1 \cap v_{2_1}^1| = 1$ , then  $m^2$  is the minimal label among all arcs of  $\{w_{j_1}^2 \mid 1 \leq j \leq n \text{ and } j \neq m^1\}$  on  $S^2$  with  $|w_{m^2}^2 \cap v_{2_1}^2| = 1$ .*

*Proof.* By Lemma 2.11, if  $m^1 = \emptyset$ , then  $\Delta v_{2_1}^1 = \Delta v_{2_1}^2$ ; if  $m^1 \neq \emptyset$ , then  $\Delta v_{2_1}^1 \subseteq \Delta v_{2_1}^2$ , and if  $j \in \Delta v_{2_1}^1 - \Delta v_{2_1}^2$ , then  $j$  is not the minimal label among all arcs of  $\{w_l^1 \mid 1 \leq l \leq n \text{ and } l \neq m^1\}$  on  $S^1$  with  $v_{2_1}^1 \cap w_{j_1}^1 \neq \emptyset$ . So, if  $m^2$  is minimal

in  $\Delta v_{2_1}^1$ , then  $m^2$  is minimal in  $\Delta v_{2_1}^2$ . By Remark 2.7,  $|w_{m_1^2}^1 \cap v_{2_1}^1| = 1$ . By Remark 2.9,  $|w_{m_1^2}^2 \cap v_{2_1}^2| = 1$ .  $\square$

By the same proof as above (see Remark 2.10), if  $m^2 = \emptyset$  and the label  $v_2$  on  $F_V$  is “+”, then we label  $v_{2_1}^1$  on  $S^1$  with “ $\times$ ” and label  $v_{2_1}^2$  on  $S^2$  with “ $\circ$ ”; if  $m^2 = \emptyset$  and the label  $v_2$  on  $F_V$  is “-”, then we label  $v_{2_1}^1$  on  $S^1$  with “ $\circ$ ” and label  $v_{2_1}^2$  on  $S^2$  with “ $\times$ ”. For convenience, for each arc  $v_{i_1}^k$  ( $3 \leq i \leq p$ ) and  $w_{j_1}^k$  ( $1 \leq j \leq n$  and  $j \neq m^1$ ) on  $S^k$  ( $k = 1, 2$ ), we denote them by  $v_{i_2}^k$  and  $w_{j_2}^k$ . We may assume that  $v_{i_2}^k$  and  $v_{i_1}^k$  have the same label, and  $w_{j_2}^k$  and  $w_{j_1}^k$  have the same label. For each disk  $V_{i_1}$  ( $3 \leq i \leq p$  or  $i = x$ ) and  $W_{j_1}$  ( $1 \leq j \leq n$  and  $j \neq m^1$ , or  $j = x$ ), we denote them by  $V_{i_2}$  and  $W_{j_2}$ . If  $v_{2_1}^k$  ( $k = 1, 2$ ) is retained, we also denote it by  $v_{2_2}^k$ . But in the future banding sum process, we do not consider  $v_{2_2}^k$ .

If  $m^2 \neq \emptyset$ , the label  $v_2$  on  $F_V$  is “+”, by Lemma 2.5, we may assume that the label  $w_{m^2}$  on  $F_V$  is “-”, then  $V_{2_1}$  is a properly embedded disk in  $V^1$  and  $W_{m_1^2}$  is a properly embedded disk in  $W^2$ . For each disk  $V_{i_1}$  (the label  $v_i$  on  $F_V$  is “+”;  $3 \leq i \leq p$  or  $i = x$ ) in  $V^1$ , if  $V_{i_1} \cap w_{m_1^2}^1 = \emptyset$ , then we do nothing and denote it by  $V_{i_2}$ ; if  $V_{i_1} \cap w_{m_1^2}^1 \neq \emptyset$ , then by the same argument as in  $(I_1)$ , we band sum  $V_{i_1}$  and  $V_{2_1}$  along  $w_{m_1^2}^1$ , after banding sum, we denote it by  $V_{i_2}$ , such that  $V_{i_2} \cap V_{2_1} = \emptyset$  and  $\partial V_{i_2} \cap (v_{2_1}^1 \cup w_{m_1^2}^1) = \emptyset$ . For each disk  $W_{j_1}$  (the label  $w_j$  on  $F_V$  is “+”;  $1 \leq j \leq n$  and  $j \neq m^1, m^2$ , or  $j = x$ ) in  $W^1$ , we do nothing and denote it by  $W_{j_2}$ . For each disk  $W_{j_1}$  (the label  $w_j$  is “-”;  $1 \leq j \leq n$  and  $j \neq m^1, m^2$ ) in  $W^2$ , if  $W_{j_1} \cap v_{2_1}^2 = \emptyset$ , then we do nothing and denote it by  $W_{j_2}$ ; if  $W_{j_1} \cap v_{2_1}^2 \neq \emptyset$ , then by the same argument as in  $(I_2)$ , we band sum  $W_{j_1}$  and  $W_{m_1^2}$  along  $v_{2_1}^2$ , after banding sum, we denote it by  $W_{j_2}$ , such that  $W_{j_2} \cap W_{m_1^2} = \emptyset$  and  $\partial W_{j_2} \cap (v_{2_1}^2 \cup w_{m_1^2}^2) = \emptyset$ . For each disk  $V_{i_1}$  (the label  $v_i$  is “-”;  $3 \leq i \leq p$ ) in  $V^2$ , we do nothing and denote it by  $V_{i_2}$ .

