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ON THE COHOMOLOGICAL DIMENSION OF FINITELY GENERATED MODULES

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Abstract. Let $(R,\mathfrak{m})$ be a commutative Noetherian local ring and I be an ideal of R. In this paper it is shown that if $\mathrm{cd}(I,R)=t>0$ and the R-module $\mathrm{Hom}_R(R/I,H_I^t(R))$ is finitely generated, then

$$t=\sup\{\dim \widehat{\widehat{R}_{\mathfrak{P}}}/\mathfrak{Q} : \mathfrak{P}\in V(I\widehat{R}),\ \mathfrak{Q}\in \mathrm{mAss}_{\widehat{\widehat{R}_{\mathfrak{P}}}}\widehat{\widehat{R}_{\mathfrak{P}}} \text{ and } \mathfrak{P}\widehat{\widehat{R}_{\mathfrak{P}}}=\mathrm{Rad}(I\widehat{\widehat{R}_{\mathfrak{P}}}+\mathfrak{Q})\}.$$

Moreover, some other results concerning the cohomological dimension of ideals with respect to the rings extension $R\subset R[X]$ will be included.

1. Introduction

Throughout this paper, let R denote a commutative Noetherian ring (with identity) and I be an ideal of R. For each R-module L, we denote by $\mathrm{mAss}_R L$ the set of minimal elements of $\mathrm{Ass}_R L$ with respect to inclusion. For any ideal $\mathfrak{a}$ of R, we denote $\{\mathfrak{p}\in \mathrm{Spec}\,R : \mathfrak{p}\supseteq\mathfrak{a}\}$ by $V(\mathfrak{a})$. For any ideal $\mathfrak{b}$ of R, *the radical of* $\mathfrak{b}$, denoted by $\mathrm{Rad}(\mathfrak{b})$, is defined to be the set $\{x\in R : x^n\in\mathfrak{b}$ for some $n\in\mathbb{N}\}$. Also, for each R-module M, we denote by $\mathrm{Att}_R M$ the set of all *attached prime ideals of* M. For any ideal I of R, the *I-adic completion* of R is denoted by $\widehat{R}$. For any unexplained notation and terminology we refer the reader to [15].

The local cohomology modules $H_I^i(M)$, $i=0,1,2,\dots$, of an R-module M with respect to I were introduced by Grothendieck, [9]. They arise as the derived functors of the left exact functor $\Gamma_I(-)$, where for an R-module M, $\Gamma_I(M)$ is the submodule of M consisting of all elements annihilated by some power of I, i.e., $\bigcup_{n=1}^{\infty}(0:_M I^n)$. There is a natural isomorphism:

$$H_I^i(M)\cong \varinjlim_{n\geq 1}\ \mathrm{Ext}_R^i(R/I^n,M).$$

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We refer the reader to [5] or [9] for more details about local cohomology.

Recall that, for an R-module M, the *cohomological dimension* of M with respect to I, denoted by $\operatorname{cd}(I, M)$, is defined as

$$\operatorname{cd}(I, M) := \sup\{i \in \mathbb{Z} : H_I^i(M) \neq 0\}.$$

One of the important problems in commutative algebra is determining the last non-zero local cohomology modules. In particular, it is very important to determine the cohomological dimension of modules with respect to ideals. The notions of cohomological dimension have produced some interesting results in local algebra. This notion have been studied by several authors; see, for example, Faltings [7], Hartshorne [10], Huneke-Lyubeznik [14], Divaani-Aazar et al. [6], Hellus [12], Hellus-Stückrad [13], Mehrvarz et al. [16] and Ghasemi et al. [8].

The main aim of this paper is to find new relation between the cohomological dimension and the Krull's dimension, under the assumption that $\operatorname{Hom}_R(R/I, H_I^t(R))$ is finitely generated, where $t = \operatorname{cd}(I, R) > 0$.

2. Cohomological dimension of modules

Let I be an ideal of a Noetherian ring R. Recall that, an R-module M is said to be *I-cofinite* if $\operatorname{Supp} M \subseteq V(I)$ and $\operatorname{Ext}_R^i(R/I, M)$ is finitely generated for all $i \geq 0$. We refer the reader to [11] for more details about cofinite modules.

