

## A FOURTH-ORDER ACCURATE FINITE DIFFERENCE SCHEME FOR THE EXTENDED-FISHER-KOLMOGOROV EQUATION

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**ABSTRACT.** In this paper, a nonlinear high-order difference scheme is proposed to solve the Extended-Fisher-Kolmogorov equation. The existence, uniqueness of difference solution and priori estimates are obtained. Furthermore, the convergence of the difference scheme is proved by utilizing the energy method to be of fourth-order in space and second-order in time in the discrete  $L^\infty$ -norm. Some numerical examples are given in order to validate the theoretical results.

### 1. Introduction

In this article, we consider the following periodic initial value problem of the Extended Fisher-Kolmogorov (EFK) equation

$$(1.1) \quad \frac{\partial u}{\partial t} + \gamma \frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 u}{\partial x^2} + f(u) = 0, \quad x \in \Omega, \quad t \in (0, T],$$

subject to the initial condition

$$(1.2) \quad u(x, 0) = u_0(x), \quad x \in \Omega,$$

and periodic condition

$$(1.3) \quad u(x + L, t) = u(x, t), \quad x \in \Omega, \quad t \in (0, T],$$

where  $f(u) = u^3 - u$ ,  $\Omega = (0, 1)$ ,  $T > 0$ ,  $\gamma$  is a positive constant and  $u_0$  is a given  $L$ -periodic function. When  $\gamma = 0$  in (1.1), we obtain the standard Fisher-Kolmogorov (FK) equation. However, by adding a stabilizing fourth-order derivative term to the Fisher-Kolmogorov equation, Couillet et al. [5] proposed (1.1) and called as the EFK equation. Problem (1.1) arises in a variety of applications such as pattern formation in bi-stable systems [6], propagation of domain walls in liquid crystals [27], travelling waves in reaction diffusion system [1, 3] and mesoscopic model of a phase transition in a binary system near Lipschitz point [11]. For computational studies, L. J. Tarcus Doss and A.

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P. Nandini has proposed a  $H^1$ -Galerkin mixed Finite Element Method for the EFK equation in [21]. Related to fourth and quintic order evolution equations, a second order difference scheme for the Sivashinsky equation is analyzed by Omrani in [16, 17], for Rosenau equation by Omrani [18], for Cahn-Hilliard equation by Khiari et al. [12], for Kuramoto-Sivashinsky equation by Akrivis [2], for Rosenau-RLW equation by Pan et al [19, 20], for Rosenau-Kawahara equation by D. He et al. [9, 10].

In [13], Khiari and Omrani have proposed a nonlinear finite difference scheme for the EFK equation (1.1)-(1.3). This scheme is second-order accurate in space. For many application problems, it is desirable to use high-order numerical algorithms to compute accurate solutions.

To obtain satisfactory higher order numerical results with reasonable computational cost, there have been attempts to develop higher order compact (HOC) schemes [7, 8, 14, 15, 22–25]. However, because the discretization of nonlinear term in compact scheme is more complicated than that in second-order one, a priori estimate in the discrete  $L^\infty$ -norm is hard to be obtained, so the unconditional convergence of any compact difference scheme for nonlinear equation is difficult to be proved.

In this article, we establish a new high-order difference scheme for the EFK equation and prove that the scheme is convergent with the convergence of  $O(h^4 + k^2)$  in the discrete  $L^\infty$ -norm.

The remainder of the article is arranged as follows. In Section 2, a nonlinear difference scheme is derived. In Section 3, existence, a priori estimations and uniqueness for numerical solutions are shown. The  $L^\infty$ -convergence for the difference scheme is proved in Section 4. In the last section, some numerical experiments are presented to support our theoretical results.

Throughout this article,  $C$  denotes a generic positive constant which is independent of the discretisation parameters  $h$  and  $k$ , but may have different values at different places.

## 2. High-order conservative difference scheme

In this section, we propose a two-level nonlinear Crank-Nicolson-type finite difference scheme for the problem (1.1)-(1.3). For convenience, the following notations are used. For a positive integer  $N$ , let time-step  $k = \frac{T}{N}$ ,  $t^n = nk$ ,  $n = 0, 1, \dots, N$ . For a positive integer  $M$ , let space-step  $h = \frac{L}{M}$ ,  $x_i = ih$ ,  $i = 0, 1, \dots, M$ , and let

$$\mathbb{R}_{per}^M = \{V = (V_i)_{i \in \mathbb{Z}} / V_i \in \mathbb{R} \text{ and } V_{i+M} = V_i, \quad i \in \mathbb{Z}\}.$$

We define the difference operators for  $U^n \in \mathbb{R}_{per}^M$  as

$$(U_i^n)_x = \frac{U_{i+1}^n - U_i^n}{h}, \quad (U_i^n)_{\bar{x}} = \frac{U_i^n - U_{i-1}^n}{h}, \quad (U_i^n)_{\hat{x}} = \frac{U_{i+1}^n - U_{i-1}^n}{2h},$$

$$U_i^{n+\frac{1}{2}} = \frac{1}{2}(U_i^{n+1} + U_i^n), \quad \partial_t U_i^n = \frac{1}{k}(U_i^{n+1} - U_i^n).$$

