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THE CAUCHY PROBLEM FOR AN INTEGRABLE GENERALIZED CAMASSA-HOLM EQUATION WITH CUBIC NONLINEARITY

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Abstract. This paper studies the Cauchy problem and blow-up phenomena for a new generalized Camassa-Holm equation with cubic nonlinearity in the nonhomogeneous Besov spaces. First, by means of the Littlewood-Paley decomposition theory, we investigate the local well-posedness of the equation in $B^s_{p,r}$ with $s > \max\{\frac{1}{p}, \frac{1}{2}, 1 - \frac{1}{p}\}$, $p, r \in [0, \infty]$. Second, we prove that the equation is locally well-posed in $B^s_{2,r}$ with the critical index $s = \frac{1}{2}$ by virtue of the logarithmic interpolation inequality and the Osgood's Lemma, and it is shown that the data-to-solution mapping is Hölder continuous. Finally, we derive two kinds of blow-up criteria for the strong solution by using induction and the conservative property of m along the characteristics.

1. Introduction

In this paper, we consider the Cauchy problem and blow-up phenomena for the following generalized Camassa-Holm equation with cubic nonlinearity:

$$\begin{cases}(1-\partial_x^2)u_t = u_x^2 u_{xxx} + u_x u_{xx}^2 + 2uu_x u_{xxx} + uu_{xx}^2 + u_x^2 u_{xx} \\ \quad +u^2 u_{xxx} - u_x^3 - u^2 u_{xx} - 3uu_x^2 - 2u^2 u_x,\ x \in \mathbb{R},\ t \geq 0, \\ u(x,0) = u_0(x),\ x \in \mathbb{R}.\end{cases} \tag{1.1}$$

The Equ. (1.1) was recently proposed by Novikov in [38], in which it is shown that the higher symmetries of this equation are quasi-local and the first one reads

$$(1 + D_x)u_\tau = m^{-7}(mm_{xx} - 3m_x^2 - 2mm_x),\ m = u - u_{xx}.$$

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Denoting $m = u - u_{xx}$, one can transform the Equ. (1.1) into the equivalent form:

$$\begin{cases} m_t = -u_x^2 m_x - 2muu_x + m^2 u_x \\ \; -2uu_x m_x + mu(m-u) - mu_x^2 - u^2 m_x, \; x \in \mathbb{R}, \; t \geq 0, \\ m(x,0) = m_0(x), \; x \in \mathbb{R}. \end{cases} \tag{1.2}$$

In [38], the author proved that the Equ. (1.2) possesses an infinite hierarchy of local higher symmetries in m, and the first such symmetry is given by

$$m_\tau = (1 - D_x) m^{-7} (mm_{xx} - 3m_x^2 - 2mm_x), \; m = u - u_{xx}.$$

It is worth pointing out that the Equ. (1.2) is a cubic integrable equation, which is a special case of the integrable non-evolutionary partial differential equations of the form

$$(1 - D_x^2) u_t = F(u, u_x, u_{xx}, u_{xxx}, \ldots), \; u = u(x,t), \; D_x = \frac{\partial}{\partial_x}, \tag{1.3}$$

where F is some function of u and its derivatives with respect to x (see Novikov [38] for more details). The Equ. (1.3) contains another two integrable equations with cubic nonlinearity, which have attracted much attention in the past few years.

The first one is the following Novikov equation:

$$m_t + u^2 m_x + 3uu_x m = 0, \; m = u - u_{xx}. \tag{1.4}$$

After its derivation, many papers were devoted to the studying of the Novikov equation. For instance, Himonas and Holliman considered the Cauchy problem of (1.4) in Sobolev space $H^s(\mathbb{R})(s > \frac{3}{2})$ on both $\mathbb{R}$ and $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ [25]. In [22], Grayshan investigated the non-periodic and periodic Cauchy problems for equation (1.4) in $H^s(\mathbb{R})$ with $s < \frac{3}{2}$. With some sign condition on the initial data, Lai etc. established the existence of global weak solution for Novikov equation in the Sobolev space $H^s(\mathbb{R})$ for $1 \leq s \leq \frac{3}{2}$ [32]. However, without sign condition on the initial data, Lai also obtained the existence of global weak solutions for the Novikov equation in the space $C([0,\infty);\mathbb{R}) \cap L^\infty([0,\infty); H^1(\mathbb{R}))$ [31]. In [37], Ni and Zhou proved the locally well-posedness for the Novikov equation in the critical Besov spaces $B^{\frac{3}{2}}_{2,1}$, and they also studied the well-posedness in $H^s(s > \frac{3}{2})$ by using the Kato's semigroup theory. For $s \geq 3$, it is shown that the orbit invariants can be applied to investigate the existence of periodic global strong solution [41]. In [33], Lenells and Wunsch proved that the weakly dissipative versions of the Novikov equation are equivalent to their non-dissipative counterparts up to a simple change of variables. For the other works about Equ. (1.4), see [29, 36, 44–46] and the references therein.

The second one is the cubic Fokas-Olver-Rosenau-Qiao (FORQ) equation:

$$m_t + (u^2 - u_x^2) m_x + 2u_x m^2 = 0, \; m = u - u_{xx}. \tag{1.5}$$

The Equ. (1.5) was introduced by Fuchssteiner [21] and Olver and Rosenau [39] as a new generalization of integrable system by implemention a simple explicit algorithm based on the bi-Hamiltonian representation of the classical integrable system. The FORQ equation possesses the Lax pair, and it is shown that the Cauchy problem can be solved by the inverse scattering transform method. In [23], Gui etc. proved that the FORQ equation admits single peakons in the form of $u(x,t)=\sqrt{\frac{3c}{2}}e^{-|x-ct|}(c>0)$. Moreover, it is shown in [20] that the FORQ equation does not have any nontrivial smooth traveling wave solutions. Being inspired by the approach developed in [14], the authors also established the local well-posedness of the FORQ equation in $B^s_{p,r}(\mathbb{R})$ with $s>\max\{\frac{5}{2},2+\frac{1}{p}\}$ and $p,r\in[1,\infty]$ [20]. And the well-posedness in the Besov space $B^s_{2,1}(\mathbb{R})$ with the critical index $s=\frac{5}{2}$ is investigated [20]. In [26], Himonas and Mantzavinos studied the well-posedness of (1.5) in $H^s(\mathbb{R})(s>\frac{5}{2})$, and they proved that the solution mapping is continuous but not uniformly continuous. For more papers concerning the FORQ equation, we refer the readers to [27, 47, 48].

The most celebrated integrable member of Equ. (1.3) is the Camassa-Holm (CH) equation which has quadratic nonlinearity:

$$m_t+um_x+2u_xm=0,\ \ m=u-u_{xx}. \tag{1.6}$$

It was originally derived by Camassa and Holm [4] to describe the undirectional propagation of shallow water waves over a flat bottom. It possesses a bi-Hamilton structure and has infinite conservation laws [4, 8]. One of the most interesting properties of the CH equation is that it has peakon solutions $ce^{-|x-ct|}$ for $c>0$, which describes an fundamental characteristic of the traveling waves of largest amplitude [13]. The local well-posedness and blow-up phenomena of the CH equation in the Sobolev and the Besov spaces are investigated in [7, 9–11, 14, 40]. For the global existence of the weak and strong solutions, we refer the readers to [7, 9, 10, 12, 42]. For the global conservative and dissipative solutions to the CH equation, see for example [2, 3, 28].

One of the closest relatives of Equ. (1.6) is the Degasperis-Procesi (DP) equation [17]:

$$u_t-u_{xxt}=4uu_x-3u_xu_{xx}-uu_{xxx}. \tag{1.7}$$

The DP equation is similar to the CH equation in several aspects, such as the asymptotic accuracy, the completely integrability and the bi-Hamiltonian structure [16]. However, the two equations are truly different. One of the novel features of the DP equation is that it not only has peakon solutions [16] and periodic peakon solutions [43], but also the periodic shock waves [19] and shock peakons [35]. For the other works, see for example [6, 18, 24, 34, 49] and the references therein.

To our best knowledge, the Cauchy problem and blow-up phenomena for the Equ. (1.1) (or (1.2)) has not been studied yet. In this paper, by using the Littlewood-Paley theory, we first establish the local existence and uniqueness of the solution in $B^s_{p,r}$ with $s > \max\{\frac{1}{p}, \frac{1}{2}, 1-\frac{1}{p}\}$, $p, r \in [0, \infty]$. Second, we investigate the local well-posedness in $B^s_{2,r}$ with the critical index $s = \frac{1}{2}$. Unfortunately, compared with the noncritical case, it seems impossible to find any convergence subsequence of the approximation solutions in $C([0,T]; B^{\frac{1}{2}}_{2,1}(\mathbb{R}))$ directly, because the priori estimates in noncritical case depend strongly on the inequality $\|fg\|_{B^s_{p,r}} \le C\|f\|_{B^s_{p,r}}\|g\|_{B^{s+1}_{p,r}}$, which does not hold any more for $s = -\frac{1}{2}, p = 2$ and $r = 1$. However, by taking advantage of a new Moser-type interpolation inequality (see Lemma 2.8 below), we can overcome this problem caused by the low regularity of $B^{-\frac{1}{2}}_{2,1}(\mathbb{R})$, and then prove the convergence of approximation solutions in $C([0,T]; B^{-\frac{1}{2}}_{2,\infty}(\mathbb{R}))$ with the help of the logarithmic interpolation inequality and the Osgood's Lemma. Moreover, we show that the solution mapping is Hölder continuous. Finally, we give two blow-up criteria for the strong solutions by using induction and the method of characteristics.

The paper is organized as follows. In Section 2, we recall some facts on the Littwood-Paley theory and the transport theory in Besov spaces. Section 3 is devoted to the local well-posedness of the Equ. (1.1) in the Besov spaces. In Section 4, we derive two kinds of blow-up criteria for the strong solution by using mathematical induction and the characteristics method.

Notation. All function spaces are considered in $\mathbb{R}$, and we shall drop them in our notation if there is no ambiguity. We denote by C the estimates that hold up to some universal constant which may change from line to line but whose meaning is clear throughout the context.