The following two well known lemmata will be quite useful in this section.

Lemma 2.1. *Let $(R, \mathfrak{m})$ be a Noetherian complete local ring and let I be an ideal of R such that $\operatorname{cd}(I, R) = t$. If $H_I^t(R)$ is Artinian and I-cofinite, then*

$$\operatorname{Att}_R H_I^t(R) = \{\mathfrak{p} \in \operatorname{mAss}_R R \ : \ \dim R/\mathfrak{p} = t \text{ and } \operatorname{Rad}(I + \mathfrak{p}) = \mathfrak{m}\}.$$

Proof. See [1, Lemma 2.3]. □

Lemma 2.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be an ideal of R such that $\operatorname{cd}(I, R) = t$. Assume that $\operatorname{Supp} H_I^t(R) \subseteq \{\mathfrak{m}\}$ and the R-module $\operatorname{Hom}_R(R/I, H_I^t(R))$ is finitely generated. Then the R-module $H_I^t(R)$ is Artinian and I-cofinite. In particular, the $\widehat{R}$-module $H_{I\widehat{R}}^t(\widehat{R})$ is Artinian and $I\widehat{R}$-cofinite.*

Proof. See [1, Lemma 2.5]. □

Now we are ready to state and prove the main result of this paper.

Theorem 2.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be an ideal of R such that $\operatorname{cd}(I, R) = t > 0$. If the R-module $\operatorname{Hom}_R(R/I, H_I^t(R))$ is finitely generated, then*

$$t = \sup\{\dim \widehat{\widehat{R}_{\mathfrak{P}}}/\mathfrak{Q} : \mathfrak{P} \in V(I\widehat{R}),\ \mathfrak{Q} \in \operatorname{mAss}_{\widehat{\widehat{R}_{\mathfrak{P}}}} \widehat{\widehat{R}_{\mathfrak{P}}} \text{ and } \mathfrak{P}\widehat{\widehat{R}_{\mathfrak{P}}} = \operatorname{Rad}(I\widehat{\widehat{R}_{\mathfrak{P}}} + \mathfrak{Q})\}.$$

Proof. Set

$$k := \sup\{\dim \widehat{\widehat{R}_{\mathfrak{P}}}/\mathfrak{Q} : \mathfrak{P} \in V(I\widehat{R}),\ \mathfrak{Q} \in \operatorname{mAss}_{\widehat{\widehat{R}_{\mathfrak{P}}}} \widehat{\widehat{R}_{\mathfrak{P}}} \text{ and } \mathfrak{P}\widehat{\widehat{R}_{\mathfrak{P}}} = \operatorname{Rad}(I\widehat{\widehat{R}_{\mathfrak{P}}} + \mathfrak{Q})\}.$$

Since $\widehat{R}$ is a faithfully flat R-algebra, it follows that $\operatorname{cd}(I\widehat{R}, \widehat{R}) = t$ and the $\widehat{R}$-module $\operatorname{Hom}_{\widehat{R}}(\widehat{R}/I\widehat{R}, H^t_{I\widehat{R}}(\widehat{R}))$ is finitely generated. Let $\mathfrak{P} \in \operatorname{mAss}_{\widehat{R}} H^t_{I\widehat{R}}(\widehat{R})$. Then, $\mathfrak{P} \in V(I\widehat{R})$ and by Lemma 2.2, the non-zero $\widehat{R}_{\mathfrak{P}}$-module $H^t_{I\widehat{R}_{\mathfrak{P}}}(\widehat{R}_{\mathfrak{P}})$ is Artinian and $I\widehat{R}_{\mathfrak{P}}$-cofinite. Thus, the non-zero $\widehat{\widehat{R}_{\mathfrak{P}}}$-module $H^t_{I\widehat{\widehat{R}_{\mathfrak{P}}}}(\widehat{\widehat{R}_{\mathfrak{P}}})$ is Artinian and $I\widehat{\widehat{R}_{\mathfrak{P}}}$-cofinite. Therefore, in view of Lemma 2.1, there exists an element $\mathfrak{Q} \in \operatorname{mAss}_{\widehat{\widehat{R}_{\mathfrak{P}}}} \widehat{\widehat{R}_{\mathfrak{P}}}$ such that $\dim \widehat{\widehat{R}_{\mathfrak{P}}}/\mathfrak{Q} = t$ and

$$\operatorname{Rad}(I\widehat{\widehat{R}_{\mathfrak{P}}} + \mathfrak{Q}) = \mathfrak{P}\widehat{\widehat{R}_{\mathfrak{P}}}.$$

So, we have $k \geq t$.