We introduce a discrete inner product as

$$(U^n, V^n)_h = h \sum_{i=1}^M U_i^n V_i^n, \quad U^n, V^n \in \mathbb{R}_{per}^M.$$

The discrete  $L^2$ -norm  $\|\cdot\|_h$ ,  $H^1$  seminorm  $|\cdot|_{1,h}$ ,  $H^2$  seminorm  $|\cdot|_{2,h}$ ,  $L^p$  norm  $\|\cdot\|_{p,h}$  and the discrete maximum-norm  $\|\cdot\|_{\infty,h}$  are defined, respectively as

$$\begin{aligned} \|U^n\|_h &= \sqrt{(U^n, U^n)_h}, \quad |U^n|_{1,h} = \sqrt{h \sum_{i=1}^M |(U_i^n)_x|^2}, \\ |U^n|_{2,h} &= h \sqrt{\sum_{i=1}^M |(U_i^n)_{x\bar{x}}|^2}, \quad \|U^n\|_{p,h} = \left[ h \sum_{i=1}^M |U_i^n|^p \right]^{\frac{1}{p}}, \quad p \geq 1, \\ \|U^n\|_{\infty,h} &= \max_{1 \leq i \leq M} |U_i^n|. \end{aligned}$$

Denote  $H_{per}^2(\Omega)$  the periodic Sobolev space of order 2.

We discretize problem (1.1)-(1.3) by the following finite difference scheme:

$$(2.1) \quad \begin{aligned} \partial_t U_i^n + \gamma \left[ \frac{5}{3} (U_i^{n+\frac{1}{2}})_{xx\bar{x}\bar{x}} - \frac{2}{3} (U_i^{n+\frac{1}{2}})_{x\bar{x}\hat{x}\hat{x}} \right] - \left[ \frac{4}{3} (U_i^{n+\frac{1}{2}})_{x\bar{x}} - \frac{1}{3} (U_i^{n+\frac{1}{2}})_{\hat{x}\hat{x}} \right] \\ + f(U_i^{n+\frac{1}{2}}) = 0, \quad i = 1, \dots, M, \quad n = 1, \dots, N, \end{aligned}$$

$$(2.2) \quad U_{i+M}^n = U_i^n, \quad i = 1, \dots, M, \quad n = 1, \dots, N,$$

$$(2.3) \quad U_i^0 = u_0(x_i), \quad i = 1, \dots, M.$$

In the next, we collect some auxiliary results. Using summation by parts and the discrete boundary condition, we can easily check the following Lemma.

**Lemma 1.** For any grid functions  $U, V \in \mathbb{R}_{per}^M$ , we have

$$(2.4) \quad (U_{\bar{x}}, V)_h = -(U, V_x)_h,$$

$$(2.5) \quad (U_{\hat{x}}, V)_h = -(U, V_{\hat{x}})_h,$$

$$(2.6) \quad (U_{x\bar{x}}, V)_h = -(U_x, V_x)_h,$$

$$(2.7) \quad (U_{\hat{x}\hat{x}}, V)_h = (U, V_{\hat{x}\hat{x}})_h.$$

Therefore, we may easily see that the following relations hold, for  $U \in \mathbb{R}_{per}^M$ :

$$(2.8) \quad (U_{x\bar{x}}, U)_h = -\|U_x\|_h^2,$$

$$(2.9) \quad (U_{\hat{x}\hat{x}}, U)_h = -\|U_{\hat{x}}\|_h^2,$$

$$(2.10) \quad (U_{xx\bar{x}\bar{x}}, U)_h = \|U_{xx}\|_h^2,$$

$$(2.11) \quad (U_{x\bar{x}\hat{x}\hat{x}}, U)_h = \|U_{x\hat{x}}\|_h^2.$$

**Lemma 2.** For any  $U \in \mathbb{R}_{per}^M$ , there is

$$(2.12) \quad \|U_{\hat{x}}\|_h^2 \leq \|U_x\|_h^2,$$

$$(2.13) \quad \|U_{x\hat{x}}\|_h^2 \leq \|U_{xx}\|_h^2.$$

*Proof.* For  $U \in \mathbb{R}_{per}^M$ , we have from (2.6)

$$(2.14) \quad \|U_x\|_h^2 = \|U_{\bar{x}}\|_h^2.$$

Using the periodic boundary condition (2.2) condition, we get

$$\begin{aligned} \|U_{\hat{x}}\|_h^2 &= h \sum_{j=1}^M \left( \frac{U_{j+1} - U_{j-1}}{2h} \right)^2 \\ &= h \sum_{j=1}^M \left( \frac{U_{j+1} - U_j}{2h} + \frac{U_j - U_{j-1}}{2h} \right)^2 \\ &\leq 2h \sum_{j=1}^M \left( \frac{U_{j+1} - U_j}{2h} \right)^2 + 2h \sum_{j=1}^M \left( \frac{U_j - U_{j-1}}{2h} \right)^2 \\ &= \frac{1}{2} \|U_x\|_h^2 + \frac{1}{2} \|U_{\bar{x}}\|_h^2. \end{aligned}$$

Therefore, (2.12) follows immediately from (2.14).