Correspondingly, for each arc  $v_{i_1}^k$  and  $w_{j_1}^k$  ( $3 \leq i \leq p; 1 \leq j \leq n$  and  $j \neq m^1, m^2; k = 1, 2$ ) on  $S^k$  before banding sum, we denote them by  $v_{i_2}^k$  and  $w_{j_2}^k$  after banding sum. Now we label  $v_{2_1}^1$  on  $S^1$  with “ $\times$ ”, label  $w_{m_1^2}^1$  on  $S^1$  with “ $\circ$ ”, label  $v_{2_1}^2$  on  $S^2$  with “ $\circ$ ” and label  $w_{m_1^2}^2$  on  $S^2$  with “ $\times$ ”. Since both  $w_{m_1^2}^1$  and  $v_{2_1}^2$  are retained, for convenience, we denote them by  $w_{m_2}^1$  and  $v_{2_2}^2$ . But in the future banding sum process, we do not need to consider them. So, we obtain four sets of pairwise disjoint properly embedded disks  $\{V_{i_2} \mid \text{the label } v_i \text{ is “+” and } 3 \leq i \leq p\} \cup \{V_{x_2}\}$  in  $V^1$ ,  $\{W_{j_2} \mid \text{the label } w_j \text{ is “+” and } 1 \leq j \leq n \text{ and } j \neq m^1, m^2\} \cup \{W_{x_2}\}$  in  $W^1$ ,  $\{V_{i_2} \mid \text{the label } v_i \text{ is “-” and } 3 \leq i \leq p\}$  in  $V^2$ , and  $\{W_{j_2} \mid \text{the label } w_j \text{ is “-” and } 1 \leq j \leq n \text{ and } j \neq m^1, m^2\}$  in  $W^2$ , satisfying the same properties as in Lemmas 2.6 and 2.8.

If  $m^2 \neq \emptyset$ , the label  $v_2$  on  $F_V$  is “-”, by Lemma 2.5, we may assume that the label  $w_{m^2}$  on  $F_V$  is “+”, then  $V_{2_1}$  is a properly embedded disk in  $V^2$  and

$W_{m_1^2}$  is a properly embedded disk in  $W^1$ . For each disk  $W_{j_1}$  (the label  $w_j$  is “+”;  $1 \leq j \leq n$  and  $j \neq m^1, m^2$ , or  $j = x$ ) in  $W^1$ , if  $W_{j_1} \cap v_{2_1}^1 = \emptyset$ , then we do nothing and denote it by  $W_{j_2}$ ; if  $W_{j_1} \cap v_{2_1}^1 \neq \emptyset$ , then by the same argument as in  $(I_2)$ , we band sum  $W_{j_1}$  and  $W_{m_1^2}$  along  $v_{2_1}^1$ , after banding sum, we denote it by  $W_{j_2}$ , such that  $W_{j_2} \cap W_{m_1^2} = \emptyset$  and  $\partial W_{j_2} \cap (v_{2_1}^1 \cup w_{m_1^2}^1) = \emptyset$ . For each disk  $V_{i_1}$  (the label  $v_i$  is “+”;  $3 \leq i \leq p$  or  $i = x$ ) in  $V^1$ , we do nothing and denote it by  $V_{i_2}$ . For each disk  $V_{i_1}$  (the label  $v_i$  is “-”;  $3 \leq i \leq p$ ) in  $V^2$ , if  $V_{i_1} \cap w_{m_1^2}^2 = \emptyset$ , then we do nothing and denote it by  $V_{i_2}$ ; if  $V_{i_1} \cap w_{m_1^2}^2 \neq \emptyset$ , then by the same argument as in  $(I_1)$ , we band sum  $V_{i_1}$  and  $V_{2_1}$  along  $w_{m_1^2}^2$ , after banding sum, we denote it by  $V_{i_2}$ , such that  $V_{i_2} \cap V_{2_1} = \emptyset$  and  $\partial V_{i_2} \cap (v_{2_1}^2 \cup w_{m_1^2}^2) = \emptyset$ . For each disk  $W_{j_1}$  (the label  $w_j$  is “-”;  $1 \leq j \leq n$  and  $j \neq m^1, m^2$ ) in  $W^2$ , we do nothing and denote it by  $W_{j_2}$ .

