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# ASYMPTOTIC STABILITY OF STRONG SOLUTIONS FOR EVOLUTION EQUATIONS WITH NONLOCAL INITIAL CONDITIONS

Pengyu Chen, Yibo Kong, and Yongxiang Li

Abstract. This paper is concerned with the global asymptotic stability of strong solutions for a class of semilinear evolution equations with nonlocal initial conditions on infinite interval. The discussion is based on analytic semigroups theory and the gradually regularization method. The results obtained in this paper improve and extend some related conclusions on this topic.

## 1. Introduction and main results

The theory of nonlocal Cauchy problem for abstract evolution equations was motivated by physical problems. Indeed, it is demonstrated that the nonlocal problems have better effects in applications than the classical Cauchy problems. For instance, nonlocal Cauchy problems have been used to represent mathematical models for evolution of various phenomena, such as nonlocal neural networks, nonlocal pharmacokinetics, nonlocal pollution and nonlocal combustion, see McKibben [13] for the details. Due to nonlocal problems have a wide range of applications in real world applications, differential or integro-differential equations with nonlocal initial conditions were studied by many authors and some basic results on nonlocal problems have been obtained, see [2, 3, 6, 8, 12, 16, 19] and the references therein. But we observed that all of the existing articles are only devoted to investigate the local existence of solutions for evolution equations with nonlocal initial conditions on finite interval, we haven't seen the relevant paper to study the global existence of solutions for nonlocal evolution equations on infinite interval. In addition, to the best of the authors' knowledge, in most of the existing articles, such as [6, 8, 12, 16], the existence of mild solutions for nonlocal evolution equations have been studied extensively, but there are very few paper studied the regularity for evolution equations with nonlocal initial conditions. Only [2, 3, 19] studied the local

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regularity of solutions for evolution equations with nonlocal initial conditions on finite interval and the assumptions are very strong.

On the other hand, the dynamical characteristics (including stable, unstable, attract, oscillatory and chaotic behavior) of differential equations have become a subject of intense research activities. For the details of this field, we refer the reader to the monographs of Burton [1], Hale [9] and the papers of Caicedo et al. [4], Chen and Guo [5], Li and Wang [11], Wang, Liu and Liu [17], Zhu, Liu and Li [20]. As far as we know, no work has been done for the asymptotic stability of strong solutions for nonlocal evolution equations. Motivated by the above-mentioned aspects, in this work we discuss the global asymptotic stability of strong solutions for a class of semilinear evolution equations with nonlocal initial conditions

$$u'(t) + Au(t) = f(t, u(t)), \quad t \geq 0, \tag{1.1}$$

$$u(0) = \sum_{k=1}^{\infty} \gamma_k u(t_k) \tag{1.2}$$

on infinite interval $\mathbb{R}_+ = [0, +\infty)$, where $\mathbb{H}$ is a Hilbert space, $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is a positive definite self-adjoint operator, $f : \mathbb{R}_+ \times \mathbb{H} \to \mathbb{H}$ is a nonlinear function satisfying some assumptions, $0 < t_1 < t_2 < \cdots < t_k < \cdots$, $t_k \to \infty \, (k \to \infty)$, $\gamma_k$ are real numbers, $\gamma_k \neq 0$, $k = 1, 2, \ldots$.

Our main results are as follows:

**Theorem 1.1.** *Let $A$ be a positive definite self-adjoint operator in Hilbert space $\mathbb{H}$ and it have compact resolvent. Assume that the nonlinear function $f\colon \mathbb{R}_+ \times \mathbb{H} \to \mathbb{H}$ is continuous and $f(\cdot, u(\cdot)) \in L^2(\mathbb{R}_+, \mathbb{H})$ for any $u \in \mathbb{H}$, $\|f(\cdot, \theta)\| \in L^1(\mathbb{R}_+)$. If the following conditions*

- $(C_1)$ $\sum\limits_{k=1}^{\infty} |\gamma_k| < e^{\lambda_1 t_1}$, *where $\lambda_1 > 0$ is the first eigenvalue of operator $A$;*
- $(C_2)$ *There exists a constant* $0 \leq L < \frac{\lambda_1 (1 - e^{-\lambda_1 t_1} \sum_{k=1}^{\infty} |\gamma_k|)}{1 + (1 - e^{-\lambda_1 t_1}) \sum_{k=1}^{\infty} |\gamma_k|}$ *such that*

$$\|f(t, u(t)) - f(t, v(t))\| \leq L \|u(t) - v(t)\|, \quad \forall \, t \in \mathbb{R}_+, \, u, v \in \mathbb{H},$$