For  $U \in \mathbb{R}_{per}^M$ , we have

$$\begin{aligned} \|U_{x\hat{x}}\|_h^2 &= h \sum_{j=1}^M \left( \frac{U_{j+2} - U_{j+1} - U_j + U_{j-1}}{2h^2} \right)^2 \\ &= h \sum_{j=1}^M \left( \frac{U_{j+2} - 2U_{j+1} + U_j}{2h^2} + \frac{U_{j+1} - 2U_j + U_{j-1}}{2h^2} \right)^2 \\ &\leq 2h \sum_{j=1}^M \left( \frac{U_{j+2} - 2U_{j+1} + U_j}{2h^2} \right)^2 + 2h \sum_{j=1}^M \left( \frac{U_{j+1} - 2U_j + U_{j-1}}{2h^2} \right)^2 \\ &= \frac{1}{2} \|U_{xx}\|_h^2 + \frac{1}{2} \|U_{x\bar{x}}\|_h^2. \end{aligned}$$

Further from (2.6)

$$(2.15) \quad \|U_{x\bar{x}}\|_h^2 = \|U_{xx}\|_h^2,$$

and (2.13) follows.  $\square$

### 3. Existence and uniqueness

#### 3.1. Existence

The existence of the approximate solution of the system (2.1)-(2.3) is proved by the use of the Browder theorem [4].

**Lemma 3.** Let  $(H, (\cdot, \cdot)_h)$  be a finite dimensional inner product space,  $\|\cdot\|_h$  be the associated norm, suppose that  $g : H \rightarrow H$  is continuous and there exists  $\lambda > 0$  such that  $(g(x), x)_h > 0$  for all  $x \in H$  with  $\|x\|_h = \lambda$ . Then, there exists  $x^* \in H$  such that  $g(x^*) = 0$  and  $\|x^*\|_h \leq \lambda$ .

**Theorem 1.** The numerical solution of discrete scheme (2.1)-(2.3) exists.

*Proof.* In order to prove the theorem by the mathematical induction, we assume that  $U^0, U^1, \dots, U^n$  exists for  $n < N$ . Let  $g$  be a function defined by

$$(3.1) \quad g(V) = 2V - 2U^n + \frac{5k\gamma}{3}V_{xx\bar{x}\bar{x}} - \frac{2k\gamma}{3}V_{x\bar{x}\hat{x}\hat{x}} - \frac{4k}{3}V_{x\bar{x}} + \frac{k}{3}V_{\hat{x}\hat{x}} + kf(V).$$

Then  $g$  is obviously continuous. Taking the inner product of (3.1) by  $V$  and using Lemma 1, we have

$$\begin{aligned} (g(V), V)_h &= 2\|V\|_h^2 - 2(U^n, V)_h + \frac{5k\gamma}{3}\|V_{xx}\|_h^2 - \frac{2k\gamma}{3}\|V_{x\hat{x}}\|_h^2 \\ &\quad + \frac{4k}{3}\|V_x\|_h^2 - \frac{k}{3}\|V_{\hat{x}}\|_h^2 + k(f(V), V)_h. \end{aligned}$$

Applying Lemma 2, we get

$$(g(V), V)_h \geq 2\|V\|_h^2 - 2\|U^n\|_h\|V\|_h + k\gamma\|V_{xx}\|_h^2 + k\|V_x\|_h^2 + k\|V\|_{4,h}^4 - k\|V\|_h^2.$$

Therefore for  $\gamma > 0$ ,

$$(g(V), V)_h \geq 2\left[\left(1 - \frac{k}{2}\right)\|V\|_h^2 - \|U^n\|_h\|V\|_h\right].$$

For  $k < 2$  and  $\|V\|_h = \frac{2}{2-k}\|U^n\|_h + 1$ , obviously  $(g(V), V)_h > 0$  and via Lemma 3 we deduce the existence of  $V^* \in \mathbb{R}_{per}^M$  such that  $g(V^*) = 0$ . If we take  $U^{n+1} = 2V^* - U^n$ , then  $U^{n+1}$  satisfies the equation (2.1).  $\square$

### 3.2. Priori estimates

**Lemma 4** (Discrete Gronwall inequality [26]). Assume  $\{G^n/n \geq 0\}$  is non-negative sequences and satisfies

$$G^0 \leq A, \quad G^n \leq A + Bk \sum_{i=0}^{n-1} G^i \quad n = 1, 2, \dots,$$

where  $A$  and  $B$  are non negative constants. Then  $G$  satisfies

$$G^n \leq Ae^{Bnk}, \quad n = 0, 1, 2, \dots$$

For the difference solution of scheme (2.1)-(2.3), we have the following priori bound.