Correspondingly, for each arc  $v_{i_1}^k$  and  $w_{j_1}^k$  ( $3 \leq i \leq p; 1 \leq j \leq n$  and  $j \neq m^1, m^2; k = 1, 2$ ) on  $S^k$  before banding sum, we denote them by  $v_{i_2}^k$  and  $w_{j_2}^k$  after banding sum. Now we label  $v_{2_1}^1$  on  $S^1$  with “o”, label  $w_{m_1^2}^1$  on  $S^1$  with “x”, label  $v_{2_1}^2$  on  $S^2$  with “x” and label  $w_{m_1^2}^2$  on  $S^2$  with “o”. Since both  $v_{2_1}^1$  and  $w_{m_1^2}^2$  are retained, we denote them by  $v_{2_2}^1$  and  $w_{m_2^2}^2$ . But in the future banding sum process, we do not need to consider them. So, we obtain four sets of pairwise disjoint properly embedded disks  $\{V_{i_2} \mid \text{the label } v_i \text{ is “+” and } 3 \leq i \leq p\} \cup \{V_{x_2}\}$  in  $V^1$ ,  $\{W_{j_2} \mid \text{the label } w_j \text{ is “+”, } 1 \leq j \leq n \text{ and } j \neq m^1, m^2\} \cup \{W_{x_2}\}$  in  $W^1$ ,  $\{V_{i_2} \mid \text{the label } v_i \text{ is “-” and } 3 \leq i \leq p\}$  in  $V^2$ , and  $\{W_{j_2} \mid \text{the label } w_j \text{ is “-”, } 1 \leq j \leq n \text{ and } j \neq m^1, m^2\}$  in  $W^2$ , satisfying the same properties as in Lemmas 2.6 and 2.8.

We continue this procedure as above, there are  $p$  steps. For each step  $l$  ( $1 \leq l \leq p$ ), by the same argument as above, before banding sum, there are four sets of pairwise disjoint arcs  $\{v_{i_{l-1}}^k \mid l \leq i \leq p\} \cup \{v_{i_{l-1}}^k \mid 1 \leq i \leq l-1 \text{ and } v_{i_{l-1}}^k \text{ is labelled with “o”}\}$  and  $\{w_{j_{l-1}}^k \mid 1 \leq j \leq n \text{ and } j \neq m^1, \dots, m^{l-1}\} \cup \{w_{j_{l-1}}^k \mid j = m^1, \dots, m^{l-1} \text{ and } w_{j_{l-1}}^k \text{ is labelled with “o”}\}$  on  $S^k$  ( $k = 1, 2$ ). Let  $\Delta v_{i_{l-1}}^k = \{j \mid v_{i_{l-1}}^k \cap w_{j_{l-1}}^k \neq \emptyset\}$  ( $l \leq i \leq p; 1 \leq j \leq n$  and  $j \neq m^1, \dots, m^{l-1}; k = 1, 2$ ). Then, we have:

**Lemma 2.14.** *For  $l \leq i \leq p$ , if  $j \in \Delta v_{i_{l-1}}^1 - \Delta v_{i_{l-1}}^2$ , then  $j$  is not the minimal label among all arcs of  $\{w_{h_{l-1}}^1 \mid 1 \leq h \leq n \text{ and } h \neq m^1, \dots, m^{l-1}\}$  on  $S^1$  with  $v_{i_{l-1}}^1 \cap w_{j_{l-1}}^1 \neq \emptyset$ ; if  $j \in \Delta v_{i_{l-1}}^2 - \Delta v_{i_{l-1}}^1$ , then  $j$  is not the minimal label among all arcs of  $\{w_{h_{l-1}}^2 \mid 1 \leq h \leq n \text{ and } h \neq m^1, \dots, m^{l-1}\}$  on  $S^2$  with  $v_{i_{l-1}}^2 \cap w_{j_{l-1}}^2 \neq \emptyset$ .*