On the other hand by the definition of k, there exists $\mathfrak{P}_1 \in V(I\widehat{R})$ with $\dim \widehat{\widehat{R}_{\mathfrak{P}_1}}/\mathfrak{Q}_1 = k$, for some $\mathfrak{Q}_1 \in \operatorname{mAss}_{\widehat{\widehat{R}_{\mathfrak{P}_1}}} \widehat{\widehat{R}_{\mathfrak{P}_1}}$ with $\mathfrak{P}_1\widehat{\widehat{R}_{\mathfrak{P}_1}} = \operatorname{Rad}(I\widehat{\widehat{R}_{\mathfrak{P}_1}} + \mathfrak{Q}_1)$. Then by the Grothendieck's Vanishing and Non-vanishing and the Independence Theorems it follows that

$$\begin{aligned}
t &= \operatorname{cd}(I, R)\\
&= \operatorname{cd}(I\widehat{R}, \widehat{R})\\
&\geq \operatorname{cd}(I\widehat{R}_{\mathfrak{P}_1}, \widehat{R}_{\mathfrak{P}_1})\\
&= \operatorname{cd}(I\widehat{\widehat{R}_{\mathfrak{P}_1}}, \widehat{\widehat{R}_{\mathfrak{P}_1}})\\
&\geq \operatorname{cd}(I\widehat{\widehat{R}_{\mathfrak{P}_1}}, \widehat{\widehat{R}_{\mathfrak{P}_1}}/\mathfrak{Q}_1)\\
&= \operatorname{cd}(I\widehat{\widehat{R}_{\mathfrak{P}_1}} + \mathfrak{Q}_1, \widehat{\widehat{R}_{\mathfrak{P}_1}}/\mathfrak{Q}_1)\\
&= \operatorname{cd}(\mathfrak{P}_1\widehat{\widehat{R}_{\mathfrak{P}_1}}, \widehat{\widehat{R}_{\mathfrak{P}_1}}/\mathfrak{Q}_1)\\
&= \dim \widehat{\widehat{R}_{\mathfrak{P}_1}}/\mathfrak{Q}_1\\
&= k.
\end{aligned}$$

So, we have $t = k$. □

The following four results are some consequences of Theorem 2.3.

Corollary 2.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be an ideal of R such that $\dim R/I = 1$. If $\operatorname{cd}(I, R) = t > 0$, then the R-module $\operatorname{Hom}_R(R/I, H^t_I(R))$ is finitely generated and so*

$$t = \sup\{\dim \widehat{\widehat{R}_{\mathfrak{P}}}/\mathfrak{Q} : \mathfrak{P} \in V(I\widehat{R}),\ \mathfrak{Q} \in \operatorname{mAss}_{\widehat{\widehat{R}_{\mathfrak{P}}}} \widehat{\widehat{R}_{\mathfrak{P}}} \text{ and } \mathfrak{P}\widehat{\widehat{R}_{\mathfrak{P}}} = \operatorname{Rad}(I\widehat{\widehat{R}_{\mathfrak{P}}} + \mathfrak{Q})\}.$$

Proof. The assertion follows from [3, Corollary 2.7] and Theorem 2.3. □

Let R be a Noetherian ring, I be an ideal of R and let M be a finitely generated R-module. Recall that following [4], for any non-negative integer n, the *n-th finiteness dimension* $f_I^n(M)$ of M relative to I is defined as

$$f_I^n(M) := \inf\{f_{IR_\mathfrak{p}}(M_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Supp}(M/IM) \text{ and } \dim R/\mathfrak{p} \geq n\}.$$

Note that $f_I^n(M)$ is either a positive integer or ∞ and that $f_I^0(M) = f_I(M)$, where the *finiteness dimension* $f_I(M)$ of M relative to I, is defined as

$$\begin{aligned} f_I(M) &:= \inf\{i \in \mathbb{N}_0 \mid H_I^i(M) \text{ is not finitely generated}\} \\ &= \inf\{f_{IR_\mathfrak{p}}(M_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}\, R\}. \end{aligned}$$

With the usual convention that the infimum of the empty set of integers is interpreted as ∞.