2. Preliminaries.

Unless otherwise specified, all the results presented in this section have been proved in [1, 5, 15]. Fix a function $\psi(x) \in C_0^\infty(\mathcal{B}_{4/3})$ with $\mathcal{B}_{4/3} = \{x \in \mathbb{R}; |x| \le \frac{4}{3}\}$ and a function $\varphi(x) \in C_0^\infty(\mathcal{C})$ with $\mathcal{C} = \{x \in \mathbb{R}; \frac{3}{4} \le |x| \le \frac{8}{3}\}$ such that

$$\begin{aligned}
&\psi(x) + \sum_{j\ge 0}\varphi(2^{-j}x) = 1, \ \forall x \in \mathbb{R},\\
&|i-j| \ge 2 \Longrightarrow \text{supp } \varphi(2^{-j}\cdot) \cap \text{supp } \varphi(2^{-i}\cdot) = \emptyset,\\
&j \ge 1 \Longrightarrow \text{supp } \psi(\cdot) \cap \text{supp } \varphi(2^{-j}\cdot) = \emptyset.
\end{aligned}$$

The nonhomogeneous Littlewood-Paley decomposition operators $\{\Delta_q\}_{q\ge -1}$ are defined by

$$\Delta_q u = 0, \text{ if } q < -1; \ \Delta_{-1}u \triangleq \psi(D)u; \ \Delta_q u \triangleq \varphi(2^{-q}D)u, \text{ if } q \ge 0,$$

where $f(D)$ stands for the pseudo-differential operator $u \to \mathscr{F}^{-1}(f\mathscr{F}u)$. Since $\varphi(\xi) = \psi(\frac{\xi}{2}) - \psi(\xi)$, we also introduce the following low frequency cut-off operator S_q:

$$S_q u \triangleq \varphi(2^{-q}D)u = \sum_{1\le p\le q-1}\Delta_p u.$$

Definition 2.1. Let $1 \leq p, r \leq \infty$, $s \in \mathbb{R}$, the 1-D nonhomogeneous Besov space $B^s_{p,r}$ is defined by

$$B^s_{p,r} \triangleq \{u \in \mathscr{S}';\ \|u\|_{B^s_{p,r}} = \Big(\sum_{j\in\mathbb{N}} 2^{rsj}\|\Delta_j u\|^r_{L^p}\Big)^{\frac{1}{r}} < \infty\}.$$

If $r = \infty$, $B^\infty_{p,r} \triangleq \bigcap_{s>0} B^s_{p,r}$.

Lemma 2.2. *Let $\mathscr{C}$ be an annulus and B a ball, there exists a constant $C > 0$ such that for $\forall k \in \mathbb{N}^+$, $u \in L^p$ and $(p,q) \in [1,\infty]^2$ with $q \geq p \geq 1$, the following estimates hold*:

$$Supp\ \widehat{u} \subseteq \lambda B \Rightarrow \|D^k u\|_{L^q} \triangleq \sup_{|\alpha|\leq k}\|\partial^\alpha u\|_{L^q} \leq C^{k+1}\lambda^{k+d(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p},$$

$$Supp\ \widehat{u} \subseteq \lambda\mathscr{C} \Rightarrow C^{-k-1}\lambda^k\|u\|_{L^p} \leq \|D^k u\|_{L^p} \leq C^{k+1}\lambda^k\|u\|_{L^p}.$$

Definition 2.3. Let u and v be two temperate distributions. The nonhomogeneous paraproduct of v by u is defined by

$$T_u v \triangleq \sum_j S_{j-1}u\Delta_j v.$$

The nonhomogeneous remainder of u and v is defined by

$$R(u,v) \triangleq \sum_{|k-j|\leq 1} \Delta_k u\Delta_j v.$$

The Bony decomposition of uv is given by

$$uv \triangleq T_u v + T_v u + R(u,v).$$

Lemma 2.4. *For any $(s,t) \in \mathbb{R}\times(-\infty,0)$ and any $(p,r_1,r_2) \in [1,\infty]^3$, there exists a positive constant C such that, for $r = \min\{1, \frac{1}{r_1}+\frac{1}{r_2}\}$, the following estimates hold*:

$$\|T_u v\|_{B^s_{p,r}} \leq C\|u\|_{L^\infty}\|D^k v\|_{B^{s-k}_{p,r}},$$

$$\|T_u v\|_{B^{s+t}_{p,r}} \leq C\|u\|_{B^t_{\infty,r_1}}\|D^k v\|_{B^{s-k}_{p,r_2}}.$$

Lemma 2.5. *Let $(s_1,s_2) \in \mathbb{R}^2$ and $(p_1,p_2,r_1,r_2) \in [1,\infty]^4$. Assume that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2},\ \ \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}.$$

Then there exists a constant $C > 0$ such that the following estimates hold:

(1) *If $s_1 + s_2 > 0$, for any (u,v) in $B^{s_2}_{p_1,r_1} \times B^{s_2}_{p_2,r_2}$, we have*

$$\|R(u,v)\|_{B^{s_1+s_2}_{p,r}} \leq C\|u\|_{B^{s_1}_{p,r}}\|v\|_{B^{s_2}_{p,r}}.$$

(2) *If $r = 1$ and $s_1 + s_2 = 0$, for any (u,v) in $B^{s_2}_{p_1,r_1} \times B^{s_2}_{p_2,r_2}$, we have*

$$\|R(u,v)\|_{B^0_{p,\infty}} \leq C\|u\|_{B^{s_1}_{p,r}}\|v\|_{B^{s_2}_{p,r}}.$$

Lemma 2.6. *Let $s \in \mathbb{R}$, $1 \le p, r, p_1, r_i \le \infty$, $i = 1, 2$, then*

(1) *Algebraic properties: if $s > 0$, $B^s_{p,r} \bigcap L^\infty$ is an algebra. Furthermore, $B^s_{p,r}$ is an algebra provided that $s > 1/p$ or $s = 1/p$ and $r = 1$.*

(2) *Embedding: If $p_1 \le p_2$ and $r_1 \le r_2$, then $B^s_{p_1,r_1} \hookrightarrow B^{s-(1/p_1-1/p_2)}_{p_2,r_2}$. If $s_1 < s_2$, the embedding $B^{s_2}_{p,r_2} \hookrightarrow B^{s_1}_{p,r_1}$ is locally compact.*

(3) *Fatou's lemma: if $\{u_n\}_{n\in\mathbb{N}^+}$ is bounded in $B^s_{p,r}$ and $u_n \to u$ in $\mathscr{S}'$, then $u \in B^s_{p,r}$ and*

$$\|u\|_{B^s_{p,r}} \le \liminf_{n\to\infty} \|u^n\|_{B^s_{p,r}}.$$

(4) *If $s_1 \le \frac{1}{p} < s_2$ ($s_2 \ge \frac{1}{p}$ if $r = 1$) and $s_1 + s_2 > 0$, then we have*

$$\|uv\|_{B^{s_1}_{p,r}} \le C\|u\|_{B^{s_1}_{p,r}} \|v\|_{B^{s_2}_{p,r}}.$$

(5) *A smooth function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be an S^m-multiplier: if $\forall \alpha \in \mathbb{N}^n$, there exists a constant $C_\alpha > 0$ such that $|\partial^\alpha f(\xi)| \le C_\alpha(1 + |\xi|)^{m-|\alpha|}$ for all $\xi \in \mathbb{R}^d$. The operator $f(D)$ is continuous from $B^s_{p,r}$ to $B^{s-m}_{p,r}$ for all $s \in \mathbb{R}$ and $1 \le p, r \le \infty$.*

Lemma 2.7. (1) *Let $s_2 > s_1$, $\theta \in (0, 1)$, we have*

$$\|u\|_{B^{\theta s_1+(1-\theta)s_2}_{p,1}} \le \frac{C}{s_2 - s_1}\left(\frac{1}{\theta} + \frac{1}{1-\theta}\right)\|u\|^\theta_{B^{s_1}_{p,\infty}} \|u\|^{1-\theta}_{B^{s_2}_{p,\infty}}.$$

(2) *For $\forall s \in \mathbb{R}$, $\epsilon > 0$ and $1 \le p \le \infty$, there exists a constant $C > 0$ such that*

$$\|u\|_{B^s_{p,1}} \le C\frac{\epsilon + 1}{\epsilon}\|u\|_{B^s_{p,\infty}}\left(1 + \log\frac{\|u\|_{B^{s+\epsilon}_{p,\infty}}}{\|u\|_{B^s_{p,\infty}}}\right).$$

Lemma 2.8. *For any $f \in B^{-\frac{1}{2}}_{2,\infty}$ and $g \in B^{\frac{1}{2}}_{2,\infty} \cap L^\infty$, there exists a constant $C > 0$ such that*

$$\|fg\|_{B^{-\frac{1}{2}}_{2,\infty}} \le C\|f\|_{B^{-\frac{1}{2}}_{2,\infty}} \|g\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty}.$$

Lemma 2.9. *Let $1 \le p, r \le \infty$, $s \ge -\min(\frac{1}{p}, 1-\frac{1}{p})$, and consider the following transport equation:*

$$\partial_t f + v\partial_x f = g, \;\; f(x, 0) = f_0(x). \tag{2.1}$$

Assume that $f_0 \in B^s_{p,r}$ and $g \in L^1([0, T]; B^s_{p,r})$. For any solution

$$f \in L^\infty([0, T]; B^s_{p,r})$$

of (2.1) with $v_x \in L^1([0, T]; B^{s-1}_{p,r})$ if $s > 1 + \frac{1}{p}$ or $v_x \in L^1([0, T]; B^{\frac{1}{p}}_{p,r} \bigcap L^\infty)$ otherwise.

(1) *If $r = 1$ or $s \ne 1 + \frac{1}{p}$, there exists $C > 0$ depending only on s, p and r such that*

$$\|f\|_{B^s_{p,r}} \le \exp\{CV_p(t)\}\|f_0\|_{B^s_{p,r}} + \int_0^t \exp\{CV_p(t) - CV_p(s)\}\|g(s)\|_{B^s_{p,r}} ds, \tag{2.2}$$

with

$$V_p(t) \triangleq \begin{cases} \int_0^t \|v_x(s)\|_{B^{\frac{1}{p}}_{p,\infty}\bigcap L^\infty} ds, & \text{if } s < 1+\frac{1}{p}; \\ \int_0^t \|v_x(s)\|_{B^{s-1}_{p,r}} ds, & \text{if } s > 1+\frac{1}{p} \text{ or } s = 1+\frac{1}{p}, r = 1. \end{cases}$$

(2) *If* $r < \infty$, *then* $f \in C([0,T]; B^s_{p,r})$. *If* $r = \infty$, *then* $f \in C([0,T]; B^{s'}_{p,1})$, *for all* $s' < s$.

(3) *If* $v = f$ *and* $s > 0$, *the inequality* (2.2) *holds true with* $V_p(t) := \int_0^t \|v_x(s)\|_{L^\infty} ds$.