**Theorem 2.** Let  $U^n$  be the solution of the difference scheme (2.1)-(2.3). Assume that  $u_0 \in L^2(\Omega)$ . Then, there exists a positive constant  $C$  independent of  $h$  and  $k$  such that

$$(3.2) \quad \|U^n\|_h \leq C.$$

*Proof.* Taking in (2.1) the inner product with  $U^{n+\frac{1}{2}}$  and using Lemma 1, we obtain:

$$\begin{aligned} & \frac{1}{2} \partial_t \|U^n\|_h^2 + \frac{5\gamma}{3} \|U_{xx}^{n+\frac{1}{2}}\|_h^2 - \frac{2\gamma}{3} \|U_{x\hat{x}}^{n+\frac{1}{2}}\|_h^2 + \frac{4}{3} \|U_x^{n+\frac{1}{2}}\|_h^2 - \frac{1}{3} \|U_{\hat{x}}^{n+\frac{1}{2}}\|_h^2 \\ &= - \|U^{n+\frac{1}{2}}\|_{4,h}^4 + \|U^{n+\frac{1}{2}}\|_h^2. \end{aligned}$$

By lemma 2 we get

$$\frac{1}{2} \partial_t \|U^n\|_h^2 + \gamma \|U_{xx}^{n+\frac{1}{2}}\|_h^2 + \|U_x^{n+\frac{1}{2}}\|_h^2 \leq \|U^{n+\frac{1}{2}}\|_h^2.$$

Therefore for  $\gamma > 0$

$$\frac{1}{2k} (\|U^{n+1}\|_h^2 - \|U^n\|_h^2) \leq \frac{1}{2} (\|U^{n+1}\|_h^2 + \|U^n\|_h^2).$$

This yields

$$(1-k) \|U^{n+1}\|_h^2 \leq (1+k) \|U^n\|_h^2,$$

when  $k \leq \frac{1}{3}$ , which gives

$$\|U^{n+1}\|_h^2 \leq (1+3k) \|U^n\|_h^2, \quad 0 \leq n \leq N-1.$$

Summing the above inequality from 0 to  $n-1$ , we have

$$\|U^n\|_h^2 \leq \|U^0\|_h^2 + 3k \sum_{l=0}^{n-1} \|U^l\|_h^2.$$

It follows from Lemma 4 that

$$\|U^n\|_h^2 \leq e^{3T} \|U^0\|_h^2, \quad 0 \leq n \leq N.$$

This completes the proof.  $\square$

Next we use the following lemmas (see [26]).

**Lemma 5.** For  $v \in \mathbb{R}_{per}^M$ , we have for  $p \geq 2$

$$(3.3) \quad \|v\|_{p,h} \leq C \left[ \|v\|_h^{1-q} \|v\|_{1,h}^q + \|v\|_h \right],$$

where  $q = \frac{1}{2} - \frac{1}{p}$  and  $C$  is a positive constant independent of  $p$  and  $h$ .

**Lemma 6.** For any  $x \geq 0, y \geq 0$ , there is

$$(3.4) \quad (x+y)^6 \leq 2^5 (x^6 + y^6).$$

**Theorem 3.** Assume that  $u_0 \in H_{per}^2(\Omega)$ . Then, the solution of the difference scheme (2.1)-(2.3) is estimated as follows

$$(3.5) \quad \|U^n\|_{\infty,h} \leq C.$$

*Proof.* Taking in (2.1) the inner product with  $2\partial_t U^n$ , and using Schwarz inequality, we obtain

$$\begin{aligned} & 2\|\partial_t U^n\|_h^2 + \frac{5\gamma}{3}\partial_t\|U_{xx}^n\|_h^2 - \frac{2\gamma}{3}\partial_t\|U_{x\hat{x}}^n\|_h^2 + \frac{4}{3}\partial_t\|U_x^n\|_h^2 - \frac{1}{3}\partial_t\|U_{\hat{x}}^n\|_h^2 \\ &= 2(-(U^{n+\frac{1}{2}})^3 + U^{n+\frac{1}{2}}, \partial_t U^n)_h \\ &\leq \|U^{n+\frac{1}{2}}\|_{6,h}^6 + \|\partial_t U^n\|_h^2 + \|U^{n+\frac{1}{2}}\|_h^2 + \|\partial_t U^n\|_h^2. \end{aligned}$$

It follows from Theorem 2 that

$$(3.6) \quad \frac{5\gamma}{3}\partial_t\|U_{xx}^n\|_h^2 - \frac{2\gamma}{3}\partial_t\|U_{x\hat{x}}^n\|_h^2 + \frac{4}{3}\partial_t\|U_x^n\|_h^2 - \frac{1}{3}\partial_t\|U_{\hat{x}}^n\|_h^2 \leq \|U^{n+\frac{1}{2}}\|_{6,h}^6 + C.$$

Applying Lemma 5 with  $p = 6$ , we obtain

$$(3.7) \quad \|U^{n+\frac{1}{2}}\|_{6,h}^6 \leq C[\|U^{n+\frac{1}{2}}\|_h^{\frac{2}{3}}\|U_x^{n+\frac{1}{2}}\|_h^{\frac{1}{3}} + \|U^{n+\frac{1}{2}}\|_h].$$

From Lemma 6, we have

$$\|U^{n+\frac{1}{2}}\|_{6,h}^6 \leq 2^5 C^6 [\|U^{n+\frac{1}{2}}\|_h^4 \|U_x^{n+\frac{1}{2}}\|_h^2 + \|U^{n+\frac{1}{2}}\|_h^6].$$

It follows from (3.2) that

$$(3.8) \quad \|U^{n+\frac{1}{2}}\|_{6,h}^6 \leq C\|U_x^{n+\frac{1}{2}}\|_h^2 + C,$$

where  $C$  is a generic constant depending on  $\|U^0\|_h$  and  $T$ .