Corollary 2.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be an ideal of R. If $\mathrm{cd}(I, R) = f_I^1(R) = t > 0$, then the R-module $\mathrm{Hom}_R(R/I, H_I^t(R))$ is finitely generated and so*

$$t = \sup\{\dim \widehat{\widehat{R}_\mathfrak{P}}/\mathfrak{Q} : \mathfrak{P} \in V(I\widehat{R}),\ \mathfrak{Q} \in \mathrm{mAss}_{\widehat{\widehat{R}_\mathfrak{P}}} \widehat{\widehat{R}_\mathfrak{P}} \text{ and, } \mathfrak{P}\widehat{\widehat{R}_\mathfrak{P}} = \mathrm{Rad}(I\widehat{\widehat{R}_\mathfrak{P}} + \mathfrak{Q})\}.$$

Proof. The assertion follows from [4, Theorem 2.3] and Theorem 2.3. □

Corollary 2.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be an ideal of R. If $\mathrm{cd}(I, R) = f_I^2(R) = t > 0$, then the R-module $\mathrm{Hom}_R(R/I, H_I^t(R))$ is finitely generated and so*

$$t = \sup\{\dim \widehat{\widehat{R}_\mathfrak{P}}/\mathfrak{Q} : \mathfrak{P} \in V(I\widehat{R}),\ \mathfrak{Q} \in \mathrm{mAss}_{\widehat{\widehat{R}_\mathfrak{P}}} \widehat{\widehat{R}_\mathfrak{P}} \text{ and } \mathfrak{P}\widehat{\widehat{R}_\mathfrak{P}} = \mathrm{Rad}(I\widehat{\widehat{R}_\mathfrak{P}} + \mathfrak{Q})\}.$$

Proof. The assertion follows from [4, Theorem 3.2] and Theorem 2.3. □

Recall that following [18], an R-module M is called *minimax*, if there exists a finitely generated submodule N of M, such that M/N is Artinian. The class of minimax modules thus includes all finitely generated and all Artinian modules.

Corollary 2.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and I be an ideal of R with $\mathrm{cd}(I, R) = t > 0$. If the R-modules $H_I^i(R)$ are minimax for all $i < t$, then*

$$t = \sup\{\dim \widehat{\widehat{R}_\mathfrak{P}}/\mathfrak{Q} : \mathfrak{P} \in V(I\widehat{R}),\ \mathfrak{Q} \in \mathrm{mAss}_{\widehat{\widehat{R}_\mathfrak{P}}} \widehat{\widehat{R}_\mathfrak{P}} \text{ and } \mathfrak{P}\widehat{\widehat{R}_\mathfrak{P}} = \mathrm{Rad}(I\widehat{\widehat{R}_\mathfrak{P}} + \mathfrak{Q})\}.$$

Proof. The assertion follows from [2, Theorem 2.3] and Theorem 2.3. □

We need the following lemma in the proof of Lemma 2.9.

Lemma 2.8. *Let R be a Noetherian ring and let M be a non-zero R-module. Let X be an indeterminate over R. Then any monic polynomial $f = a_0 + \cdots + a_{n-1}X^{n-1} + X^n \in R[X]$ of positive degree is an $M \otimes_R R[X]$-regular sequence.*

Proof. Using the isomorphism of $R[X]$-modules $M \otimes_R R[X] \simeq M[X]$, without loss of generality it is enough to prove the element f is an $M[X]$-regular sequence. It is clear that $M[X]/fM[X] \neq 0$. Also, if $0 \neq g \in M[X]$ then there are elements $m_0, \ldots, m_k \in M$ such that $g = m_0 + m_1X + \cdots + m_kX^k$ with $m_k \neq 0$, for some integer $k \geq 0$. Then there are elements $m'_0, \ldots, m'_{k+n-1} \in M$ such that $fg = m'_0X + m'_1X^2 + \cdots + m'_{k+n-1}X^{k+n-1} + m_kX^{n+k} \neq 0$. So, the element $f \in R[X]$ is an $M[X]$-regular sequence. □

The following lemma plays a key role in the proof of Theorem 2.10.