Lemma 2.10. *Let* p, r, s, f_0 *and* g *be as in Lemma 2.9. Suppose that* $v \in L^\rho([0,T]; B^{-M}_{\infty,\infty})$ *for some* $\rho > 1$, $M > 0$ *and* $v_x \in L^1([0,T]; B^{s-1}_{p,r})$ *if* $s > 1+\frac{1}{p}$ *or* $s = 1+\frac{1}{p}$ *and* $r = 1$, *and* $v_x \in L^1([0,T]; B^{\frac{1}{p}}_{p,\infty}\bigcap L^\infty)$ *if* $s < 1+\frac{1}{p}$. *Then the transport equation* (2.1) *admits a unique solution* u *in the space* $C([0,T]; B^s_{p,r})$ *if* $r < \infty$ *or* $L^\infty([0,T]; B^s_{p,r})\bigcap(\bigcap_{s'<s} C([0,T]; B^{s'}_{p,1})$ *if* $r < \infty$. *Moreover, the inequalities of Lemma 2.9 hold true.*

Lemma 2.11. *Let* $\rho \geq 0$ *be a measurable function,* $\gamma > 0$ *be a locally integrable function and* μ *be a continuous and increasing function. For some* $a \geq 0$, *if*

$$\rho(t) \leq a + \int_{t_0}^t \gamma(s)\mu(\rho(s))ds.$$

(1) *If* $a > 0$, *then* $-\mathcal{M}(\rho(t)) + \mathcal{M}(a) \leq \int_{t_0}^t \gamma(s)ds$, *where* $\mathcal{M}(x) \triangleq \int_x^1 \frac{1}{\mu(r)}dr$.

(2) *If* $a = 0$ *and* μ *satisfies the condition* $\int_0^1 \frac{dr}{\mu(r)}dr = +\infty$, *then* $\rho \equiv 0$.

3. Local well-posedness in the Besov spaces

In this section, we shall investigate the local well-posedness of the initial value problem (1.2) in the nonhomogeneous Besov spaces.

3.1. Local well-posedness in $B^s_{p,r}$ *with* $s > \max\{\frac{1}{p}, \frac{1}{2}, 1-\frac{1}{p}\}$, $p, r \in [1,\infty]$.

Noting that the Equ. (1.2) can be reformulated in the form of

$$\begin{cases} m_t + (u+u_x)^2 m_x = m^2(u+u_x) - m(u+u_x)^2, \ x \in \mathbb{R}, \ t \geq 0, \\ m(x,0) = m_0(x), \ x \in \mathbb{R}, \end{cases} \tag{3.1}$$

where $m = u - u_{xx}$ is the momentum variable in the physical sense. To give the main result in this subsection, we introduce the following spaces for convenience.

Definition 3.1. For $\forall T > 0$, $s \in \mathbb{R}$ and $1 \leq p \leq +\infty$, we set

$$E^s_{p,r}(T) \triangleq \begin{cases} C([0,T]; B^s_{p,r}) \cap C^1([0,T]; B^{s-1}_{p,r}), \ \text{if } r < \infty, \\ L^\infty([0,T]; B^s_{p,\infty}) \cap \text{Lip}([0,T]; B^{s-1}_{p,\infty}), \ \text{if } r = \infty, \end{cases}$$

and $E^s_{p,r} = \bigcap_{T>0} E^s_{p,r}(T)$.

The uniqueness and existence of the solution is ensured by the following theorem.

Theorem 3.2. *Let $m_0 \in B^s_{p,r}$ be the initial data, where $s > \max\{\frac{1}{p}, \frac{1}{2}, 1-\frac{1}{p}\}$, $p, r \in [1,\infty]$. Then there exists a time $T \triangleq T(\|m_0\|_{B^s_{p,r}}) > 0$ such that the Equ.* (3.1) *admits a unique solution $m \in E^s_{p,r}(T)$. Moreover, the solution mapping $m_0 \mapsto m$ is continuous from $B^s_{p,r}$ into $C([0,T]; B^{s'}_{p,r}) \cap C^1([0,T]; B^{s'-1}_{p,r})$ for all $s' < s$ if $r = \infty$, and $s' = s$ if $1 \le r < \infty$.*

By using the Littlewood-Paley decomposition and the Plancherel's formula, the Besov space $B^s_{2,2}$ is equivalent to the Sobolev spaces H^s. Therefore, Theorem 3.2 implies the following result.

Corollary 3.3. *Let $m_0 \in H^s$ with $s > \frac{1}{2}$, there exists a $T \triangleq T(\|m_0\|_{H^s}) > 0$ such that the Equ.* (3.1) *admits a unique solution*

$$m \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-1}).$$

Moreover, the solution mapping $m_0 \to m$ is continuous.

The uniqueness and continuity with respect to the initial data in some sense can be obtained by the following priori estimates.

Lemma 3.4. *Assume that $s > \max\{\frac{1}{p}, \frac{1}{2}, 1-\frac{1}{p}\}$ with $p, r \in [1,\infty]$. Let $m^{(1)}, m^{(2)} \in L^\infty([0,T]; B^s_{p,r}) \cap C([0,T]; \mathcal{S}')$ be two solutions to the Equ.* (3.1) *with respect to the initial datum $m^{(1)}_0, m^{(2)}_0 \in B^s_{p,r}$, respectively. Denoting*

$$m^{(12)} \triangleq m^{(1)} - m^{(2)} \text{ and } m^{(12)}_0 \triangleq m^{(1)}(0) - m^{(2)}(0),$$

then for $\forall t \in [0,T]$, the following priori estimates hold:

(1) *If $s > \max\{1-\frac{1}{p}, \frac{1}{p}, \frac{1}{2}\}$ but $s \neq 2+\frac{1}{p}$, we have*

(3.2)

$$\|m^{(12)}(t)\|_{B^{s-1}_{p,r}} \le \|m^{(12)}_0\|_{B^{s-1}_{p,r}} \exp\left\{C\int_0^t (\|m^{(2)}(s)\|^2_{B^s_{p,r}} + \|m^{(1)}(s)\|^2_{B^s_{p,r}}) ds\right\}.$$

(2) *If $s = 2+\frac{1}{p}$, for any $\theta \in (0,1)$, we have*

$$\begin{aligned} \|m^{(12)}\|_{B^{1+\frac{1}{p}}_{p,r}} \le\, & C\|m^{(12)}_0\|^\theta_{B^{1+\frac{1}{p}}_{p,r}} (\|m^{(1)}\|^{1-\theta}_{B^{2+\frac{1}{p}}_{p,r}} + \|m^{(2)}\|^{1-\theta}_{B^{2+\frac{1}{p}}_{p,r}}) \\ & \times \exp\left\{C\theta\int_0^t (\|m^{(2)}(s)\|^2_{B^{2+\frac{1}{p}}_{p,r}} + \|m^{(1)}(s)\|^2_{B^{2+\frac{1}{p}}_{p,r}}) ds\right\}. \end{aligned} \tag{3.3}$$

Proof. Obviously, the function $m^{(12)}(x,t)$ satisfies the following transport equation:

$$\begin{cases} m^{(12)}_t + (u^{(1)} + u^{(1)}_x)^2 m^{(12)}_x = \mathcal{F}^{(12)}(x,t), \\ m^{(12)}(x,0) = m^{(12)}_0(x) \triangleq m^{(1)}_0 - m^{(2)}_0, \end{cases} \tag{3.4}$$

where $\mathcal{F}^{(12)}(x,t)=[(u^{(2)}+u_x^{(2)})^2-(u^{(1)}+u_x^{(1)})^2]m_x^{(2)}+m^{(12)}(m^{(1)}+m^{(2)})(u^{(1)}+u_x^{(1)})+(m^{(2)})^2(u^{(12)}+u_x^{(12)})-m^{(12)}(u^{(2)}+u_x^{(2)})^2-m^{(1)}(u^{(2)}+u_x^{(2)}+u^{(1)}+u_x^{(1)})(u^{(12)}+u_x^{(12)})$.

For $s>\max\{\frac{1}{p},\frac{1}{2},1-\frac{1}{p}\}$ but $s\neq 2+\frac{1}{p}$, by applying the Lemma 2.9 to the first equation of (3.1), we deduce that

$$
\begin{aligned}
\|m^{(12)}(t)\|_{B^{s-1}_{p,r}} \leq \|m_0^{(12)}\|_{B^{s-1}_{p,r}}+\int_0^t \|\mathcal{F}^{(12)}(\cdot,t)\|_{B^{s-1}_{p,r}}ds \\
+C\int_0^t V_p'(s)\|m^{(12)}(s)\|_{B^{s-1}_{p,r}}ds,
\end{aligned} \tag{3.5}
$$

where

$$
V_p(t)\triangleq\int_0^t (\|\partial_x(u^{(1)}+u_x^{(1)})^2\|_{B^{\frac{1}{p}}_{p,r}\cap L^\infty}+\|\partial_x(u^{(1)}+u_x^{(1)})^2\|_{B^{s-2}_{p,r}})d\tau.
$$

Noting that $(1-\partial_x^2)^{-1}$ is a multiplier of degree -2, it follows from the (5) of Lemma 2.6 that

$$
C_1\|u^{(i)}\|_{B^{s+2}_{p,r}}\lesssim\|m^{(i)}\|_{B^s_{p,r}}\lesssim C_2\|u^{(i)}\|_{B^{s+2}_{p,r}},\ \text{for } \forall s\in\mathbb{R},\ i=12,1,2, \tag{3.6}
$$

where C_1 and C_2 are positive constants independent of the index i.