*Bour's theorem hold, then the nonlocal problem* (1.1)-(1.2) *has a unique global strong solution $u^\star \in W^{1,2}(\mathbb{R}_+, \mathbb{H}) \cap L^2(\mathbb{R}_+, D(A))$ and it is globally asymptotically stable.*

If we replace the condition $(C_2)$ by the following condition

- $(C_3)$ There exists a nonnegative function $\alpha$, which is Lebesgue integrable and satisfying $\int_0^\infty \alpha(s) ds < \frac{1 - e^{-\lambda_1 t_1} \sum_{k=1}^{\infty} |\gamma_k|}{1 + (1 - e^{-\lambda_1 t_1}) \sum_{k=1}^{\infty} |\gamma_k|}$ such that

$$\|f(t, u(t)) - f(t, v(t))\| \leq \alpha(t) \|u(t) - v(t)\|, \quad \forall \, t \in \mathbb{R}_+, \, u, v \in \mathbb{H},$$

then we can obtain the following result.

**Theorem 1.2.** *Let $A$ be a positive definite self-adjoint operator in Hilbert space $\mathbb{H}$ and it have compact resolvent. Assume that the nonlinear function $f$: $\mathbb{R}_+\times\mathbb{H}\to\mathbb{H}$ is continuous and $f(\cdot,u(\cdot))\in L^2(\mathbb{R}_+,\mathbb{H})$ for any $u\in\mathbb{H}$, $\|f(\cdot,\theta)\|\in L^1(\mathbb{R}_+)$. If the conditions $(C_1)$ and $(C_3)$ are satisfied, then the nonlocal problem (1.1)-(1.2) has a unique global strong solution $u\in W^{1,2}(\mathbb{R}_+,\mathbb{H})\cap L^2(\mathbb{R}_+,D(A))$.*

## 2. Preliminaries

Let $\mathbb{H}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$, then $\|\cdot\|=\sqrt{\langle\cdot,\cdot\rangle}$ is the norm on $\mathbb{H}$ induced by inner product $\langle\cdot,\cdot\rangle$. Denote

$$C_b(\mathbb{R}_+,\mathbb{H})=\{u\mid u:\mathbb{R}_+\to\mathbb{H}\text{ is continuous and }u(t)\text{ is bounded for all }t\in\mathbb{R}_+\}.$$

Then it is easy to verify that $C_b(\mathbb{R}_+,\mathbb{H})$ is a Banach space endowed with the norm

$$\|u\|_b=\sup_{t\in\mathbb{R}_+}\|u(t)\|,\quad \forall\, u\in C_e(\mathbb{R}_+,\mathbb{H}).$$

We denote by $\mathcal{L}(\mathbb{H})$ the Banach space of all linear and bounded operators in $\mathbb{H}$, and by $\theta$ the zero element in $\mathbb{H}$.

Throughout this paper, we assume that $A:D(A)\subset\mathbb{H}\to\mathbb{H}$ is a positive definite self-adjoint operator in Hilbert space $\mathbb{H}$ and it have compact resolvent. By the spectral resolution theorem of self-adjoint operator, the spectrum $\sigma(A)$ only consists of real eigenvalues and it can be arrayed in sequences as

$$0<\lambda_1\le\lambda_2\le\cdots\le\lambda_k\le\cdots,\qquad \lim_{k\to+\infty}\lambda_k=\infty.$$

By the positive definite property of $A$, the first eigenvalue $\lambda_1>0$. From Henry [10] and Pazy [14], we know that $-A$ generates an analytic operator semigroup $S(t)$ $(t\ge0)$ on $\mathbb{H}$, which is exponentially stable and satisfies $\|S(t)\|_{\mathcal{L}(\mathbb{H})}\le e^{-\lambda_1 t}$ for $t\ge0$. Since the positive definite self-adjoint operator $A$ has compact resolvent, the embedding $D(A)\hookrightarrow\mathbb{H}$ is compact, and therefore $S(t)$ $(t\ge0)$ is also a compact semigroup.