Lemma 2.9. *Let R be a Noetherian ring and let M be a non-zero R-module. Let X be an indeterminate over R and set $S := R[X]$. Then for every monic polynomial $f \in S$ of positive degree the following statements hold:*

(i) $\Gamma_{fS}(M \otimes_R S) = 0$.

(ii) *For every positive integer n we have*

$$(0 :_{H^1_{fS}(M \otimes_R S)} f^n) \simeq M[X]/f^nM[X].$$

In particular, we have $H^1_{fS}(M \otimes_R S) \neq 0$.

Proof. (i) By Lemma 2.8, f is an $M \otimes_R S$-regular sequence and hence $\Gamma_{fS}(M \otimes_R S) = 0$.

(ii) The exact sequence

$$0 \longrightarrow M[X] \xrightarrow{f^n} M[X] \longrightarrow M[X]/f^nM[X] \longrightarrow,$$

induces the following exact sequence

$$\Gamma_{fS}(M[X]) \longrightarrow \Gamma_{fS}(M[X]/f^nM[X]) \longrightarrow H^1_{fS}(M[X]) \xrightarrow{f^n} H^1_{fS}(M[X]).$$

Now the last exact sequence induces the isomorphisms

$$(0 :_{H^1_{fS}(M \otimes_R S)} f^n) \simeq (0 :_{H^1_{fS}(M[X])} f^n) \simeq M[X]/f^nM[X].$$ □

The following theorem is our second main result in this paper.

Theorem 2.10. *Let R be a Noetherian ring and let M be a non-zero R-module. Let X be an indeterminate over R and set $S := R[X]$. Let $f \in S$ be a monic polynomial of positive degree. Then for every integer $i \geq 0$ and any proper ideal I of R we have $H^i_I(R) \neq 0$ if and only if $H^i_{IS}(S) \neq 0$. Also, for every integer $i \geq 0$ and any proper ideal I of R we have $H^i_I(R) \neq 0$ if and only if $H^{i+1}_{IS+fS}(S) \neq 0$. In particular,*

$$\operatorname{cd}(IS, S) = \operatorname{cd}(I, R) \quad \text{and} \quad \operatorname{cd}((I, f)S, S) = \operatorname{cd}(I, R) + 1.$$

Moreover, for each integer $i \geq 0$ we have $H^i_{IS+fS}(S) = fH^i_{IS+fS}(S)$ and the S-module $H^i_{IS+fS}(S)$ is finitely generated if and only if $H^i_{IS+fS}(S) = 0$.

Proof. Since S is a faithfully flat R-algebra and $H^j_{IS}(S) \simeq H^j_I(R) \otimes_R S$, for each integer $j \geq 0$, it follows that $H^i_I(R) \neq 0$ if and only if $H^i_{IS}(S) \neq 0$ and hence $\mathrm{cd}(IS, S) = \mathrm{cd}(I, R)$.

On the other hand in view of [17, Corollary 3.5], there exists an exact sequence

$$0 \longrightarrow H^1_{fS}(H^i_{IS}(S)) \longrightarrow H^{i+1}_{IS+fS}(S) \longrightarrow \Gamma_{fS}(H^{i+1}_{IS}(S)) \longrightarrow 0 \tag{2.10.1}$$

for every integer $i \geq 0$. By Lemma 2.9 we have

$$\Gamma_{fS}(H^{i+1}_{IS}(S)) \simeq \Gamma_{fS}(H^{i+1}_I(R) \otimes_R S) = 0$$

and hence the exact sequence (2.10.1) yields an isomorphism of S-modules

$$H^{i+1}_{IS+fS}(S) \simeq H^1_{fS}(H^i_{IS}(S)).$$

Thus by Lemma 2.9, we have $H^i_I(R) \neq 0$ if and only if $H^{i+1}_{IS+fS}(S) \neq 0$. Now, it is clear that