Since $s>\max\{1-\frac{1}{p},\frac{1}{p},\frac{1}{2}\}$, the Besov space $B^s_{p,r}$ is a Banach algebra, so we have

$$
\begin{aligned}
V_p'(t)&=\|\partial_x(u^{(1)}+u_x^{(1)})^2\|_{B^{\frac{1}{p}}_{p,r}\cap L^\infty}+\|\partial_x(u^{(1)}+u_x^{(1)})^2\|_{B^{s-2}_{p,r}}\\
&\leq C\|\partial_x(u^{(1)}+u_x^{(1)})^2\|_{B^s_{p,r}}+\|(u^{(1)}+u_x^{(1)})^2\|_{B^s_{p,r}}\\
&\leq C(\|u^{(1)}\|_{B^{s+1}_{p,r}}+\|u_x^{(1)}\|_{B^{s+1}_{p,r}})^2\leq C\|m^{(1)}\|^2_{B^s_{p,r}}.
\end{aligned} \tag{3.7}
$$

Next, let us estimate the term $\|\mathcal{F}^{(12)}(\cdot,t)\|_{B^{s-1}_{p,r}}$ in (3.5). If $\max\{\frac{1}{p},\frac{1}{2}\}<s\leq 1+\frac{1}{p}$, it is obvious that $s-1\leq\frac{1}{p}\leq s$ and $(s-1)+s>0$, by using the interpolation inequality ((4) of Lemma 2.6) and the fact that $B^s_{p,r}$ is a Banach algebra, we have

$$
\begin{aligned}
&\|[(u^{(2)}+u_x^{(2)})^2-(u^{(1)}+u_x^{(1)})^2]m_x^{(2)}\|_{B^{s-1}_{p,r}}\\
&\leq C\|(u^{(2)}+u_x^{(2)}+u^{(1)}+u_x^{(1)})(u^{(12)}+u_x^{(12)})\|_{B^s_{p,r}}\|m_x^{(2)}\|_{B^{s-1}_{p,r}}\\
&\leq C\|u^{(2)}+u_x^{(2)}+u^{(1)}+u_x^{(1)}\|_{B^s_{p,r}}\|u^{(12)}+u_x^{(12)}\|_{B^s_{p,r}}\|m^{(2)}\|_{B^s_{p,r}}\\
&\leq C(\|m^{(1)}\|_{B^s_{p,r}}+\|m^{(2)}\|_{B^s_{p,r}})\|m^{(2)}\|_{B^s_{p,r}}\|u^{(12)}\|_{B^{s+1}_{p,r}}\\
&\leq C(\|m^{(1)}\|^2_{B^s_{p,r}}+\|m^{(2)}\|^2_{B^s_{p,r}})\|m^{(12)}\|_{B^{s-1}_{p,r}}.
\end{aligned} \tag{3.8}
$$

By taking the similar argument, we can also estimate that

$$
\begin{aligned}
&\|m^{(12)}(m^{(1)}+m^{(2)})(u^{(1)}+u_x^{(1)})\|_{B^{s-1}_{p,r}}\\
&\leq C\|m^{(12)}\|_{B^{s-1}_{p,r}}\|(m^{(1)}+m^{(2)})(u^{(1)}+u_x^{(1)})\|_{B^s_{p,r}}
\end{aligned} \tag{3.9}
$$

$$\begin{aligned}
&\le C\|m^{(12)}\|_{B^{s-1}_{p,r}}(\|m^{(1)}\|_{B^{s}_{p,r}}+\|m^{(2)}\|_{B^{s}_{p,r}})(\|u^{(1)}\|_{B^{s}_{p,r}}+\|u^{(1)}\|_{B^{s+1}_{p,r}})\\
&\le C\|m^{(12)}\|_{B^{s-1}_{p,r}}(\|m^{(1)}\|^2_{B^{s}_{p,r}}+\|m^{(2)}\|^2_{B^{s}_{p,r}}),
\end{aligned}$$

$$\begin{aligned}
&\|(m^{(2)})^2(u^{(12)}+u^{(12)}_x)-m^{(12)}(u^{(2)}+u^{(2)}_x)^2\|_{B^{s-1}_{p,r}}\\
&\le C(\|m^{(2)}\|^2_{B^{s}_{p,r}}\|u^{(12)}+u^{(12)}_x\|_{B^{s-1}_{p,r}}+\|m^{(12)}\|_{B^{s-1}_{p,r}}\|(u^{(2)}+u^{(2)}_x)^2\|_{B^{s-1}_{p,r}})\\
&\le C(\|m^{(2)}\|^2_{B^{s}_{p,r}}\|u^{(12)}\|_{B^{s}_{p,r}}+\|m^{(12)}\|_{B^{s-1}_{p,r}}\|u^{(2)}+u^{(2)}_x\|^2_{B^{s}_{p,r}})\\
&\le C\|m^{(12)}\|_{B^{s-1}_{p,r}}\|m^{(2)}\|^2_{B^{s}_{p,r}},
\end{aligned}\tag{3.10}$$

$$\begin{aligned}
&\|m^{(1)}(u^{(2)}+u^{(2)}_x+u^{(1)}+u^{(1)}_x)(u^{(12)}+u^{(12)}_x)\|_{B^{s-1}_{p,r}}\\
&\le C\|m^{(1)}(u^{(2)}+u^{(2)}_x+u^{(1)}+u^{(1)}_x)\|_{B^{s}_{p,r}}\|u^{(12)}+u^{(12)}_x\|_{B^{s-1}_{p,r}}\\
&\le C\|m^{(1)}\|_{B^{s}_{p,r}}(\|u^{(2)}\|_{B^{s+1}_{p,r}}+\|u^{(1)}\|_{B^{s+1}_{p,r}})\|u^{(12)}\|_{B^{s+1}_{p,r}}\\
&\le C\|m^{(12)}\|_{B^{s-1}_{p,r}}(\|m^{(2)}\|^2_{B^{s}_{p,r}}+\|m^{(1)}\|^2_{B^{s}_{p,r}}).
\end{aligned}\tag{3.11}$$

Putting the estimates (3.8)-(3.11) together, we obtain

$$\|\mathcal{F}^{(12)}(\cdot,t)\|_{B^{s-1}_{p,r}}\le C(\|m^{(2)}\|^2_{B^{s}_{p,r}}+\|m^{(1)}\|^2_{B^{s}_{p,r}})\|m^{(12)}\|_{B^{s-1}_{p,r}}.\tag{3.12}$$

Therefore, it follows from (3.5), (3.7) and (3.12) that

$$\begin{aligned}
\|m^{(12)}(t)\|_{B^{s-1}_{p,r}}\le{}&\|m^{(12)}_0\|_{B^{s-1}_{p,r}}\\
&+C\int_0^t\|m^{(12)}(\tau)\|_{B^{s-1}_{p,r}}(\|m^{(2)}\|^2_{B^{s}_{p,r}}+\|m^{(1)}\|^2_{B^{s}_{p,r}})d\tau\\
&+C\int_0^t\|m^{(1)}\|^2_{B^{s}_{p,r}}\|m^{(12)}(\tau)\|_{B^{s-1}_{p,r}}d\tau\\
\le{}&\|m^{(12)}_0\|_{B^{s-1}_{p,r}}\\
&+C\int_0^t\|m^{(12)}(\tau)\|_{B^{s-1}_{p,r}}(\|m^{(2)}\|^2_{B^{s}_{p,r}}+\|m^{(1)}\|^2_{B^{s}_{p,r}})d\tau.
\end{aligned}$$

By applying the Gronwall inequality, we get

$$\|m^{(12)}(t)\|_{B^{s-1}_{p,r}}\le\|m^{(12)}_0\|_{B^{s-1}_{p,r}}\exp\left\{C\int_0^t(\|m^{(2)}(\tau)\|^2_{B^{s}_{p,r}}+\|m^{(1)}(\tau)\|^2_{B^{s}_{p,r}})d\tau\right\}.\tag{3.13}$$

On the other hand, for $s>1+\frac{1}{p}$ but $s\neq 2+\frac{1}{p}$, the Besov space $B^{s-1}_{p,r}$ is a Banach algebra, by virtue of the property of (3.6), it is easier to obtain the desired inequality (3.2), and we shall omit the details here. Hence we have proved (1).

To deal with the criteria case $s=2+\frac{1}{p}$, we use the interpolation method. Indeed, by choosing $\theta=\frac{1}{2}(1-\frac{1}{2p})\in(0,1)$, it is clear that $s-1=1+\frac{1}{p}=\frac{1}{2p}\theta+(2+\frac{1}{2p})(1-\theta)$. By utilizing the interpolation inequality ((4) of Lemma

2.6), we deduce that

$$
\begin{aligned}
\|m^{(12)}\|_{B^{s-1}_{p,r}} &\le C\|m^{(12)}\|^{\theta}_{B^{\frac{1}{2p}}_{p,r}}\|m^{(12)}\|^{1-\theta}_{B^{2+\frac{1}{2p}}_{p,r}}\\
&\le C\|m^{(12)}_0\|^{\theta}_{B^{\frac{1}{2p}}_{p,r}}\exp\Big\{C\theta\int_0^t(\|m^{(2)}(\tau)\|^2_{B^{1+\frac{1}{2p}}_{p,r}}+\|m^{(1)}(\tau)\|^2_{B^{1+\frac{1}{2p}}_{p,r}})d\tau\Big\}\\
&\quad \|m^{(12)}\|^{1-\theta}_{B^{2+\frac{1}{2p}}_{p,r}}\\
&\le C\|m^{(12)}_0\|^{\theta}_{B^{\frac{1}{2p}}_{p,r}}\exp\Big\{C\theta\int_0^t(\|m^{(2)}(\tau)\|^2_{B^{1+\frac{1}{2p}}_{p,r}}+\|m^{(1)}(\tau)\|^2_{B^{1+\frac{1}{2p}}_{p,r}})d\tau\Big\}\\
&\quad \times(\|m^{(1)}\|^{1-\theta}_{B^{2+\frac{1}{2p}}_{p,r}}+\|m^{(2)}\|^{1-\theta}_{B^{2+\frac{1}{2p}}_{p,r}})\\
&\le C\|m^{(12)}_0\|^{\theta}_{B^{s-1}_{p,r}}(\|m^{(1)}\|^{1-\theta}_{B^{s}_{p,r}}+\|m^{(2)}\|^{1-\theta}_{B^{s}_{p,r}})\\
&\quad \times\exp\Big\{C\theta\int_0^t(\|m^{(2)}(\tau)\|^2_{B^{s}_{p,r}}+\|m^{(1)}(\tau)\|^2_{B^{s}_{p,r}})d\tau\Big\}. \qquad (3.14)
\end{aligned}
$$

This completes the proof of the Lemma 3.4. □

Next, we shall construct the approximation solution to the Equ. (3.1) by means of the classical Friedrich's regularization method.

Lemma 3.5. *Let p, r be the same as in the statement Lemma 3.4. Let $s > \max\{\frac{1}{p}, \frac{1}{2}, 1-\frac{1}{p}\}$ and $m_0 \in B^s_{p,r}$ be the initial data. Assume that $m^{(0)} = 0$, then*

(1) *there exists a sequence of smooth functions $\{m^{(n)}\}_{n\in\mathbb{N}^+} \in C([0,\infty); B^{\infty}_{p,r})$ which solves the following linear transport equation:*

(3.15)
$$
\begin{cases}
m^{(n+1)}_t+(u^{(n)}+u^{(n)}_x)^2 m^{(n+1)}_x=(m^{(n)})^2(u^{(n)}+u^{(n)}_x)-m^{(n)}(u^{(n)}+u^{(n)}_x)^2,\\
m^{(n+1)}(x,0)=m^{(n+1)}_0(x)\triangleq S_{n+1}m_0,
\end{cases}
$$

where S_{n+1} is the low frequency cut-off of m_0 given by $S_{n+1}m_0 = \sum_{q\ge -1}\Delta_q m_0$.