Next, we give some concepts and conclusions on the fractional powers of $A$. For $\alpha>0$, $A^{-\alpha}$ is defined by

$$A^{-\alpha}=\frac{1}{\Gamma(\alpha)}\int_0^\infty s^{\alpha-1}S(s)ds,$$

where $\Gamma(\cdot)$ is the Gamma function. $A^{-\alpha}\in\mathcal{L}(\mathbb{H})$ is injective, and $A^\alpha$ can be defined by $A^\alpha=(A^{-\alpha})^{-1}$ with the domain $D(A^\alpha)=A^{-\alpha}(\mathbb{H})$. For $\alpha=0$, let $A^\alpha=I$. We endow an inner product $\langle\cdot,\cdot\rangle_\alpha=\langle A^\alpha\cdot,A^\alpha\cdot\rangle$ to $D(A^\alpha)$. Since $A^\alpha$ is a closed linear operator, it follows that $(D(A^\alpha),\langle\cdot,\cdot\rangle_\alpha)$ is a Hilbert space. We denote by $\mathbb{H}_\alpha$ the Hilbert space $(D(A^\alpha),\langle\cdot,\cdot\rangle_\alpha)$. Especially, $\mathbb{H}_0=\mathbb{H}$ and $\mathbb{H}_1=D(A)$. For $0\le\alpha<\beta$, $\mathbb{H}_\beta$ is densely embedded into $\mathbb{H}_\alpha$ and the embedding $\mathbb{H}_\beta\hookrightarrow\mathbb{H}_\alpha$ is compact. For the details, we refer to [10] and [18].

From [14, Chapter 4, Corollary 2.5], we know that for any $u_0 \in D(A)$, if the linear function $h$ is continuously differentiable on $\mathbb{R}_+$, then the initial value problem of linear evolution equation (LIVP)

$$\begin{cases} u'(t) + Au(t) = h(t), \quad t \in \mathbb{R}_+, \\ u(0) = u_0 \end{cases} \tag{2.1}$$

exists a unique classical solution $u \in C^1((0, +\infty), \mathbb{H}) \cap C((0, +\infty), D(A)) \cap C(\mathbb{R}_+, \mathbb{H})$ expressed by

$$u(t) = S(t)u_0 + \int_0^t S(t-s)h(s)ds. \tag{2.2}$$

If $u_0 \in \mathbb{H}$ and $h \in L^1(\mathbb{R}_+, \mathbb{H})$, the function $u$ given by (2.2) belongs to $C(\mathbb{R}_+, \mathbb{H})$, which is known as a mild solution of the LIVP (2.1). If a mild solution $u$ of the LIVP (2.1) belongs to $W^{1,1}(\mathbb{R}_+, \mathbb{H}) \cap L^1(\mathbb{R}_+, D(A))$ and satisfies the equation for a.e. $t \in \mathbb{R}_+$, we call it a strong solution. By [14, Chapter 4, Corollary 2.10], we know that for any $u_0 \in D(A)$, if the linear function $h$ is differentiable on $\mathbb{R}_+$, then LIVP (2.1) exists a unique strong solution.

Define an operator $\mathbb{B}$ by

$$\mathbb{B} = \Big(I - \sum_{k=1}^{\infty} \gamma_k S(t_k)\Big)^{-1}.$$

Then by condition $(C_1)$, we know that

$$\Big\| \sum_{k=1}^{\infty} \gamma_k S(t_k) \Big\| \le \sum_{k=1}^{\infty} |\gamma_k| e^{-\lambda_1 t_1} < 1.$$

Therefore, from operator spectrum theorem we know that the operator $\mathbb{B}$ exists and it is bounded. Furthermore, by Neumann expression, $\mathbb{B}$ can be expressed by

$$\mathbb{B} = \sum_{n=0}^{\infty} \Big( \sum_{k=1}^{\infty} \gamma_k S(t_k) \Big)^n.$$