Using (3.6) and (3.8), we have for  $\gamma > 0$

$$\begin{aligned} & \frac{5\gamma}{3}\partial_t\|U_{xx}^n\|_h^2 - \frac{2\gamma}{3}\partial_t\|U_{x\hat{x}}^n\|_h^2 + \frac{4}{3}\partial_t\|U_x^n\|_h^2 - \frac{1}{3}\partial_t\|U_{\hat{x}}^n\|_h^2 \\ (3.9) \quad & \leq C\|U_x^{n+\frac{1}{2}}\|_h^2 + C \\ & \leq C(\|U_x^{n+\frac{1}{2}}\|_h^2 + \gamma\|U_{xx}^{n+\frac{1}{2}}\|_h^2) + C. \end{aligned}$$

Let  $B^n = \frac{5\gamma}{3}\|U_{xx}^n\|_h^2 - \frac{2\gamma}{3}\|U_{x\hat{x}}^n\|_h^2 + \frac{4}{3}\|U_x^n\|_h^2 - \frac{1}{3}\|U_{\hat{x}}^n\|_h^2$  and summing up (3.9) from 0 to  $n-1$ , we obtain

$$(3.10) \quad B^n \leq B^0 + Ck \sum_{l=0}^n (\|U_x^l\|_h^2 + \gamma\|U_{xx}^l\|_h^2) + C(T).$$

It follows from Lemma 2 that

$$(3.11) \quad B^n \geq \gamma\|U_{xx}^n\|_h^2 + \|U_x^n\|_h^2.$$

Then using (3.10) and (3.11), we have

$$(3.12) \quad \gamma\|U_{xx}^n\|_h^2 + \|U_x^n\|_h^2 \leq B^0 + Ck \sum_{l=0}^n (\|U_x^l\|_h^2 + \gamma\|U_{xx}^l\|_h^2) + C(T).$$

Letting now  $G^n = \gamma\|U_{xx}^n\|_h^2 + \|U_x^n\|_h^2$  and using the above inequality we find

$$(3.13) \quad G^n \leq B^0 + Ck \sum_{l=0}^n G^l + C(T).$$

This yields

$$(3.14) \quad (1 - Ck)G^n \leq C' + Ck \sum_{l=0}^{n-1} G^l,$$

where  $C'$  is a positive constant depending on  $U^0$  and  $T$ .

Applying Lemma 4, we obtain

$$(3.15) \quad \gamma \|U_{xx}^n\|_h^2 + \|U_x^n\|_h^2 \leq C(U^0, T),$$

as  $\gamma > 0$ , it follows that

$$(3.16) \quad \|U_x^n\|_h \leq C(U^0, T).$$

Using Lemma 5 with  $p = \infty$  and (3.2) and (3.16), we have

$$\|U^n\|_{\infty, h} \leq C,$$

where  $C = C(U^0, T)$ , which completes the proof.  $\square$

### 3.3. Uniqueness

The uniqueness of the solution of difference scheme (2.1) – (2.3) is proved in the next theorem.

**Theorem 4.** *For  $k$  small enough, the solution of the difference scheme (2.1)-(2.3) is uniquely solvable.*

*Proof.* Assume that  $U^n$  and  $V^n$  both satisfy the scheme (2.1)-(2.3), let  $\theta^n = U^n - V^n$ , then we obtain

$$(3.17) \quad \begin{aligned} & \partial_t \theta^n + \frac{5\gamma}{3} (\theta^{n+\frac{1}{2}})_{xx\bar{x}\bar{x}} - \frac{2\gamma}{3} (\theta^{n+\frac{1}{2}})_{x\bar{x}\hat{x}\hat{x}} - \frac{4}{3} (\theta^{n+\frac{1}{2}})_{x\bar{x}} + \frac{1}{3} \\ & (\theta^{n+\frac{1}{2}})_{\hat{x}\hat{x}} + f(U^{n+\frac{1}{2}}) - f(V^{n+\frac{1}{2}}) = 0, \quad i = 1, \dots, M, \quad n = 1, \dots, N, \end{aligned}$$

$$(3.18) \quad \theta_{i+M}^n = \theta_i^n, \quad i = 1, \dots, M, \quad n = 1, \dots, N,$$

$$(3.19) \quad \theta_i^0 = 0, \quad i = 1, \dots, M.$$

Taking in (3.17) the inner product with  $\theta^{n+\frac{1}{2}}$  and using Lemma 1, we have

$$(3.20) \quad \begin{aligned} & \frac{1}{2} \partial_t \|\theta^n\|_h^2 + \frac{5\gamma}{3} \|\theta_{xx}^{n+\frac{1}{2}}\|_h^2 - \frac{2\gamma}{3} \|\theta_{x\hat{x}}^{n+\frac{1}{2}}\|_h^2 + \frac{4}{3} \|\theta_x^{n+\frac{1}{2}}\|_h^2 - \frac{1}{3} \|\theta_{\hat{x}}^{n+\frac{1}{2}}\|_h^2 \\ & + (f(U^{n+\frac{1}{2}}) - f(V^{n+\frac{1}{2}}), \theta^{n+\frac{1}{2}})_h = 0. \end{aligned}$$