(2) *there exists a $T \triangleq T(\|m_0\|_{B^s_{p,r}}) > 0$ such that the sequence $\{m^{(n)}\}_{n\in\mathbb{N}^+}$ is uniformly bounded in $E^s_{p,r}(T)$, and it is also a Cauchy sequence in*

$$C([0,T]; B^{s-1}_{p,r})$$

and converges to some function $m \in C([0,T]; B^{s-1}_{p,r})$.

Proof. Noting that $S_{n+1}m_0 \in B^{\infty}_{p,r} \triangleq \bigcap_{s\in\mathbb{R}} B^s_{p,r}$, by virtue of the Lemma 2.10 and induction with respect to the index n, we deduce that the Equ. (3.15) admits a unique global solution $m^{(n+1)} \in C([0,T]; B^{\infty}_{p,r})$ for any $T > 0$. It remains to prove (2).

To this end, let us first show that the sequence $\{m^{(n)}\}_{n\in\mathbb{N}^+}$ is uniformly bounded in $C([0,T]; B^s_{p,r})$ for some $T > 0$ which depends on the initial data.

By applying the Lemma 2.9 to (3.15), we get

$$\|m^{(n+1)}(t)\|_{B^s_{p,r}} \leq \exp\{CV(t)\}\|S_{n+1}m_0\|_{B^s_{p,r}} + \int_0^t \exp\{CV(t)-CV(\tau)\} \times \Big(\|(m^{(n)})^2(u^{(n)}+u^{(n)}_x)\|_{B^s_{p,r}}+\|m^{(n)}(u^{(n)}+u^{(n)}_x)^2\|_{B^s_{p,r}}\Big)d\tau, \tag{3.16}$$

where the $V(t)$ is defined as follows

$$\begin{aligned} V(t) &\triangleq \int_0^t (\|\partial_x(u^{(n)}+u^{(n)}_x)^2\|_{B^{s-1}_{p,r}} + \|\partial_x(u^{(n)}+u^{(n)}_x)^2\|_{B^{\frac{1}{p}}_{p,r}\cap L^\infty})d\tau \\ &\leq C\int_0^t \|u^{(n)}+u^{(n)}_x\|^2_{B^s_{p,r}}d\tau \leq C\int_0^t \|m^{(n)}(\tau)\|^2_{B^s_{p,r}}d\tau. \end{aligned} \tag{3.17}$$

Since $s>\max\{\frac{1}{p},\frac{1}{2},1-\frac{1}{p}\}$, the Besov space $B^s_{p,r}$ is a Banach algebra, we can estimate that

$$\begin{aligned} &\|(m^{(n)})^2(u^{(n)}+u^{(n)}_x)\|_{B^s_{p,r}} + \|m^{(n)}(u^{(n)}+u^{(n)}_x)^2\|_{B^s_{p,r}} \\ &\leq C\|m^{(n)}\|^2_{B^s_{p,r}}\|u^{(n)}+u^{(n)}_x\|_{B^s_{p,r}} + C\|m^{(n)}\|_{B^s_{p,r}}\|u^{(n)}+u^{(n)}_x\|^2_{B^s_{p,r}} \\ &\leq C\|m^{(n)}\|^2_{B^s_{p,r}}(\|u^{(n)}\|_{B^s_{p,r}}+\|u^{(n)}\|_{B^{s+1}_{p,r}}) \\ &\quad + C\|m^{(n)}\|_{B^s_{p,r}}(\|u^{(n)}\|^2_{B^s_{p,r}}+\|u^{(n)}_x\|^2_{B^s_{p,r}}) \\ &\leq C\|m^{(n)}\|^3_{B^s_{p,r}} + C\|m^{(n)}\|_{B^s_{p,r}}(\|m^{(n)}\|^2_{B^{s-2}_{p,r}}+\|m^{(n)}\|^2_{B^{s-1}_{p,r}}) \\ &\leq C\|m^{(n)}\|^3_{B^s_{p,r}}. \end{aligned} \tag{3.18}$$

Plugging the estimates (3.17)-(3.18) into (3.16), we obtain

$$\begin{aligned} \|m^{(n+1)}(t)\|_{B^s_{p,r}} &\leq C\Big(\exp\{CV(t)\}\|m_0\|_{B^s_{p,r}} \\ &\qquad + \int_0^t \exp\{CV(t)-CV(\tau)\}\|m^{(n)}(\tau)\|^3_{B^s_{p,r}}d\tau\Big) \\ &\leq C\Big(\exp\Big\{C\int_0^t \|m^{(n)}(\tau)\|^2_{B^s_{p,r}}d\tau\Big\}\|m_0\|_{B^s_{p,r}} \\ &\qquad + \int_0^t \exp\Big\{C\int_\tau^t \|m^{(n)}(\tau')\|^2_{B^s_{p,r}}d\tau'\Big\}\|m^{(n)}(\tau)\|^3_{B^s_{p,r}}d\tau\Big). \end{aligned} \tag{3.19}$$

Here we have used the fact that $\|S_{n+1}m_0\|_{B^s_{p,r}} \leq C\|m_0\|_{B^s_{p,r}}$, where the positive constant C is independent of the index n.

If $\|m^{(n)}\|_{B^s_{p,r}}<1$, it follows from (3.19) that $\|m^{(n+1)}\|_{B^s_{p,r}} < C(\|m_0\|_{B^s_{p,r}}+t)$, which implies that the sequence $\{m^{(n)}\}_{n\in\mathbb{N}^+}$ is uniformly bounded. On the other hand, if $\|m^{(n)}\|_{B^s_{p,r}}\geq 1$, let us fix a $T>0$ such that $4TC^3\|m_0\|^2_{B^s_{p,r}}<1$, and

$$\|m^{(n)}(t)\|_{B^s_{p,r}} \leq \frac{C\|m_0\|_{B^s_{p,r}}}{\sqrt{1-4C^3\|m_0\|^2_{B^s_{p,r}}t}} \tag{3.20}$$

$$\leq \frac{C\|m_0\|_{B^s_{p,r}}}{\sqrt{1-4C^3\|m_0\|^2_{B^s_{p,r}}T}} \triangleq \overline{H} \text{ for } \forall t\in[0,T].$$

In view of the definition of $V(t)$, we can deduce from (3.20) that

$$\begin{aligned}\exp\{CV(t)-CV(\tau)\} &= \exp\left\{C\int_\tau^t \|m^{(n)}(\tau')\|^2_{B^s_{p,r}}d\tau'\right\}\\ &\leq \exp\left\{\int_\tau^t \frac{C^3\|m_0\|^2_{B^s_{p,r}}}{1-4C^3\|m_0\|^2_{B^s_{p,r}}t}d\tau'\right\}\\ &= \sqrt[4]{\frac{1-4C^3\|m_0\|^2_{B^s_{p,r}}\tau}{1-4C^3\|m_0\|^2_{B^s_{p,r}}t}}.\end{aligned} \tag{3.21}$$

Especially, by taking $\tau=0$, we have

$$\exp\{CV(t)\}\leq \frac{1}{\sqrt[4]{1-4C^3\|m_0\|^2_{B^s_{p,r}}t}} \text{ for } \forall t\in[0,T]. \tag{3.22}$$

By (3.20)-(3.22), the inequality (3.19) is equivalent to

$$\begin{aligned}&\|m^{(n+1)}(t)\|_{B^s_{p,r}}\\ &\leq C\left(\frac{\|m_0\|_{B^s_{p,r}}}{\sqrt[4]{1-4C^3\|m_0\|^2_{B^s_{p,r}}t}}+\int_0^t\sqrt[4]{\frac{1-4C^3\|m_0\|_{B^s_{p,r}}\tau}{1-4C^3\|m_0\|_{B^s_{p,r}}t}}\|m^{(n)}(\tau)\|^3_{B^s_{p,r}}d\tau\right)\\ &\leq C\left(\frac{\|m_0\|_{B^s_{p,r}}}{\sqrt[4]{1-4C^3\|m_0\|^2_{B^s_{p,r}}t}}+\frac{\|m_0\|_{B^s_{p,r}}}{\sqrt[4]{1-4C^3\|m_0\|^2_{B^s_{p,r}}t}}\int_0^t\frac{C^3\|m_0\|^2_{B^s_{p,r}}d\tau}{(1-4C^3\|m_0\|^2_{B^s_{p,r}}t)^{\frac{5}{4}}}\right)\\ &= \frac{C\|m_0\|_{B^s_{p,r}}}{\sqrt{1-4C^3\|m_0\|^2_{B^s_{p,r}}t}}\end{aligned}$$

for $\forall t\in[0,T]$. Hence we obtain that the sequence $\{m^{(n)}\}_{n\in\mathbb{N}^+}$ is uniformly bounded in the case of $\|m^{(n)}\|_{B^s_{p,r}}\geq 1$ in the interval $[0,T]$. On this base, utilizing the Equ. (3.15), it is not difficult to see that $\{\partial_t m^{(n)}\}_{n\in\mathbb{N}^+}$ is uniformly bounded in $C([0,T];B^{s-1}_{p,r})$. Therefore, we have proved the uniformly boundedness of the sequence $\{m^{(n)}\}_{n\in\mathbb{N}^+}$ in $E^s_{p,r}(T)$.

Next, we shall show that the sequence $\{m^{(n)}\}_{n\in\mathbb{N}^+}$ is a Cauchy sequence in $C([0,T];B^{s-1}_{p.r})$. To this end, let us consider the following equation:

(3.23)
$$\begin{cases} \quad \partial_t(m^{(n+k+1)}-m^{(n+1)})+(u^{(n+k)}+u_x^{(n+k)})^2\partial_x(m^{(n+k+1)}-m^{(n+k)}) \\ =[(u^{(n)}+u_x^{(n)})^2-(u^{(n+k)}+u_x^{(n+k)})^2]m_x^{(n+1)} \\ \quad +[(m^{(n+k)})^2-(m^{(n)})^2](u^{(n+k)}+u_x^{(n+k)}) \\ \quad +(m^{(n)})^2(u^{(n+k)}+u_x^{(n+k)}-u^{(n)}-u_x^{(n)}) \\ \quad +(m^{(n)}-m^{(n+k)})(u^{(n+k)}+u_x^{(n+k)})^2 \\ \quad +m^{(n+k)}[(u^{(n)}+u_x^{(n)})^2-(u^{(n+k)}+u_x^{(n+k)})^2] \\ \triangleq F(u^{(n)},u^{(n+k)},m^{(n)},m^{(n+k)}), \\ \quad (m^{(n+k+1)}-m^{(n+1)})(x,0)=S_{n+k+1}m_0-S_{n+1}m_0. \end{cases}$$

For $s>\max\{\frac{1}{p},\frac{1}{2},1-\frac{1}{p}\}$ but $s\neq 2+\frac{1}{p}$, by means of the Lemma 2.9, we obtain

(3.24)
$$\begin{aligned} &\|(m^{(n+k+1)}-m^{(n+1)})(t)\|_{B^{s-1}_{p,r}} \\ \leq{}& \|S_{n+k+1}m_0-S_{n+1}m_0\|_{B^{s-1}_{p,r}}+C\int_0^t V'(s)\|(m^{(n+k+1)}-m^{(n+1)})(t)\|_{B^{s-1}_{p,r}}ds \\ &+\int_0^t\|F(u^{(n)},u^{(n+k)},m^{(n)},m^{(n+k)})\|_{B^{s-1}_{p,r}}ds. \end{aligned}$$

Thanks to the uniformly boundedness of $\{m^{(n)}\}_{n\in\mathbb{N}^+}$ in the space $C([0,T];B^{s-1}_{p.r})$, we have

$$\begin{aligned} V'(s)&=\|\partial_x(u^{(n)}+u_x^{(n)})^2\|_{B^{s-1}_{p,r}}+\|\partial_x(u^{(n)}+u_x^{(n)})^2\|_{B^{\frac{1}{p}}_{p,r}\cap L^\infty} \\ &\leq C\|m^{(n)}\|^2_{B^s_{p,r}}\leq C \end{aligned} \tag{3.25}$$

for some positive constant C which is independent of n.