$$\mathrm{cd}((I, f)S, S) = \mathrm{cd}(I, R) + 1.$$

Since $\Gamma_{fS}(S) = 0$ it follows that $H^0_{IS+fS}(S) = 0 = fH^0_{IS+fS}(S)$. Also, for every integer $i \geq 0$, by [5, Remark 2.2.7 and Theorem 2.2.16], there is an exact sequence of S-modules

$$(H^i_{IS}(S))_f \longrightarrow H^1_{fS}(H^i_{IS}(S)) \longrightarrow 0,$$

which implies that

$$H^1_{fS}(H^i_{IS}(S)) = fH^1_{fS}(H^i_{IS}(S)).$$

Hence

$$H^{i+1}_{IS+fS}(S) = fH^{i+1}_{IS+fS}(S).$$

Therefore $H^i_{IS+fS}(S) = fH^i_{IS+fS}(S)$ for each integer $i \geq 0$. Now since the S-module $H^i_{IS+fS}(S)$ is fS-torsion it follows that if the S-module $H^i_{IS+fS}(S)$ is finitely generated then there exists a positive integer n such that $f^n H^i_{IS+fS}(S)$ $= 0$. Hence it follows from the hypothesis $H^i_{IS+fS}(S) = fH^i_{IS+fS}(S)$ that $H^i_{IS+fS}(S) = f^n H^i_{IS+fS}(S) = 0$. □

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References

[1] K. Bahmanpour, *A note on Lynch's conjecture*, Comm. Algebra **45** (2017), no. 6, 2738–2745.

[2] K. Bahmanpour and R. Naghipour, *On the cofiniteness of local cohomology modules*, Proc. Amer. Math. Soc. **136** (2008), no. 7, 2359–2363.

[3] ———, *Cofiniteness of local cohomology modules for ideals of small dimension*, J. Algebra **321** (2009), no. 7, 1997–2011.

[4] K. Bahmanpour, R. Naghipour, and M. Sedghi, *Minimaxness and cofiniteness properties of local cohomology modules*, Comm. Algebra **41** (2013), no. 8, 2799–2814.
[5] M. P. Brodmann and R. Y. Sharp, *Local Cohomology: an algebraic introduction with geometric applications*, Cambridge University Press, Cambridge,1998.
[6] K. Divaani-Aazar, R. Naghipour, and M. Tousi, *Cohomological dimension of certain algebraic varieties*, Proc. Amer. Math. Soc. **130** (2002), no. 12, 3537–3544.
[7] G. Faltings, *Über lokale Kohomologiegruppen höher Ordnung*, J. Reine Angew. Math. **313** (1980), 43–51.
[8] G. Ghasemi, K. Bahmanpour, and J. Azami, *Upper bounds for the cohomological dimensions of finitely generated modules over a commutative Noetherian ring*, Colloq. Math. **137** (2014), no. 2, 263–270.
[9] A. Grothendieck, *Local Cohomology*, Notes by R. Hartshorne, Lecture Notes in Math. 862, Springer, New York, 1966.
[10] R. Hartshorne, *Cohomological dimension of algebraic varieties*, Ann. of Math. **88** (1968), 403–450.
[11] ———, *Affine duality and cofiniteness*, Invent. Math. **9** (1970), 145–164.
[12] M. Hellus, *Matlis duals of top local cohomology modules and the arithmetic rank of an ideal*, Comm. Algebra **35** (2007), no. 4, 1421–1432.
[13] M. Hellus and J. Stückrad, *Matlis duals of top local cohomology modules*, Proc. Amer. Math. Soc. **136** (2008), no. 2, 489–498.
[14] C. Huneke and G. Lyubezink, *On the vanishing of local cohomology modules*, Invent. Math. **102** (1990), no. 1, 73–93.
[15] H. Matsumura, *Commutative Ring Theory*, Cambridge Univ. Press, Cambridge, UK, 1986.
[16] A. A. Mehrvarz, K. Bahmanpour, and R. Naghipour, *Arithmetic rank, cohomologal dimension and filter regular sequences*, J. Algebra Appl. **8** (2009), no. 6, 855–862.
[17] P. Schenzel, *Proregular sequences, local cohomology, and completion*, Math. Scand. **92** (2003), no. 2, 161–180.
[18] H. Zöschinger, *Minimax modules*, J. Algebra **102** (1986), no. 1, 1–32.

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