Hence

$$\|\mathbb{B}\| \le \sum_{n=0}^{\infty} \Big\| \sum_{k=1}^{\infty} \gamma_k S(t_k) \Big\|^n = \frac{1}{1 - \Big\| \sum\limits_{k=1}^{\infty} \gamma_k S(t_k) \Big\|} \le \frac{1}{1 - e^{-\lambda_1 t_1} \sum\limits_{k=1}^{\infty} |\gamma_k|}.$$

To prove our main results, for any $h \in C_b(\mathbb{R}_+, \mathbb{H})$, we consider the linear evolution equation nonlocal problem (LEENP)

$$u'(t) + Au(t) = h(t), \quad t \in \mathbb{R}_+, \tag{2.3}$$

$$u(0) = \sum_{k=1}^{\infty} \gamma_k u(t_k). \tag{2.4}$$

**Lemma 2.1.** *If the condition* $(C_1)$ *is satisfied, then the LEENP* (2.3)-(2.4) *exists a unique mild solution* $u \in C_b(\mathbb{R}_+, \mathbb{H})$ *which is given by*

$$u(t) = \sum_{k=1}^{\infty} \gamma_k S(t)\mathbb{B} \int_0^{t_k} S(t_k - s)h(s)ds + \int_0^t S(t-s)h(s)ds, \quad t \in \mathbb{R}_+. \tag{2.5}$$

*Proof.* By the above discussion, (2.1) and (2.2), we know that the evolution equation (2.3) exists a unique mild solution $u \in C_b(\mathbb{R}_+, \mathbb{H})$ which can be expressed by

$$u(t) = S(t)u(0) + \int_0^t S(t-s)h(s)ds. \tag{2.6}$$

By (2.6), we know that for every $k = 1, 2, \ldots,$

$$u(t_k) = S(t_k)u(0) + \int_0^{t_k} S(t_k - s)h(s)ds. \tag{2.7}$$

From (2.4) and (2.7), we have

$$u(0) = \sum_{k=1}^{\infty} \gamma_k S(t_k)u(0) + \sum_{k=1}^{\infty} \gamma_k \int_0^{t_k} S(t_k - s)h(s)ds. \tag{2.8}$$

Since $I - \sum_{k=1}^{\infty} \gamma_k S(t_k)$ has a bounded inverse operator $\mathbb{B}$, by (2.8) we know that

$$u(0) = \sum_{k=1}^{\infty} \gamma_k \mathbb{B} \int_0^{t_k} S(t_k - s)h(s)ds. \tag{2.9}$$

Combining (2.6) and (2.9), we get that the mild solution $u$ satisfies (2.5).

Inversely, we can verify directly that the function $u \in C_b(\mathbb{R}_+, \mathbb{H})$ given by (2.5) is a mild solution of LEENP (2.3)-(2.4). □

The following two lemmas will be used in the proof of our main results.

**Lemma 2.2** ([15, Chapter II, Theorem 3.3])**.** *Assume that* $\mathbb{V}$ *and* $\mathbb{H}$ *are two Hilbert space,* $\mathbb{V} \subset \mathbb{H}$*,* $\mathbb{V}$ *denses in* $\mathbb{H}$*, the injection is continuous and compact,* $A : D(A) \subset \mathbb{H} \to \mathbb{V}$ *is a positive definite self-adjoint operator in* $\mathbb{H}$*. Then for any* $u_0 \in \mathbb{V}$ *and* $h \in L^2(\mathbb{R}_+, \mathbb{V})$*, the mild solution of the LIVP* (2.1) *has the regularity*

$$u \in W^{1,2}(\mathbb{R}_+, \mathbb{H}) \cap L^2(\mathbb{R}_+, D(A)) \cap C(\mathbb{R}_+, \mathbb{V}).$$

**Lemma 2.3** ([7])**.** *If*

$$m(t) \le g(t) + \int_0^t k(s)m(s)ds, \qquad t \in [0, T),$$

*where all the functions involved are continuous on* $[0, T)$*,* $T \le +\infty$*, and* $k(t) \ge 0$*,* $g(t)$ *is nondecreasing. Then*

$$m(t) \le g(t) \exp\Big(\int_0^t k(s)ds\Big), \qquad t \in [0, T).$$

## 3. Proof of the main results

*Proof of Theorem 1.1.* Firstly, we prove the uniqueness of the global strong solution for nonlocal problem (1.1)-(1.2). Consider the operator $\mathbb{Q}$ on $C_b(\mathbb{R}_+, \mathbb{H})$ defined by

$$
\begin{aligned}
(\mathbb{Q}u)(t) = & \sum_{k=1}^{\infty} \gamma_k S(t)\mathbb{B} \int_0^{t_k} S(t_k - s) f(s, u(s)) ds \\
& + \int_0^t S(t-s) f(s, u(s)) ds, \quad t \in \mathbb{R}_+.
\end{aligned} \tag{3.1}
$$