By differentiability of  $f$  and (3.5) we find

$$(3.21) \quad |(f(U^{n+\frac{1}{2}}) - f(V^{n+\frac{1}{2}}), \theta^{n+\frac{1}{2}})_h| \leq C \|\theta^{n+\frac{1}{2}}\|_h^2.$$

It follows from (3.20), (3.21) and Lemma 2 that

$$\frac{1}{2} \partial_t \|\theta^n\|_h^2 + \gamma \|\theta_{xx}^{n+\frac{1}{2}}\|_h^2 + \|\theta_x^{n+\frac{1}{2}}\|_h^2 \leq C \|\theta^{n+\frac{1}{2}}\|_h^2.$$



This yields for  $\gamma > 0$

$$\frac{1}{2k}(\|\theta^{n+1}\|_h^2 - \|\theta^n\|_h^2) \leq C(\|\theta^{n+1}\|_h^2 + \|\theta^n\|_h^2).$$

Consequently,

$$(1 - 2kC)\|\theta^{n+1}\|_h^2 \leq (1 + 2kC)\|\theta^n\|_h^2.$$

For  $k < \frac{1}{2C}$ , applying Lemma 4 and (3.19), we obtain

$$\|\theta^{n+1}\|_h = 0.$$

This completes the proof of uniqueness.  $\square$

#### 4. Convergence

According to Taylor expansion, we obtain the following Lemma

**Lemma 7.** *For any smooth function  $\psi$  we have*

$$(4.1) \quad \frac{4}{3}(\psi_i)_{x\bar{x}} - \frac{1}{3}(\psi_i)_{\hat{x}\hat{x}} = \frac{d^2\psi}{dx^2}(x_i) + O(h^4),$$

$$(4.2) \quad \frac{5}{3}(\psi_i)_{xx\bar{x}\bar{x}} - \frac{2}{3}(\psi_i)_{x\bar{x}\hat{x}\hat{x}} = \frac{d^4\psi}{dx^4}(x_i) + O(h^4).$$

Define the net function  $u_i^n = u(x_i, t^n)$ ,  $1 \leq i \leq M$ ,  $0 \leq n \leq N$ . The truncation error of the scheme (2.1)-(2.3) is

$$(4.3) \quad \begin{aligned} \partial_t u_i^n + \gamma \left[ \frac{5}{3}(u_i^{n+\frac{1}{2}})_{xx\bar{x}\bar{x}} - \frac{2}{3}(u_i^{n+\frac{1}{2}})_{x\bar{x}\hat{x}\hat{x}} \right] - \left[ \frac{4}{3}(u_i^{n+\frac{1}{2}})_{x\bar{x}} - \frac{1}{3}(u_i^{n+\frac{1}{2}})_{\hat{x}\hat{x}} \right] \\ + f(u_i^{n+\frac{1}{2}}) = r_i^n. \end{aligned}$$

According to Taylor's expansion and Lemma 7, we obtain

**Lemma 8.** *Suppose that  $u(x, t) \in C_{x,t}^{8,3}([0, L] \times [0, T])$ , then the truncation error of the difference scheme (2.1)-(2.3) satisfies*

$$(4.4) \quad |r_i^n| = O(h^4 + k^2) \quad \text{as } k \rightarrow 0, h \rightarrow 0.$$

The convergence of the difference scheme (2.1)-(2.3) will be given in the following theorem.

**Theorem 5.** *Assume that the solution of (1.1)-(1.3)  $u(x, t) \in C_{x,t}^{8,3}([0, L] \times [0, T])$ . Then the solution of the difference scheme (2.1)-(2.3) converges to the solution of the problem (1.1)-(1.3) in the discrete  $L^\infty$ -norm and the rate of convergence is the order of  $O(h^4 + k^2)$ , when  $h$  and  $k$  are sufficiently small.*

*Proof.* Denote

$$c_0 = \max_{0 \leq x \leq L, 0 \leq t \leq T} |u(x, t)|.$$

Let  $e_i^n = u_i^n - U_i^n$ ,  $1 \leq i \leq M$ ,  $0 \leq n \leq N$ . Subtracting (2.1) from (4.4), we obtain

$$(4.5) \quad \begin{aligned} & \partial_t e_i^n + \gamma \left[ \frac{5}{3} (e_i^{n+\frac{1}{2}})_{xx\bar{x}\bar{x}} - \frac{2}{3} (e_i^{n+\frac{1}{2}})_{x\bar{x}\hat{x}\hat{x}} \right] - \left[ \frac{4}{3} (e_i^{n+\frac{1}{2}})_{x\bar{x}} - \frac{1}{3} (e_i^{n+\frac{1}{2}})_{\hat{x}\hat{x}} \right] \\ & + f(u_i^{n+\frac{1}{2}}) - f(U_i^{n+\frac{1}{2}}) = r_i^n, \quad i = 1, 2, \dots, M, \quad n = 0, 1, \dots, N-1. \end{aligned}$$