Similar to the estimates in the proof of Lemma 3.4, we can deduce that

$$\begin{aligned} &\|F(u^{(n)},u^{(n+k)},m^{(n)},m^{(n+k)})\|_{B^{s-1}_{p,r}} \\ \leq{}& C\|m^{(n+k)}-m^{(n)}\|_{B^{s-1}_{p,r}}\Big(\|m^{(n)}\|^2_{B^s_{p,r}}+\|m^{(n+k)}\|^2_{B^s_{p,r}}\Big). \end{aligned} \tag{3.26}$$

Noting that

$$\begin{aligned} \|m_0^{(n+k)}-m_0^{(n)}\|_{B^{s-1}_{p,r}}&=\|S_{n+k+1}m_0-S_{n+1}m_0\|_{B^{s-1}_{p,r}} \\ &\leq\left(\sum_{q\geq-1}2^{q(s-1)r}\left\|\Delta_q\left(\sum_{q=n+1}^{n+k}\Delta_q m_0\right)\right\|^r_{L^p}\right)^{\frac{1}{r}} \\ &\leq C\left(\sum_{q=n+1}^{n+k}2^{-qr}2^{qrs}\|\Delta_q m_0)\|^r_{L^p}\right)^{\frac{1}{r}}\leq C2^{-n}\|m_0\|_{B^s_{p,r}}, \end{aligned} \tag{3.27}$$

which combined with (3.24), (3.25) and (3.26) yield that

$$
\begin{aligned}
&\|(m^{(n+k+1)}-m^{(n+1)})(t)\|_{B^{s-1}_{p,r}}\\
&\le C\bigg(2^{-n}\|m_0\|_{B^s_{p,r}}+\int_0^t\|m^{(n+k)}(\tau)-m^{(n)}(\tau)\|_{B^{s-1}_{p,r}}d\tau\bigg).
\end{aligned}
\tag{3.28}
$$

Setting

$$\mathcal{H}^{n,k}(t)=\|(m^{(n+k)}-m^{(n)})(t)\|_{B^{s-1}_{p,r}}.$$

By (3.28) and using the induction with respect to n, for $\forall k\in\mathbb{N}^+$, we have

$$
\begin{aligned}
\sup_{t\in[0,T]}\mathcal{H}^{n+1,k}(t)&\le C\bigg(2^{-n}+\frac{2^{2-n}t^2}{2!}+\cdots+\frac{t^n}{n!}\bigg)+C\int_0^t\frac{(t-\tau)^n}{n!}\mathcal{H}_{0,k}(\tau)d\tau\\
&\le C2^{-n}\bigg(1+2T+\cdots+\frac{(2T)^n}{n!}\bigg)+\frac{CT^{n+1}}{(n+1)!}\\
&\le Ce^{2T}2^{-n}+\frac{CT^{n+1}}{(n+1)!}\longrightarrow 0 \text{ as } n\to\infty.
\end{aligned}
\tag{3.29}
$$

This implies that the approximation solution $\{m^{(n)}\}_{n\in\mathbb{N}^+}$ is a Cauchy sequence in $C([0,T];B^{s-1}_{p,r})$.

For the critical case $s=2+\frac{1}{p}$, by applying the interpolation method which has been used in the proof of Lemma 3.4, we can show that $\{m^{(n)}\}_{n\in\mathbb{N}^+}$ is a Cauchy sequence in $C([0,T];B^{1+\frac{1}{p}}_{p,r})$. So there exists m such that $m^{(n)}\to m$ in $C([0,T];B^{s-1}_{p,r})$.

This completes the proof of the Lemma 3.5. □

Based on the above two lemmas, we can now prove the main result in this section.

Proof of Theorem 3.2. By Lemma 3.5, $\{m^{(n)}\}_{n\in\mathbb{N}^+}$ is a Cauchy sequence in $C([0,T];B^{s-1}_{p,r})$, there exists a function $m\in C([0,T];B^{s-1}_{p,r})$ such that $m^{(n)}\to m$ in $C([0,T];B^{s-1}_{p,r})$ as $n\to\infty$. Since $\{m^{(n)}\}_{n\in\mathbb{N}^+}$ is uniformly bounded in $C([0,T];B^s_{p,r})$, using the Fatou Lemma for Besov spaces, we get

$$m\in L^\infty([0,T];B^s_{p,r}),\ \text{and}\ \|m\|_{L^\infty([0,T];B^s_{p,r})}\le C\liminf_{n\to\infty}\|m^{(n)}\|_{B^s_{p,r}}.$$

Since $m^{(n)}\to m$ in $C([0,T];B^{s-1}_{p,r})$, an interpolation argument ensures that the convergence holds true in $C([0,T];B^{s'}_{p,r})$, for $\forall s'<s$. Then, by passing to the limit in (3.15), we conclude that m is a solution to the Equ. (3.1) in the sense of $C([0,T];B^{s'-1}_{p,r})$, for $\forall s'<s$. Since $m\in L^\infty([0,T];B^s_{p,r})$, one can verify that the right hand side of the Equ. (3.1) belong to $L^\infty([0,T];B^s_{p,r})$. Using the Equ. (3.1) again, we see that $\partial_t m\in C([0,T];B^{s-1}_{p,r})$ for $r<\infty$ and $\partial_t m\in L^\infty([0,T];B^{s-1}_{p,r})$ otherwise, hence we get that $m\in E^s_{p,r}(T)$.

The continuity of the solution mapping in $C([0,T];B^s_{p,r})\cap C^1([0,T];B^{s-1}_{p,r})$ for $r<\infty$ can be proved by using a standard sequence of viscosity approximate solutions $\{m_\epsilon\}_{\epsilon>0}$ to the Equ. (3.1), which uniformly converges in $C([0,T];B^s_{p,r})$ $\cap C^1([0,T];B^{s-1}_{p,r})$.

This finishes the proof of Theorem 3.2. □

3.2. Local well-posedness in the critical Besov space $B^{\frac{1}{2}}_{2,1}$.

This subsection is devoted to the well-posedness of the Equ. (3.1) in $B^{\frac{1}{2}}_{2,1}$. However, as we said in the Introduction, one can not obtain the convergence of the sequence $\{m^{(n)}\}_{n\in\mathbb{N}^+}$ directly, since the estimate

$$\|fg\|_{B^s_{p,r}} \le C\|f\|_{B^s_{p,r}}\|g\|_{B^{s+1}_{p,r}}$$

which was applied in the proof of Theorem 3.2 does not hold anymore. What saves the game in some sense is a new Moser-type estimate (Lemma 2.8), which together with the logarithmic interpolation inequality and the Osgood's Lemma enable us to finish the proof of the desired result.

The main result in the subsection can be stated as follows.

Theorem 3.6. *Let $m_0 \in B^{\frac{1}{2}}_{2,1}$ be the initial data, there exists a*

$$T \triangleq T(\|m_0\|_{B^{\frac{1}{2}}_{2,1}}) > 0$$

such that the Equ. (3.1) *admits a unique solution*

$$m \in C([0,T];B^{\frac{1}{2}}_{2,1}) \cap C^1([0,T];B^{-\frac{1}{2}}_{2,1}).$$

Especially, for $T = \frac{3}{16C^3\|m_0\|^2_{B^{1/2}_{2,1}}}$, we have $\|m\|_{L^\infty([0,T];B^{\frac{1}{2}}_{2,1})} \le 2C\|m_0\|_{B^{\frac{1}{2}}_{2,1}}$.

Moreover, the solution mapping $\Phi : B^{\frac{1}{2}}_{2,1} \mapsto C([0,T];B^s_{2,1})$ is Hölder continuous, precisely,

$$\|m^{(2)} - m^{(1)}\|_{L^\infty([0,T];B^s_{2,1})} \le C\|m^{(2)}(0) - m^{(1)}(0)\|^{\theta\exp(-CT)}_{B^{\frac{1}{2}}_{2,1}},$$

for $\theta = \frac{1}{2} - s \in (0,1)$, where $m^{(i)}$ is the solution of Equ. (3.1) *corresponding to $m^{(i)}(0)$, $i = 1,2$.*

Proof. The proof of the Theorem 3.6 will be divided into several steps.

Step 1. Consider the approximation equation (3.15) in the previous subsection, and assume that $m^{(n)} \in L^\infty(0,T;B^{\frac{1}{2}}_{2,1})$. Noting that $B^{\frac{1}{2}}_{2,1}$ is a Banach algebra, it is easy to see that $(m^{(n)})^2(u^{(n)} + u^{(n)}_x)$, $m^{(n)}(u^{(n)} + u^{(n)}_x)^2 \in L^\infty(0,T;B^{\frac{1}{2}}_{2,1})$. Taking advantage of the Lemma 2.10, we obtain that $m^{(n+1)} \in L^\infty(0,T;B^{\frac{1}{2}}_{2,1})$, for any $T > 0$.