By the condition $(C_1)$ and Lemma 2.1, it is easy to see that the mild solution of nonlocal problem (1.1)-(1.2) is equivalent to the fixed point of the operator $\mathbb{Q}$ defined by (3.1).

Next, we will show that the operator $\mathbb{Q}$ maps the functions in $C_b(\mathbb{R}_+, \mathbb{H})$ to $C_b(\mathbb{R}_+, \mathbb{H})$. For any $u \in C_b(\mathbb{R}_+, \mathbb{H})$, by the condition $(C_2)$, we know that

$$
\|f(t, u(t))\| \le L\|u(t)\| + \|f(t, \theta)\|, \qquad t \in \mathbb{R}_+. \tag{3.2}
$$

Therefore, by (3.1) and (3.2), we get that

$$
\begin{aligned}
\|(\mathbb{Q}u)(t)\| \le & \Big\| \sum_{k=1}^{\infty} \gamma_k S(t)\mathbb{B} \int_0^{t_k} S(t_k - s) f(s, u(s)) ds \Big\| \\
& + \Big\| \int_0^t S(t-s) f(s, u(s)) ds \Big\| \\
\le & \sum_{k=1}^{\infty} |\gamma_k| e^{-\lambda_1 t} \|\mathbb{B}\| \int_0^{t_k} e^{-\lambda_1 (t_k - s)} [L\|u(s)\| + \|f(s, \theta)\|] ds \\
& + \int_0^t e^{-\lambda_1 (t-s)} [L\|u(s)\| + \|f(s, \theta)\|] ds \\
\le & \frac{\sum_{k=1}^{\infty} |\gamma_k|}{1 - e^{-\lambda_1 t_1} \sum_{k=1}^{\infty} |\gamma_k|} \int_0^t e^{-\lambda_1 (t-s)} [L\|u\|_b + \|f(s, \theta)\|] ds \\
& + \int_0^t e^{-\lambda_1 (t-s)} [L\|u\|_b + \|f(s, \theta)\|] ds \\
\le & \frac{1 + (1 - e^{-\lambda_1 t_1}) \sum_{k=1}^{\infty} |\gamma_k|}{1 - e^{-\lambda_1 t_1} \sum_{k=1}^{\infty} |\gamma_k|} \Big[ \frac{\|u\|_b L}{\lambda_1} + \int_0^{\infty} \|f(s, \theta)\| ds \Big].
\end{aligned} \tag{3.3}
$$

By (3.3), the condition $(C_2)$ and the fact that $\|f(t, \theta)\| \in L^1(\mathbb{R}_+)$, we know that

$$
\|\mathbb{Q}u\|_b = \sup_{t \in \mathbb{R}_+} \|(\mathbb{Q}u)(t)\| < +\infty,
$$

which means that $\mathbb{Q}u \in C_b(\mathbb{R}_+, \mathbb{H})$.

For any $u, v \in C_b(\mathbb{R}_+, \mathbb{H})$, by (3.1) and the condition $(C_2)$, we get that

$$\begin{aligned}
\|(\mathbb{Q}u)(t) - (\mathbb{Q}v)(t)\| &\le \sum_{k=1}^{\infty} |\gamma_k| e^{-\lambda_1 t} \|\mathbb{B}\| \int_0^{t_k} e^{-\lambda_1(t_k - s)} \\
&\quad \times \|f(s, u(s)) - f(s, v(s))\| ds \\
&\quad + \int_0^t e^{-\lambda_1(t-s)} \|f(s, u(s)) - f(s, v(s))\| ds \\
&\le \frac{\sum_{k=1}^{\infty} |\gamma_k|}{1 - e^{-\lambda_1 t_1} \sum_{k=1}^{\infty} |\gamma_k|} \int_0^t e^{-\lambda_1(t-s)} L \|u - v\|_b ds \\
&\quad + \int_0^t e^{-\lambda_1(t-s)} L \|u - v\|_b ds \\
&\le \frac{1 + (1 - e^{-\lambda_1 t_1}) \sum_{k=1}^{\infty} |\gamma_k|}{\lambda_1 (1 - e^{-\lambda_1 t_1} \sum_{k=1}^{\infty} |\gamma_k|)} L \|u - v\|_b.
\end{aligned} \tag{3.4}$$