$$(4.6) \quad e_i^n = e_{i+M}^n, \quad i = 1, 2, \dots, M, \quad n = 0, 1, \dots, N,$$

$$(4.7) \quad e_i^0 = 0, \quad i = 1, 2, \dots, M.$$

Computing the inner product of (4.5) with  $e^{n+\frac{1}{2}}$  and using Lemma 1, we have

$$(4.8) \quad \begin{aligned} & \frac{1}{2} \partial_t \|e^{n+\frac{1}{2}}\|_h^2 + \frac{5\gamma}{3} \|e_{xx}^{n+\frac{1}{2}}\|_h^2 - \frac{2\gamma}{3} \|e_{x\hat{x}}^{n+\frac{1}{2}}\|_h^2 + \frac{4}{3} \|e_x^{n+\frac{1}{2}}\|_h^2 - \frac{1}{3} \|e_{\hat{x}}^{n+\frac{1}{2}}\|_h^2 \\ & = (f(U^{n+\frac{1}{2}}) - f(u^{n+\frac{1}{2}}), e^{n+\frac{1}{2}})_h + (r^n, e^{n+\frac{1}{2}})_h. \end{aligned}$$

It follows from Lemma 2 that

$$(4.9) \quad \begin{aligned} & \frac{1}{2} \partial_t \|e^{n+\frac{1}{2}}\|_h^2 + \gamma \|e_{xx}^{n+\frac{1}{2}}\|_h^2 + \|e_x^{n+\frac{1}{2}}\|_h^2 \\ & \leq [\|f(U^{n+\frac{1}{2}}) - f(u^{n+\frac{1}{2}})\|_h + \|r^n\|_h] \|e^{n+\frac{1}{2}}\|_h. \end{aligned}$$

For the nonlinear term  $\|f(U^{n+\frac{1}{2}}) - f(u^{n+\frac{1}{2}})\|_h$ , we use the boundedness of  $\|U^n\|_{\infty, h}$  and  $c_0$  to find that

$$(4.10) \quad \|f(U^{n+\frac{1}{2}}) - f(u^{n+\frac{1}{2}})\|_h \leq C \|e^{n+\frac{1}{2}}\|_h.$$

Substituting (4.10) in (4.9) and using Lemma 9, we obtain for  $\gamma > 0$

$$\frac{1}{2k} (\|e^{n+1}\|_h^2 - \|e^n\|_h^2) \leq C (\|e^{n+1}\|_h^2 + \|e^n\|_h^2) + (h^4 + k^2)^2.$$

This implies that

$$(1 - 2kC) \|e^{n+1}\|_h^2 \leq (1 + 2kC) \|e^n\|_h^2 + 2k(h^4 + k^2)^2.$$

For  $k$  sufficiently small, using (4.7) and an application of Lemma 4 yields the following inequality

$$(4.11) \quad \|e^n\|_h \leq C(h^4 + k^2).$$

Taking in (4.5) the inner product with  $2\partial_t e^n$ , and using Schwarz inequality, we have

$$(4.12) \quad \begin{aligned} & 2\|\partial_t e^n\|_h^2 + \frac{5\gamma}{3} \partial_t \|e_{xx}^n\|_h^2 - \frac{2\gamma}{3} \partial_t \|e_{x\hat{x}}^n\|_h^2 + \frac{4}{3} \partial_t \|e_x^n\|_h^2 - \frac{1}{3} \partial_t \|e_{\hat{x}}^n\|_h^2 \\ & = 2(f(U^{n+\frac{1}{2}}) - f(u^{n+\frac{1}{2}}), \partial_t e^n)_h + 2(r^n, \partial_t e^n)_h \\ & \leq \|f(U^{n+\frac{1}{2}}) - f(u^{n+\frac{1}{2}})\|_h^2 + \|\partial_t e^n\|_h^2 + \|r^n\|_h^2 + \|\partial_t e^n\|_h^2. \end{aligned}$$

It follows from Lemma 8, (4.10) and (4.11) that

$$(4.13) \quad \frac{5\gamma}{3} \partial_t \|e_{xx}^n\|_h^2 - \frac{2\gamma}{3} \partial_t \|e_{x\hat{x}}^n\|_h^2 + \frac{4}{3} \partial_t \|e_x^n\|_h^2 - \frac{1}{3} \partial_t \|e_{\hat{x}}^n\|_h^2 \leq C(h^4 + k^2)^2.$$