*Proof.* Note that  $\Delta v_{i_{l-1}}^1 \not\subseteq \Delta v_{i_{l-1}}^2$  and  $\Delta v_{i_{l-1}}^2 \not\subseteq \Delta v_{i_{l-1}}^1$  for  $l \geq 3$ . Recall the step 2 in Lemma 13, we do not need to consider  $\Delta v_{i_1}^1 \subseteq \Delta v_{i_1}^2$  and  $\Delta v_{i_1}^2 \subseteq \Delta v_{i_1}^1$ , if  $j \in \Delta v_{i_1}^2 - \Delta v_{i_1}^1$ , i.e.,  $j \in \Delta v_{i_1}^2$  and  $j \notin \Delta v_{i_1}^1$ , then  $w_j^2 \cap v_1^2 \neq \emptyset$  and for some  $i$ ,  $v_i^2 \cap w_{m^1}^2 = \emptyset$  and  $v_i^2 \cap (\cup_{r \in I(w_{m^1})} w_r^2) \neq \emptyset$ , see Figure 5. We may assume

that  $v_i^2 \cap w_r^2 \neq \emptyset$  for some  $r \in I(w_{m^1})$ . So,  $j > m^1 > r$ . After banding sum,  $w_{j_1}^2 \cap v_{i_1}^2 \neq \emptyset$  on  $S^2$ , see Figure 5, and  $w_{j_1}^1 \cap v_{i_1}^1 = \emptyset$  on  $S^1$ , see Figure 6.

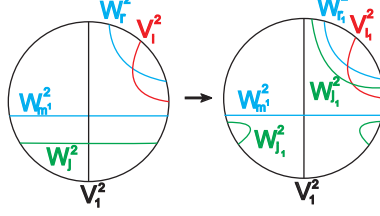


FIGURE 5.  $w_{j_1}^2 \cap v_{i_1}^2 \neq \emptyset$  on  $S^2$

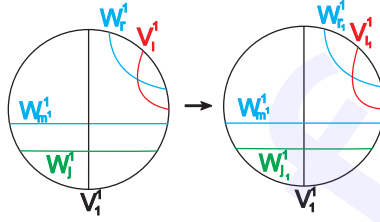
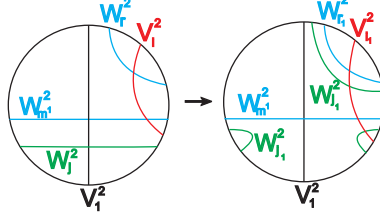
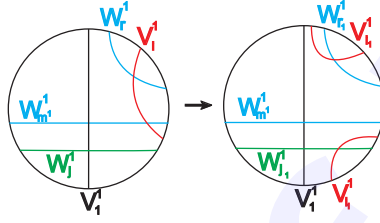