By (3.4) and condition $(C_2)$, we know that

$$\|\mathbb{Q}u - \mathbb{Q}v\|_b = \sup_{t \in \mathbb{R}_+} \|(\mathbb{Q}u)(t) - \mathbb{Q}v)(t)\| < \|u - v\|_b.$$

Hence, $\mathbb{Q} : C_b(\mathbb{R}_+, \mathbb{H}) \to C_b(\mathbb{R}_+, \mathbb{H})$ is a contraction operator, and therefore $\mathbb{Q}$ has a unique fixed point $u^\star \in C_b(\mathbb{R}_+, \mathbb{H})$, which is in turn the unique mild solution of nonlocal problem (1.1)-(1.2) on $\mathbb{R}_+$. Since $u^\star$ is the unique mild solution of LEENP (2.3)-(2.4) for $h(\cdot) = f(\cdot, u^\star(\cdot))$ and $h(\cdot) = f(\cdot, u^\star(\cdot)) \in L^2(\mathbb{R}_+, \mathbb{H})$, by the maximal regularity of linear evolution equations with positive definite operator in Hilbert spaces (see for details Lemma 2.2), when $u^\star(0) = u_0 \in \mathbb{V} := \mathbb{H}_{\frac{1}{2}}$, the mild solution of the LEENP (2.3)-(2.4) has the regularity

$$u^\star \in W^{1,2}(\mathbb{R}_+, \mathbb{H}) \cap L^2(\mathbb{R}_+, D(A)) \cap C(\mathbb{R}_+, \mathbb{H}_{\frac{1}{2}}) \tag{3.5}$$

and it is a strong solution.

We noticed that $u^\star(t)$ is the mild solution of the LEENP (2.3)-(2.4) for

$$u^\star(0) = \sum_{k=1}^{\infty} \gamma_k \mathbb{B} \int_0^{t_k} S(t_k - s) h(s) ds.$$

By the representation (2.2) of mild solution, $u^\star(t) = S(t)u^\star(0) + v(t)$, where $v(t) = \int_0^t S(t-s)h(s)ds$. Since the function $v(t)$ is a mild solution of the LEENP (2.3)-(2.4) with the null initial value $u^\star(0) = \theta$, $v$ has the regularity (3.5). By the analytic property of the semigroup $S(t)$, $S(t_k)u^\star(0) \in D(A) \subset \mathbb{H}_{1/2}$. Hence,

$$u^\star(0) = \sum_{k=1}^{\infty} \gamma_k S(t_k)u^\star(0) + \sum_{k=1}^{\infty} \gamma_k v(t_k) \in \mathbb{H}_{1/2}.$$

Using the regularity (3.5) again, we obtain that $u^\star \in W^{1,2}(\mathbb{R}_+, \mathbb{H}) \cap L^2(\mathbb{R}_+, D(A))$ and it is a strong solution of the LEENP (2.3)-(2.4), which means that the unique fixed point $u^\star$ of the operator $\mathbb{Q}$ defined by (3.1) belongs to $W^{1,2}(\mathbb{R}_+, \mathbb{H}) \cap L^2(\mathbb{R}_+, D(A))$ is the unique global strong solution of the nonlocal problem (1.1)-(1.2).