Step 2. Using the similar argument in Theorem 3.2, one can find $T > 0$ such that $4TC^3\|m_0\|^2_{B^s_{p,r}} < 1$, and the sequence $\{m^{(n)}\}_{n\in\mathbb{N}^+}$ is uniformly bounded in $B^{\frac{1}{2}}_{2,1}$ satisfying

$$\|m^{(n)}(t)\|_{B^{\frac{1}{2}}_{2,1}} \leq \frac{C\|m_0\|_{B^{\frac{1}{2}}_{2,1}}}{\sqrt{1-4C^3\|m_0\|^2_{B^{\frac{1}{2}}_{2,1}} t}} \leq \frac{C\|m_0\|_{B^{\frac{1}{2}}_{2,1}}}{\sqrt{1-4C^3\|m_0\|^2_{B^{\frac{1}{2}}_{2,1}} T}} \triangleq \overline{M} \text{ for } \forall t\in[0,T] \tag{3.30}$$

where the constant $C>0$ is independent of the n and T. Especially, if we choose $T=\frac{3}{16C^3\|m_0\|^2_{B^{1/2}_{2,1}}}$, it follows from (3.30) that $\|m^{(n)}(t)\|_{B^{\frac{1}{2}}_{2,1}} \leq 2C\|m_0\|_{B^{\frac{1}{2}}_{2,1}}$ for all $t\in[0,T]$.

Step 3. To obtain the solution, we now prove that the approximation solution $\{m^{(n)}\}_{n\in\mathbb{N}^+}$ is a Cauchy sequence in $L^\infty([0,T];B^{-\frac{1}{2}}_{2,\infty})$. To this end, applying the Lemma 2.9 to Equ. (3.23) to get

$$\begin{aligned}
&\|m^{(n+k+1)}(t)-m^{(n+1)}(t)\|_{B^{-\frac{1}{2}}_{2,\infty}}\\
\leq{}& C\Big(\|S_{n+k+1}m_0-S_{n+1}m_0\|_{B^{-\frac{1}{2}}_{2,\infty}}+\int_0^t\|F(u^{(n)},u^{(n+k)},m^{(n)},m^{(n+k)})\|_{B^{-\frac{1}{2}}_{2,\infty}}ds\Big),
\end{aligned}\tag{3.31}$$

where the functional $F(u^{(n)},u^{(n+k)},m^{(n)},m^{(n+k)})$ is defined as same as that in (3.23). And we have used the uniform boundedness of $\{m^{(n)}\}_{n\in\mathbb{N}^+}$ in $L^\infty([0,T];B^{\frac{1}{2}}_{2,1})$.

By virtue of the 1-D Moser-type estimate (see Lemma 2.8) and the embedding $B^{\frac{1}{2}}_{2,1}\hookrightarrow B^{\frac{1}{2}}_{2,\infty}\cap L^\infty$, we can estimate that

$$\begin{aligned}
&\|[(u^{(n)}+u^{(n)}_x)^2-(u^{(n+k)}+u^{(n+k)}_x)^2]m^{(n+1)}_x\|_{B^{-\frac{1}{2}}_{2,\infty}}\\
\leq{}& C\|(u^{(n)}+u^{(n)}_x+u^{(n+k)}+u^{(n+k)}_x)(u^{(n)}-u^{(n+k)}+u^{(n)}_x-u^{(n+k)}_x)m^{(n+1)}_x\|_{B^{-\frac{1}{2}}_{2,\infty}}\\
\leq{}& C(\|u^{(n)}-u^{(n+k)}\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty}+\|u^{(n)}_x-u^{(n+k)}_x\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty})\|m^{(n+1)}_x\|_{B^{-\frac{1}{2}}_{2,\infty}}\\
\leq{}& C(\|u^{(n)}-u^{(n+k)}\|_{B^{\frac{1}{2}}_{2,1}}+\|u^{(n)}-u^{(n+k)}\|_{B^{\frac{3}{2}}_{2,1}})\\
\leq{}& C\|m^{(n)}-m^{(n+k)}\|_{B^{-\frac{1}{2}}_{2,1}},
\end{aligned}\tag{3.32}$$

(3.33)

$$\|[(m^{(n+k)})^2-(m^{(n)})^2](u^{(n+k)}+u^{(n+k)}_x)\|_{B^{-\frac{1}{2}}_{2,\infty}}$$

$$\leq C\|(m^{(n+k)}-m^{(n)})(m^{(n+k)}+m^{(n)})\|_{B^{-\frac{1}{2}}_{2,\infty}}\|u^{(n+k)}+u^{(n+k)}_x\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty}$$
$$\leq C\|m^{(n+k)}+m^{(n)}\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty}\|m^{(n+k)}-m^{(n)}\|_{B^{-\frac{1}{2}}_{2,\infty}}\|m^{(n+k)}\|_{B^{\frac{1}{2}}_{2,1}}$$
$$\leq C\|m^{(n)}-m^{(n+k)}\|_{B^{-\frac{1}{2}}_{2,1}},$$

(3.34)
$$\begin{aligned}&\|(m^{(n)})^2(u^{(n+k)}-u^{(n)}+u^{(n+k)}_x-u^{(n)}_x)\\&\quad+(m^{(n)}-m^{(n+k)})(u^{(n+k)}+u^{(n+k)}_x)^2\|_{B^{-\frac{1}{2}}_{2,\infty}}\\&\leq C(\|u^{(n+k)}-u^{(n)}+u^{(n+k)}_x-u^{(n)}_x\|_{B^{-\frac{1}{2}}_{2,\infty}}+\|m^{(n)}-m^{(n+k)}\|_{B^{-\frac{1}{2}}_{2,\infty}})\\&\leq C\|m^{(n)}-m^{(n+k)}\|_{B^{-\frac{1}{2}}_{2,1}},\end{aligned}$$

(3.35)
$$\begin{aligned}&\|m^{(n+k)}[(u^{(n)}+u^{(n)}_x)^2-(u^{(n+k)}+u^{(n+k)}_x)^2]\|_{B^{-\frac{1}{2}}_{2,\infty}}\\&\leq C\|u^{(n)}+u^{(n)}_x+u^{(n+k)}+u^{(n+k)}_x\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty}\\&\quad\|u^{(n)}-u^{(n+k)}+u^{(n)}_x-u^{(n+k)}_x\|_{B^{-\frac{1}{2}}_{2,\infty}}\\&\leq C(\|u^{(n)}\|_{B^{\frac{3}{2}}_{2,1}}+\|u^{(n+k)}\|_{B^{\frac{3}{2}}_{2,1}})(\|u^{(n)}-u^{(n+k)}\|_{B^{\frac{1}{2}}_{2,1}}+\|u^{(n)}-u^{(n+k)}\|_{B^{\frac{3}{2}}_{2,1}})\\&\leq C\|m^{(n)}-m^{(n+k)}\|_{B^{-\frac{1}{2}}_{2,1}}.\end{aligned}$$

Therefore, by (3.31)-(3.35), we obtain

$$\begin{aligned}&\|m^{(n+k+1)}(t)-m^{(n+1)}(t)\|_{B^{-\frac{1}{2}}_{2,\infty}}\\&\leq C\left(2^{-n}\|m_0\|_{B^{\frac{1}{2}}_{2,1}}+\int_0^t\|m^{(n)}(s)-m^{(n+k)}(s)\|_{B^{-\frac{1}{2}}_{2,1}}ds\right),\end{aligned}\tag{3.36}$$

where we used the fact that there exists a positive constant C which is independent of n such that

$$\|S_{n+k+1}m_0-S_{n+1}m_0\|_{B^{-\frac{1}{2}}_{2,\infty}}\leq C2^{-n}\|m_0\|_{B^{\frac{1}{2}}_{2,1}}.$$

Noting that $\|m^{(n)}(t)\|_{B^{\frac{1}{2}}_{2,1}}+\|m^{(n+k)}(t)\|_{B^{\frac{1}{2}}_{2,1}}\leq 2\overline{M}$, for all $t\in[0,T)$. For convenience, setting $\mathcal{K}^{n,k}(t)=\|m^{(n+k+1)}(t)-m^{(n+1)}(t)\|_{B^{-\frac{1}{2}}_{2,\infty}}$. Thanks to the logarithmic interpolation inequality (see (2) of Lemma 2.7), we deduce from (3.36) that

$$\mathcal{K}^{n+1,k}(t)\leq C\left[2^{-n}\|m_0\|_{B^{\frac{1}{2}}_{2,1}}+\int_0^t\mathcal{K}^{n,k}(s)\log\left(e+\frac{2\overline{M}}{\mathcal{K}^{n,k}(s)}\right)ds\right].\tag{3.37}$$

Therefore, by using the monotonicity of the function $f(x) = x\log(e + \frac{C}{x})$, we have

$$\begin{aligned}&\limsup_{n\to\infty}\sup_{k\in\mathbb{N}^+}\mathcal{K}^{n+1,k}(t)\\ &\le C\int_0^t \limsup_{n\to\infty}\sup_{k\in\mathbb{N}^+}\mathcal{K}^{n,k}(s)\log\left(e+\frac{2\overline{M}}{\limsup_{n\to\infty}\sup_{k\in\mathbb{N}^+}\mathcal{K}^{n,k}(s)}\right)ds.\end{aligned} \tag{3.38}$$

By utilizing the Osgood's Lemma (see Lemma 2.11) with $a = 0$ and $\mu(x) = x\ln(e + \frac{C}{x})$, we get

$$\limsup_{n\to\infty}\sup_{k\in\mathbb{N}^+}\mathcal{K}^{n,k}(t) = 0, \text{ for } \forall t \in [0,T],$$

which implies that $\{m^{(n)}\}_{n\in\mathbb{N}^+}$ is a Cauchy sequence in $C([0,T]; B^{-\frac{1}{2}}_{2,\infty})$, and thus $m^{(n)}$ convergence to some function $m \in C([0,T]; B^{-\frac{1}{2}}_{2,\infty})$. By applying the similar argument in the proof of Theorem 3.2, we can verify that $m \in C([0,T]; B^{\frac{1}{2}}_{2,1}) \cap C^1([0,T]; B^{-\frac{1}{2}}_{2,1})$ is indeed a solution to the Equ. (3.1).