FIGURE 6.  $w_{j_1}^1 \cap v_{i_1}^1 = \emptyset$  on  $S^1$ ,

Since  $v_i^2 \cap w_r^2 \neq \emptyset$  for some  $r \in I(w_{m^1})$  and  $r < j$ , then after banding sum,  $j$  is not the minimal label among all arcs of  $\{w_{h_1}^2 \mid 1 \leq h \leq n \text{ and } h \neq m^1\}$  on  $S^2$  with  $v_{i_1}^2 \cap w_{j_1}^2 \neq \emptyset$  and  $j \notin \Delta v_{i_1}^1$ . If  $j \in (\Delta v_{i_1}^1 \cap \Delta v_{i_1}^2)$  and  $|w_{j_1}^2 \cap v_{i_1}^2| > 1$ , then by the same argument,  $j$  is not the minimal label among all arcs of  $\{w_{h_1}^2 \mid 1 \leq h \leq n \text{ and } h \neq m^1\}$  on  $S^2$  with  $v_{i_1}^2 \cap w_{j_1}^2 \neq \emptyset$ , see Figure 7, also,  $j$  is not the minimal label among all arcs of  $\{w_{h_1}^1 \mid 1 \leq h \leq n \text{ and } h \neq m^1\}$  on  $S^1$  with  $v_{i_1}^1 \cap w_{j_1}^1 \neq \emptyset$ , see Figure 8. If  $j \in (\Delta v_{i_1}^1 \cap \Delta v_{i_1}^2)$  and  $|w_{j_1}^2 \cap v_{i_1}^2| \leq 1$ , then by Remarks 9 and 11,  $|w_{j_1}^1 \cap v_{i_1}^1| = |w_{j_1}^2 \cap v_{i_1}^2| = 1$ . So,  $m^2 \in (\Delta v_{i_1}^1 \cap \Delta v_{i_1}^2)$  and  $|w_{m_1}^1 \cap v_{i_1}^1| = |w_{m_1}^2 \cap v_{i_1}^2| = 1$ . If  $j \in \Delta v_{i_1}^1 - \Delta v_{i_1}^2$ , then by the same arguments,  $j$  is not the minimal label among all arcs of  $\{w_{h_1}^1 \mid 1 \leq h \leq n \text{ and } h \neq m^1\}$  on  $S^1$  with  $v_{i_1}^1 \cap w_{j_1}^1 \neq \emptyset$  and  $j \notin \Delta v_{i_1}^2$ , and  $m^2 \in (\Delta v_{i_1}^1 \cap \Delta v_{i_1}^2)$  and  $|w_{m_1}^1 \cap v_{i_1}^1| = |w_{m_1}^2 \cap v_{i_1}^2| = 1$ .

By the same arguments, for  $l \leq i \leq p$ , if  $j \in \Delta v_{i_{l-1}}^1 - \Delta v_{i_{l-1}}^2$ , then  $j$  is not the minimal label among all arcs of  $\{w_{h_{l-1}}^1 \mid 1 \leq h \leq n \text{ and } h \neq m^1, \dots, m^{l-1}\}$  on  $S^1$  with  $v_{i_{l-1}}^1 \cap w_{j_{l-1}}^1 \neq \emptyset$  and  $j \notin \Delta v_{i_{l-1}}^2$ ; if  $j \in (\Delta v_{i_{l-1}}^1 \cap \Delta v_{i_{l-1}}^2)$  and  $|w_{j_{l-1}}^1 \cap v_{i_{l-1}}^1| > 1$ , then by the same argument,  $j$  is not the minimal label among all arcs of  $\{w_{h_{l-1}}^1 \mid 1 \leq h \leq n \text{ and } h \neq m^1, \dots, m^{l-1}\}$  on  $S^1$  with  $v_{i_{l-1}}^1 \cap w_{j_{l-1}}^1 \neq \emptyset$ , also,  $j$  is not the minimal label among all arcs of  $\{w_{h_{l-1}}^2 \mid 1 \leq h \leq n \text{ and } h \neq m^1, \dots, m^{l-1}\}$  on  $S^2$  with  $v_{i_{l-1}}^2 \cap w_{j_{l-1}}^2 \neq \emptyset$  and  $j \notin \Delta v_{i_{l-1}}^1$ .

FIGURE 7.  $|w_{j_1}^2 \cap v_{i_1}^2| > 1$  on  $S^2$ FIGURE 8.  $w_{j_1}^1$  and  $v_{i_1}^1$  on  $S^1$ 