Secondly, we investigate the global asymptotic stability of strong solution for nonlocal problem (1.1)-(1.2). By the condition $(C_2)$, Banach contraction theorem, Lemma 2.2, the method used in the proof of the regularity for $u^\star$ and the fact $f(\cdot, u(\cdot)) \in L^2(\mathbb{R}_+, \mathbb{H})$, we know that for any $u_0 \in \mathbb{H}$, the initial value problem of evolution equation

$$\begin{cases} u'(t) + Au(t) = f(t, u(t)), \quad t \in \mathbb{R}_+, \\ u(0) = u_0 \end{cases} \tag{3.6}$$

exists a unique global strong solution $u \in W^{1,2}(\mathbb{R}_+, \mathbb{H}) \cap L^2(\mathbb{R}_+, D(A))$ and it satisfies

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s, u(s))ds, \quad t \in \mathbb{R}_+. \tag{3.7}$$

From the semigroup representation of the solutions, the unique global strong solution $u^\star$ of the nonlocal problem (1.1)-(1.2) satisfies the integral equation

$$u^\star(t) = S(t)u^\star(0) + \int_0^t S(t-s)f(s, u^\star(s))ds, \quad t \in \mathbb{R}_+. \tag{3.8}$$

From (3.7), (3.8) and the condition $(C_2)$, we get that

$$\begin{aligned} \|u^\star(t) - u(t)\| &\le \|S(t)\|\|u^\star(0) - u_0\| \\ &\quad + \int_0^t \|S(t-s)\|\|f(s, u^\star(s)) - f(s, u(s))\|ds \\ &\le e^{-\lambda_1 t}\|u^\star(0) - u(0)\| + \int_0^t e^{-\lambda_1(t-s)} L\|u^\star(s) - u(s)\|ds \\ &\le e^{-\lambda_1 t}\|u^\star(0) - u(0)\| \\ &\quad + e^{-\lambda_1 t}\int_0^t Le^{\lambda_1 s}\|u^\star(s) - u(s)\|ds, \quad t \in \mathbb{R}_+. \end{aligned} \tag{3.9}$$

Let $m(t) = e^{\lambda_1 t}\|u^\star(t) - u(t)\|$, $t \in \mathbb{R}_+$. By (3.9), we know that

$$m(t) \le m(0) + \int_0^t Lm(s)ds, \quad t \in \mathbb{R}_+. \tag{3.10}$$

Therefore, by Lemma 2.3 and (3.10), we get that

$$m(t) = e^{\lambda_1 t}\|u^\star(t) - u(t)\| \le m(0)e^{\int_0^t L ds}, \quad t \in \mathbb{R}_+. \tag{3.11}$$

Set $\rho := \lambda_1 - L$, from the condition $(C_2)$, we know that $\rho > 0$. Therefore, by (3.11), we have

$$\|u^\star(t) - u(t)\| \le m(0)e^{-\rho t} \to 0 \quad (t \to +\infty).$$

Hence, the global strong solution $u^\star$ is globally asymptotically stable. Furthermore, from the proof process, we easily see that the global strong solution $u^\star$ exponentially attracts every strong solution of the initial value problem (3.6). □

*Proof of Theorem 1.2.* By Theorem 1.1, we know that the mild solution of the nonlocal problem (1.1)-(1.2) is equivalent to the fixed point of the operator $\mathbb{Q}$ defined by (3.1). For any $u \in C_b(\mathbb{R}_+, \mathbb{H})$, by the condition $(C_3)$ we know that for any $t \in \mathbb{R}_+$,

$$\|f(t, u(t))\| \le \alpha(t)\|u(t)\| + \|f(t, \theta)\|. \tag{3.12}$$

Therefore, by (3.1), (3.12) and the condition $(C_3)$, we get that

$$\begin{aligned}
\|(\mathbb{Q}u)(t)\| &\le \Big\| \sum_{k=1}^{\infty} \gamma_k S(t)\mathbb{B} \int_0^{t_k} S(t_k - s) f(s, u(s)) ds \Big\| \\
&\quad + \Big\| \int_0^t S(t-s) f(s, u(s)) ds \Big\| \\
&\le \sum_{k=1}^{\infty} |\gamma_k| e^{-\lambda_1 t}\|\mathbb{B}\| \int_0^{t_k} e^{-\lambda_1(t_k - s)}[\alpha(s)\|u(s)\| + \|f(s,\theta)\|] ds \\
&\quad + \int_0^t e^{-\lambda_1(t-s)}[\alpha(s)\|u(s)\| + \|f(s,\theta)\|] ds \\
&\le \frac{\sum_{k=1}^{\infty} |\gamma_k|}{1 - e^{-\lambda_1 t_1}\sum_{k=1}^{\infty}|\gamma_k|} \int_0^t [\alpha(s)\|u\|_b + \|f(s,\theta)\|] ds \\
&\quad + \int_0^t [\alpha(s)\|u\|_b + \|f(s,\theta)\|] ds \\
&< \|u\|_b + \frac{1 + (1 - e^{-\lambda_1 t_1})\sum_{k=1}^{\infty}|\gamma_k|}{1 - e^{-\lambda_1 t_1}\sum_{k=1}^{\infty}|\gamma_k|} \int_0^{\infty} \|f(s,\theta)\| ds.
\end{aligned} \tag{3.13}$$