Letting  $A^n = \frac{5\gamma}{3}\|e_{xx}^n\|_h^2 - \frac{2\gamma}{3}\|e_{x\hat{x}}^n\|_h^2 + \frac{4}{3}\|e_x^n\|_h^2 - \frac{1}{3}\|e_{\hat{x}}^n\|_h^2$ , and summing up (4.13) from 0 to  $n-1$ , we obtain

$$(4.14) \quad A^n \leq A^0 + Cnk(h^4 + k^2)^2.$$

From (4.7), it follows that

$$(4.15) \quad A^n \leq CT(h^4 + k^2)^2.$$

Again using Lemma 2, we obtain

$$(4.16) \quad \gamma\|e_{xx}^n\|_h^2 + \|e_x^n\|_h^2 \leq A^n \leq C(T)(h^4 + k^2)^2,$$

and hence,

$$(4.17) \quad \|e_x^n\|_h^2 \leq C(T)(h^4 + k^2)^2.$$

An application of Lemma 5 with  $p = \infty$  and (4.11) and (4.17) yields

$$(4.18) \quad \|e^n\|_{\infty, h} \leq C(h^4 + k^2),$$

where  $C = C(T)$ . This completes the proof.  $\square$

## 5. Numerical results

We give the numerical experiment that the exact solution of problem is known. Consider the following inhomogeneous periodic initial value problem of the EFK equation

$$(5.1) \quad \frac{\partial u}{\partial t} + \gamma \frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 u}{\partial x^2} + u^3 - u = g(x, t), \quad x \in (0, 1), \quad t \in (0, T],$$

subject to the initial condition

$$(5.2) \quad u(x, 0) = \sin(2\pi x), \quad x \in (0, 1),$$

where

$$g(x, y, t) = e^{-t} \sin(2\pi x) \left[ -2 + 4\pi^2 + 16\gamma\pi^4 + \sin^2(2\pi x)e^{-2t} \right].$$

The analytic solution for Equations (5.1) and (5.2) is  $u(x, t) = e^{-t} \sin(2\pi x)$ .

We compute the numerical solutions to this problem by the difference scheme (2.1)-(2.3) when  $\gamma = 0.01$  and  $T = 1$ . Take  $h = \frac{1}{M}$ ,  $t = \frac{T}{N}$ . Denote  $\{U_i^n(h, k), \quad 1 \leq i \leq M, \quad 0 \leq n \leq N\}$  the approximate solution. Suppose

$$\max_{0 \leq n \leq N} \max_{1 \leq i \leq M} |u(x_i, t^n) - U_i^n(h, k)| = O(h^q) + O(k^p).$$

If  $O(k^p)$  is sufficiently small, we have

$$\max_{1 \leq i \leq M} |u(x_i, t^N) - U_{2i}^N(\frac{h}{2}, k)| = O(h^q).$$

Let  $E(h) = \max_{1 \leq i \leq M} |u(x_i, t^N) - U_{2i}^N(\frac{h}{2}, k)|$ , then

$$\log_2 \left( \frac{E(h)}{E(\frac{h}{2})} \right) \approx q.$$

TABLE 1. The maximum norm errors and spatial convergence order with fixed time step  $k = 1/10000$ .

$h$	$E(h)$	$q$
1/20	$1.42525 \times 10^{-4}$	3.988
1/40	$8.98425 \times 10^{-6}$	3.999
1/80	$5.61664 \times 10^{-7}$	4.0437
1/160	$3.40578 \times 10^{-8}$	*

TABLE 2. The maximum norm errors and temporal convergence order with fixed space step  $h = 1/500$ .

$k$	$F(k)$	$p$
1/20	$3.5411 \times 10^{-4}$	2.28
1/40	$7.29336 \times 10^{-5}$	2.013
1/80	$1.80720 \times 10^{-5}$	2.008
1/160	$4.49395 \times 10^{-6}$	*

If  $O(h^q)$  is sufficiently small, we obtain

$$\max_{1 \leq i \leq M} |u(x_i, t^N) - U_{2i}^N(h, \frac{k}{2})| = O(k^p).$$

Denote  $F(k) = \max_{1 \leq i \leq M} |u(x_i, t^N) - U_{2i}^N(h, \frac{k}{2})|$ , thus

$$\log_2 \left( \frac{F(k)}{F(\frac{k}{2})} \right) \approx p.$$

In Table 1, the computation results are given for different space steps and when the time grid is fixed to be  $k = 1/10000$ . From this table, we conclude that the convergence order  $q$  in space is about 4, which is in accordance with the theoretical analysis.

Next, we test the time convergence order of the scheme (2.1)-(2.3). Fix spatial step  $h = 1/500$ . Table 2 shows that the convergence order is about 2 with respect to the temporal direction, which agrees with the prediction. Therefore, we conclude that the difference scheme (2.1)-(2.3) is convergent with the convergence order of  $O(h^4 + k^2)$  in maximum norm, which is in accordance Theorem 5.

## 6. Conclusion

In this article, we proposed a nonlinear finite difference scheme for solving the extended Fisher-Kolmogorov equation. Based on the extrapolation technique on the finite difference schemes, one can obtain higher-order accuracy by using certain combinations of difference solutions with various grid parameters. The existence and uniqueness were showed by the discrete energy method. Furthermore, the difference scheme is without any restrictions on the grid ratios, and the convergent order in maximum norm is two in temporal direction and four in spatial direction. Numerical results have been presented and verified the theoretical results. Future works are planned to study the maximum norm-convergence of high-order accurate difference scheme for solving the extended Fisher-Kolmogorov equation in two-dimensions.

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