$h \neq m^1, \dots, m^{l-1}$  on  $S^2$  with  $v_{i_{l-1}}^2 \cap w_{j_{l-1}}^2 \neq \emptyset$ ; if  $j \in (\Delta v_{i_{l-1}}^1 \cap \Delta v_{i_{l-1}}^2)$  and  $|w_{j_{l-1}}^2 \cap v_{i_{l-1}}^2| > 1$ , then by the same argument,  $j$  is not the minimal label among all arcs of  $\{w_{h_{l-1}}^2 \mid 1 \leq h \leq n \text{ and } h \neq m^1, \dots, m^{l-1}\}$  on  $S^2$  with  $v_{i_{l-1}}^2 \cap w_{j_{l-1}}^2 \neq \emptyset$ , also,  $j$  is not the minimal label among all arcs of  $\{w_{h_{l-1}}^1 \mid 1 \leq h \leq n \text{ and } h \neq m^1, \dots, m^{l-1}\}$  on  $S^1$  with  $v_{i_{l-1}}^1 \cap w_{j_{l-1}}^1 \neq \emptyset$ ; if  $j \in (\Delta v_{i_{l-1}}^1 \cap \Delta v_{i_{l-1}}^2)$ ,  $|w_{j_{l-1}}^1 \cap v_{i_{l-1}}^1| \leq 1$  and  $|w_{j_{l-1}}^2 \cap v_{i_{l-1}}^2| \leq 1$ , then  $|w_{j_{l-1}}^1 \cap v_{i_{l-1}}^1| = |w_{j_{l-1}}^2 \cap v_{i_{l-1}}^2| = 1$ . So,  $m^l \in (\Delta v_{i_{l-1}}^1 \cap \Delta v_{i_{l-1}}^2)$  and  $|w_{m_{l-1}}^1 \cap v_{i_{l-1}}^1| = |w_{m_{l-1}}^2 \cap v_{i_{l-1}}^2| = 1$ . If  $j \in \Delta v_{i_{l-1}}^2 - \Delta v_{i_{l-1}}^1$ , then by the same arguments,  $j$  is not the minimal label among all arcs of  $\{w_{h_{l-1}}^1 \mid 1 \leq h \leq n \text{ and } h \neq m^1, \dots, m^{l-1}\}$  on  $S^2$  with  $v_{i_{l-1}}^1 \cap w_{j_{l-1}}^1 \neq \emptyset$  and  $j \notin \Delta v_{i_{l-1}}^1$ , and  $m^l \in (\Delta v_{i_{l-1}}^1 \cap \Delta v_{i_{l-1}}^2)$  and  $|w_{m_{l-1}}^1 \cap v_{i_{l-1}}^1| = |w_{m_{l-1}}^2 \cap v_{i_{l-1}}^2| = 1$ .  $\square$

For step  $l$ , we consider the arc  $v_{l_{l-1}}^1$  on  $S^1$ . By the proof of Lemma 2.14, we have:

**Lemma 2.15.** *If  $m^l$  is the minimal label among all arcs of  $\{w_{j_{l-1}}^1 \mid 1 \leq j \leq n \text{ and } j \neq m^1, \dots, m^{l-1}\}$  on  $S^1$  with  $|w_{m_{l-1}}^1 \cap v_{l_{l-1}}^1| = 1$ , then  $m^l$  is the minimal label among all arcs of  $\{w_{j_{l-1}}^2 \mid 1 \leq j \leq n \text{ and } j \neq m^1, \dots, m^{l-1}\}$  on  $S^2$  with  $|w_{m_{l-1}}^2 \cap v_{l_{l-1}}^2| = 1$ .*

If  $m^l = \emptyset$  and the label  $v_l$  on  $F_V$  is “+”, then we label  $v_{l_{l-1}}^1$  on  $S^1$  with “ $\times$ ” and label  $v_{l_{l-1}}^2$  on  $S^2$  with “ $\circ$ ”; if  $m^l = \emptyset$  and the label  $v_l$  on  $F_V$  is “-”, then we label  $v_{l_{l-1}}^1$  on  $S^1$  with “ $\circ$ ” and label  $v_{l_{l-1}}^2$  on  $S^2$  with “ $\times$ ”. For convenience, for each arc  $v_{i_{l-1}}^k$  ( $l \leq i \leq p$ ) and  $w_{j_{l-1}}^k$  ( $1 \leq j \leq n$  and  $j \neq m^1, \dots, m^{l-1}$ ) on  $S^k$  ( $k = 1, 2$ ), we denote them by  $v_{i_l}^k$  and  $w_{j_l}^k$ . We may assume that  $v_{i_l}^k$  and  $v_{i_{l-1}}^k$  have the same label, and  $w_{j_l}^k$  and  $w_{j_{l-1}}^k$  have the same label. For each disk  $V_{i_{l-1}}$  ( $l \leq i \leq p$  or  $i = x$ ) and  $W_{j_{l-1}}$  ( $1 \leq j \leq n$  and  $j \neq m^1, \dots, m^{l-1}$ , or  $j = x$ ), we denote them by  $V_{i_l}$  and  $W_{j_l}$ . If  $v_{l_{l-1}}^k$  ( $k = 1, 2$ ) is retained, we also denote it by  $v_{l_l}^k$ . But in the future banding sum process, we do not consider  $v_{l_l}^k$ .