By (3.13) and the fact that $\|f(t,\theta)\| \in L^1(\mathbb{R}_+)$, we know that

$$\|\mathbb{Q}u\|_b = \sup_{t \in \mathbb{R}_+} \|(\mathbb{Q}u)(t)\| < +\infty,$$

which means that $\mathbb{Q}u \in C_b(\mathbb{R}_+, \mathbb{H})$. Therefore, the operator $\mathbb{Q}$ maps the functions in $C_b(\mathbb{R}_+, \mathbb{H})$ to $C_b(\mathbb{R}_+, \mathbb{H})$.

For any $u, v \in C_b(\mathbb{R}_+, \mathbb{H})$, by (3.1) and the condition $(C_3)$, we get that

$$\begin{aligned}
\|(\mathbb{Q}u)(t) - (\mathbb{Q}v)(t)\| &\le \sum_{k=1}^{\infty} |\gamma_k| e^{-\lambda_1 t} \|\mathbb{B}\| \int_0^{t_k} e^{-\lambda_1 (t_k - s)} \\
&\quad \times \|f(s, u(s)) - f(s, v(s))\| ds \\
&\quad + \int_0^t e^{-\lambda_1 (t-s)} \|f(s, u(s)) - f(s, v(s))\| ds \\
&\le \frac{\sum\limits_{k=1}^{\infty} |\gamma_k|}{1 - e^{-\lambda_1 t_1} \sum\limits_{k=1}^{\infty} |\gamma_k|} \int_0^t e^{-\lambda_1 (t-s)} \alpha(s) \|u - v\|_b ds \\
&\quad + \int_0^t e^{-\lambda_1 (t-s)} \alpha(s) \|u - v\|_b ds \\
&\le \frac{1 + (1 - e^{-\lambda_1 t_1}) \sum\limits_{k=1}^{\infty} |\gamma_k|}{1 - e^{-\lambda_1 t_1} \sum\limits_{k=1}^{\infty} |\gamma_k|} \int_0^{\infty} \alpha(s) ds \|u - v\|_b \\
&< \|u - v\|_b,
\end{aligned} \tag{3.14}$$

from which we know that

$$\|\mathbb{Q}u - \mathbb{Q}v\|_b = \sup_{t \in \mathbb{R}_+} \|(\mathbb{Q}u)(t) - \mathbb{Q}v)(t)\| < \|u - v\|_b.$$

Hence, $\mathbb{Q} : C_b(\mathbb{R}_+, \mathbb{H}) \to C_b(\mathbb{R}_+, \mathbb{H})$ is a contraction operator, and therefore $\mathbb{Q}$ has a unique fixed point $u \in C_b(\mathbb{R}_+, \mathbb{H})$, which is in turn the unique mild solution of nonlocal problem (1.1)-(1.2) on $\mathbb{R}_+$. By using a completely similar method with which used in the proof of Theorem 1.1, we can prove that $u \in W^{1,2}(\mathbb{R}_+, \mathbb{H}) \cap L^2(\mathbb{R}_+, D(A))$ is the unique global strong solution of the nonlocal problem (1.1)-(1.2). □

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Pengyu Chen
Department of Mathematics
Northwest Normal University
Lanzhou 730070, P. R. China
*E-mail address*: chpengyu123@163.com

Yibo Kong
Department of Mathematics
Northwest Normal University
Lanzhou 730070, P. R. China
*E-mail address*: kongyb@163.com

Yongxiang Li
Department of Mathematics
Northwest Normal University
Lanzhou 730070, P. R. China
*E-mail address*: liyxnwnu@163.com