If  $m^l \neq \emptyset$ , the label  $v_l$  on  $F_V$  is “+”, then by Lemma 2.5, we may assume that the label  $w_{m^l}$  on  $F_V$  is “-”. By Lemma 2.15 and the same argument as above (see  $(I_1)$ ), after banding sum, we label  $v_{l_{l-1}}^1$  on  $S^1$  with “ $\times$ ”, label  $w_{m_{l-1}}^1$  on  $S^1$  with “ $\circ$ ”, label  $v_{l_{l-1}}^2$  on  $S^2$  with “ $\circ$ ”, and label  $w_{m_{l-1}}^2$  on  $S^2$  with “ $\times$ ”; if  $m^l \neq \emptyset$ , the label  $v_l$  on  $F_V$  is “-”, then by Lemma 2.5, we may assume that the label  $w_{m^l}$  on  $F_V$  is “+”. By Lemma 2.15 and the same argument as above (see  $(I_2)$ ), after banding sum, we label  $v_{l_{l-1}}^1$  on  $S^1$  with “ $\circ$ ”, label  $w_{m_{l-1}}^1$  on  $S^1$  with “ $\times$ ”, label  $v_{l_{l-1}}^2$  on  $S^2$  with “ $\times$ ”, and label  $w_{m_{l-1}}^2$  on  $S^2$  with “ $\circ$ ”. If the arc  $v_{l_{l-1}}^k$  (resp.  $w_{m_{l-1}}^k$ ) on  $S^k$  ( $k = 1, 2$ ) is labelled with “ $\circ$ ”, then we denote it by  $v_{l_l}^k$  (resp.  $w_{m_l}^k$ ), but in the future banding sum process, we do not consider it. By the same argument as in Lemmas 2.6 and 2.8, after banding sum, we have:

**Lemma 2.16.** *There are four sets of pairwise disjoint arcs  $\{v_{i_l}^k \mid l+1 \leq i \leq p\} \cup \{v_{i_l}^k \mid 1 \leq i \leq l \text{ and } v_{i_l}^k \text{ is labelled with “}\circ\text{”}\}$  and  $\{w_{j_l}^k \mid 1 \leq j \leq n \text{ and } j \neq m^1, \dots, m^l\} \cup \{w_{j_l}^k \mid j = m^1, \dots, m^l \text{ and } w_{j_l}^k \text{ is labelled with “}\circ\text{”}\}$  on  $S^k$  ( $k = 1, 2$ ), and four sets of pairwise disjoint disks  $\{V_{i_l} \mid \text{the label } v_{i_l} \text{ is “+” and } l+1 \leq i \leq p\} \cup \{V_{x_l}\}$  in  $V^1$ ,  $\{W_{j_l} \mid \text{the label } w_{j_l} \text{ is “+”}, 1 \leq j \leq n \text{ and } j \neq m^1, \dots, m^l\} \cup \{W_{x_l}\}$  in  $W^1$ ,  $\{V_{i_l} \mid \text{the label } v_{i_l} \text{ is “-” and } l+1 \leq i \leq p\}$  in  $V^2$ , and  $\{W_{j_l} \mid \text{the label } w_{j_l} \text{ is “-”}, 1 \leq j \leq n \text{ and } j \neq m^1, \dots, m^l\}$  in  $W^2$ , satisfying the following conditions:*

- (1) *If  $V_{i_l}$  lies in  $V^1$  and  $W_{j_l}$  lies in  $W^1$ , then  $V_{i_l} \cap F_V^1 = (v_{i_l}^1 \cap F_V^1) \cup_{r \in I(v_i)} (v_{r_l}^1 \cap F_V^1)$ ,  $W_{j_l} \cap F_V^1 = (w_{j_l}^1 \cap F_V^1) \cup_{r \in I(w_j)} (w_{r_l}^1 \cap F_V^1)$ ,  $V_{x_l} \cap F_V^1 = \cup_{r \in I(v)} (v_{r_l}^1 \cap F_V^1)$ ,  $W_{x_l} \cap F_V^1 = \cup_{r \in I(w)} (w_{r_l}^1 \cap F_V^1)$ ,  $V_{i_l} \cap W_{j_l} = V_{i_l} \cap W_{j_l} \cap F_V^1$ ,  $V_{i_l} \cap W_{x_l} = V_{i_l} \cap W_{x_l} \cap F_V^1$ ,  $V_{x_l} \cap W_{j_l} = V_{x_l} \cap W_{j_l} \cap F_V^1$ ,  $V_{x_l} \cap W_{x_l} = \{x\} \cup (V_{x_l} \cap W_{x_l} \cap F_V^1)$ ;*
- (2) *If  $V_{i_l}$  lies in  $V^2$  and  $W_{j_l}$  lies in  $W^2$ , then  $V_{i_l} \cap F_V^2 = (v_{i_l}^2 \cap F_V^2) \cup_{r \in I(v_i)} (v_{r_l}^2 \cap F_V^2)$ ,  $W_{j_l} \cap F_V^2 = (w_{j_l}^2 \cap F_V^2) \cup_{r \in I(w_j)} (w_{r_l}^2 \cap F_V^2)$ ,  $V_{i_l} \cap W_{j_l} = V_{i_l} \cap W_{j_l} \cap F_V^2$ .*

For step  $p$ , as in Lemma 2.16, after banding sum, we obtain three sets of pairwise disjoint arcs  $\{v_{i_p}^k \mid 1 \leq i \leq p \text{ and } v_{i_p}^k \text{ is labelled with “}\circ\text{”}\}$  and  $\{w_{j_p}^k \mid 1 \leq j \leq n \text{ and } j \neq m^1, \dots, m^p\} \cup \{w_{j_p}^k \mid j = m^1, \dots, m^p \text{ and } w_{j_p}^k \text{ is}$



labelled with “ $\circ$ ” on  $S^k$  ( $k = 1, 2$ ), and three sets of pairwise disjoint disks  $\{V_{x_p}\}$  in  $V^1$ ,  $\{W_{j_p} \mid \text{the label } w_j \text{ is “+”}, 1 \leq j \leq n \text{ and } j \neq m^1, \dots, m^p\} \cup \{W_{x_p}\}$  in  $W^1$ , and  $\{W_{j_p} \mid \text{the label } w_j \text{ is “-”}, 1 \leq j \leq n \text{ and } j \neq m^1, \dots, m^p\}$  in  $W^2$ , satisfying the following condition:

(\*) If  $W_{j_p}$  lies in  $W^1$ , then  $W_{j_p} \cap F_V^1 = (w_{j_p}^1 \cap F_V^1) \cup_{r \in I(w_j)} (w_{r_p}^1 \cap F_V^1)$ ,  $V_{x_p} \cap F_V^1 = \cup_{r \in I(v)} (v_{r_p}^1 \cap F_V^1)$ ,  $W_{x_p} \cap F_V^1 = \cup_{r \in I(w)} (w_{r_p}^1 \cap F_V^1)$ ,  $V_{x_p} \cap W_{j_p} = V_{x_p} \cap W_{j_p} \cap F_V^1$ ,  $V_{x_p} \cap W_{x_p} = \{x\} \cup (V_{x_p} \cap W_{x_p} \cap F_V^1)$ .

For each arc  $w_{j_p}^k$  ( $1 \leq j \leq n$  and  $j \neq m^1, \dots, m^p$ ) on  $S^k$  ( $k = 1, 2$ ), if the label  $w_j$  on  $F_V$  is “+”, then we label  $w_{j_p}^1$  on  $S^1$  with “ $\times$ ”, and label  $w_{j_p}^2$  on  $S^2$  with “ $\circ$ ”; if the label  $w_j$  on  $F_V$  is “-”, then we label  $w_{j_p}^1$  on  $S^1$  with “ $\circ$ ”, and label  $w_{j_p}^2$  on  $S^2$  with “ $\times$ ”. For each  $r \in I(v)$ , by (5) in Proposition 2.3, the label  $v_{r_p}^1$  on  $S^1$  is “-”. Then,  $v_{r_p}^1$  is labelled with “ $\circ$ ”. Hence,  $v_{r_p}^1$  is retained. So,  $V_{x_p}$  is a properly embedded disk in  $V^1$ . For each  $r \in I(w)$ , by (6) in Proposition 2.3, the label  $w_{r_p}^1$  on  $S^1$  is “-”. Then,  $w_{r_p}^1$  is labeled with “ $\circ$ ”. Hence,  $w_{r_p}^1$  is retained. So,  $W_{x_p}$  is a properly embedded disk in  $W^1$ . Since both  $v_{r_p}^1$  and  $w_{r_p}^1$  are retained,  $v_{r_p}^1 \cap w_{r_p}^1 = \emptyset$ . By (\*),  $V_{x_p} \cap W_{x_p} = x$ . So,  $M^1 = V^1 \cup_{S^1} W^1$  is stabilized.  $\square$

By Proposition 2.4, Theorem 1.2 holds.  $\square$

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