Step 4. We prove the uniqueness and stability of the solutions. By applying the Lemma 2.9 to the Equ. (3.4) to get

$$\|m^{(12)}(t)\|_{B^{-\frac{1}{2}}_{2,\infty}} \le C\left(\|m_0^{(12)}\|_{B^{-\frac{1}{2}}_{2,\infty}} + \int_0^t \|\mathcal{F}^{(12)}(\cdot,s)\|_{B^{-\frac{1}{2}}_{2,\infty}}\,ds\right). \tag{3.39}$$

Similar to the estimates in Step 3, we can deal with the term $\|\mathcal{F}^{(12)}(\cdot,s)\|_{B^{-\frac{1}{2}}_{2,\infty}}$ by means of the 1-D Moser-type inequality, and it follows from (3.39) that

(3.40)

$$\begin{aligned}&\|m^{(12)}(t)\|_{B^{-\frac{1}{2}}_{2,\infty}}\\ &\le C\left(\|m_0^{(12)}\|_{B^{-\frac{1}{2}}_{2,\infty}} + \int_0^t \|m^{(12)}(\cdot,s)\|_{B^{-\frac{1}{2}}_{2,1}}\,ds\right)\\ &\le C\left(\|m_0^{(12)}\|_{B^{-\frac{1}{2}}_{2,\infty}} + \int_0^t \|m^{(12)}(\cdot,s)\|_{B^{-\frac{1}{2}}_{2,\infty}}\log\left(e+\frac{2\overline{M}}{\|m^{(12)}(\cdot,s)\|_{B^{-\frac{1}{2}}_{2,\infty}}}\right)ds\right).\end{aligned}$$

If $\|m_0^{(12)}\|_{B^{-\frac{1}{2}}_{2,\infty}} < \min\left\{1, \frac{C}{2\overline{M}}\right\}$, using the Osgood's Lemma to (3.40), for $\forall t \in [0,T)$, we get

$$\frac{\|m^{(12)}(t)\|_{B^{-1/2}_{2,\infty}}}{e} \le \left(\frac{C\|m^{(12)}(0)\|_{B^{-1/2}_{2,\infty}}}{e}\right)^{\exp(-C\ln(e+\frac{C}{2\overline{M}})T)},$$

which is equivalent to

$$\|m^{(2)}(t) - m^{(1)}(t)\|_{B^{-\frac{1}{2}}_{2,\infty}} \le C\|m^{(2)}(0) - m^{(1)}(0)\|_{B^{\frac{1}{2}}_{2,1}}^{\exp(-C\ln(e+\frac{C}{2\overline{M}})T)} \tag{3.41}$$

for all $t \in [0,T)$. This implies that if the initial datum m_0', m_0 satisfies $\|m_0' - m_0\|_{B^{\frac{1}{2}}_{2,1}} < \min\{1, \frac{C}{2\overline{M}}\}$, then the solution mapping from $B^{\frac{1}{2}}_{2,1}$ to $C([0,T); B^{-\frac{1}{2}}_{2,\infty})$ is Hölder continuous.

However, we shall show that the solution mapping is Hölder continuous from $B^{\frac{1}{2}}_{2,1}$ to $B^{s}_{2,1}$ for $|s| < \frac{1}{2}$. Indeed, by virtue of the interpolation inequality ((1) of Lemma 2.7), we have

$$
\begin{aligned}
(3.42)\quad & \|m^{(2)}(t) - m^{(1)}(t)\|_{B^{s}_{2,1}} \\
& \le C\left(\frac{1}{\theta} + \frac{1}{1-\theta}\right)\|m^{(2)}(t) - m^{(1)}(t)\|^{\theta}_{B^{-\frac{1}{2}}_{2,\infty}} \|m^{(2)}(t) - m^{(1)}(t)\|^{1-\theta}_{B^{\frac{1}{2}}_{2,\infty}} \\
& \le C\left(\frac{1}{\theta} + \frac{1}{1-\theta}\right)\|m^{(2)}(t) - m^{(1)}(t)\|^{\theta}_{B^{-\frac{1}{2}}_{2,\infty}} (\|m^{(2)}\|_{B^{\frac{1}{2}}_{2,1}} + \|m^{(1)}\|_{B^{\frac{1}{2}}_{2,1}})^{1-\theta} \\
& \le C\frac{(2\overline{M})^{1-\theta}C^{\theta}}{\theta(1-\theta)}\|m^{(2)}(0) - m^{(1)}(0)\|^{\theta\exp(-C\ln(e+\frac{C}{2\overline{M}})T)}_{B^{-\frac{1}{2}}_{2,\infty}} \\
& \le C(\theta,T)\|m^{(2)}(0) - m^{(1)}(0)\|^{\theta\exp(-C(T))}_{B^{\frac{1}{2}}_{2,1}}
\end{aligned}
$$

for $\forall\theta = \frac{1}{2} - s \in (0,1)$, where $C(a,b,\ldots)$ means that the positive constant C depends only on $a, b, \ldots$. Thus the solution mapping is Hölder continuous from $B^{\frac{1}{2}}_{2,1}$ to $C([0,T], B^{s}_{2,1})$ with the Hölder exponent depends only on θ and T.

The proof of Theorem 3.6 is completed. □

4. Blow-up phenomena

In this section, we shall establish blow-up criteria for the solutions to the Equ. (3.1).

Lemma 4.1. (1)[1] *For $s > 0$, there exists constant $C > 0$ independent of u and v such that*

$$
\begin{aligned}
&\|uv\|_{H^s} \le C(\|u\|_{H^s}\|v\|_{L^\infty} + \|v\|_{H^s}\|u\|_{H^s}),\\
&\|u\partial_x v\|_{H^s} \le C(\|u\|_{H^{s+1}}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|\partial_x v\|_{H^s}).
\end{aligned}
$$

(2) (Kato-Ponce [30]) *For $s > 0$, if $f \in H^s \cap W^{1,\infty}$, $g \in H^{s-1} \cap L^\infty$, set $\Lambda^s = (1-\partial_x^2)^{\frac{s}{2}}$, then*

$$\|\Lambda^s(uv) - u\Lambda^s v\|_{L^2} \le C(\|\Lambda^s u\|_{L^2}\|v\|_{L^\infty} + \|u_x\|_{L^\infty}\|\Lambda^{s-1}v\|_{L^2}).$$

The following lemma is a corollary of Lemma 2.9.

Lemma 4.2 ([1]). *Consider the transport equation*

$$\partial_t f + v\partial_x f = g,\ \ f(x,0) = f_0(x).$$

Let $0 \le \sigma < 1$, and suppose that $f_0 \in H^\sigma$, $g \in L^1(0,T;H^\sigma)$, $v_x \in L^1(0,T;L^\infty)$, $f \in L^\infty(0,T;H^\sigma) \cap C([0,T];\mathcal{S}')$. Then $f \in C(0,T;H^\sigma)$. More precisely, there *exists a constant C depends only on σ such that, for every $0 < t \le T$,*

$$\|f\|_{H^\sigma} \le \|f_0\|_{H^\sigma} + C\int_0^t \|g(\tau)\|_{H^\sigma} d\tau + C\int_0^t V'(\tau)\|f(\tau)\|_{H^\sigma} d\tau \tag{4.1}$$

with $V(t) = \int_0^t \|v(\tau)\|_{L^\infty} + \|\partial_x v(\tau)\|_{L^\infty} d\tau$.

The first blow-up criteria for the strong solution to Equ. (3.1) can be stated as follows.

Theorem 4.3. *Let $m_0 = (1-\partial_x^2)u_0 \in H^s$ with $s > \frac{1}{2}$, and $T > 0$ is the maximum existence time of the corresponding solution m to the equation* (3.1). *Then*

$$T < \infty \Longrightarrow \int_0^T \|m(\cdot, t)\|_{L^\infty}^2 dt = \infty. \tag{4.2}$$

Proof. Since $H^s \approx B^s_{2,2}$ for any $s > \frac{1}{2}$, the existence of the solution $m \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-1})$ is ensured by the Theorem 3.2. We now prove the Theorem 4.3 by induction with respect to the index s, which can be achieved by three steps.

Step 1. For $\frac{1}{2} < s < 1$, by using the Lemma 4.2 to the first equation of (3.1), we obtain

$$\begin{aligned} \|m(t)\|_{H^s} \le{}& \|m_0\|_{H^s} \\ &+ C\int_0^t (\|(u+u_x)^2(\tau)\|_{L^\infty} + \|\partial_x(u+u_x)^2(\tau)\|_{L^\infty})\|m(\tau)\|_{H^s} d\tau \\ &+ \int_0^t \|m^2(u+u_x)(\tau)\|_{H^s} + \|m(u+u_x)^2(\tau)\|_{H^s} d\tau \end{aligned} \tag{4.3}$$

for all $t \in [0,T)$. Noting that $u = (1-\partial_x)^{-1}m = G * m$ with $G(x) \triangleq \frac{1}{2}e^{-|x|}$ on $\mathbb{R}$, $\partial_x u = \partial_x G * m$, $\partial_x^2 u = u - m$ and $\|G\|_{L^1} = \|\partial_x G\|_{L^1} = 1$. By using the Young inequality, we have

$$\|u\|_{L^\infty}, \|u_x\|_{L^\infty},\ \|u_{xx}\|_{L^\infty} \le \|m\|_{L^\infty}. \tag{4.4}$$

Similarly, for all $s \in \mathbb{R}$, we have

$$\|u\|_{H^s}, \|u_x\|_{H^s}, \|u_{xx}\|_{H^s} \le \|m\|_{H^s}. \tag{4.5}$$

Therefore, we have

$$\begin{aligned} &\|(u+u_x)^2\|_{L^\infty} + \|\partial_x(u+u_x)^2\|_{L^\infty} \\ &\le C(\|u\|_{L^\infty} + \|u_x\|_{L^\infty})(\|u\|_{L^\infty} + \|u_x\|_{L^\infty} + \|u_{xx}\|_{L^\infty}) \le C\|m\|_{L^\infty}^2. \end{aligned} \tag{4.6}$$

Owing to the (1) of Lemma 4.1, and the properties (4.3) and (4.4), we have

$$\begin{aligned} \|m^2(u+u_x)\|_{H^s} &\le C(\|u+u_x\|_{H^s}\|m^2\|_{L^\infty} + \|u+u_x\|_{L^\infty}\|m^2\|_{H^s}) \\ &\le C(\|m\|_{H^s}\|m^2\|_{L^\infty} + \|m\|_{L^\infty}\|m^2\|_{H^s}) \\ &\le C\|m\|_{H^s}\|m\|_{L^\infty}^2, \end{aligned} \tag{4.7}$$

$$
\begin{aligned}
\|m(u+u_x)^2\|_{H^s} &\leq C(\|(u+u_x)^2\|_{L^\infty}\|m\|_{H^s} + \|m\|_{L^\infty}\|(u+u_x)^2\|_{H^s}) \\
&\leq C(\|u+u_x\|_{L^\infty}^2\|m\|_{H^s} + \|m\|_{L^\infty}\|(u+u_x)^2\|_{H^s}) \\
&\leq C(\|m\|_{L^\infty}^2\|m\|_{H^s} + \|m\|_{L^\infty}\|u+u_x\|_{L^\infty}\|u+u_x\|_{H^s}) \\
&\leq C\|m\|_{L^\infty}^2\|m\|_{H^s}. \qquad (4.8)
\end{aligned}
$$

It follows from (4.3) and (4.6) − (4.8) that

$$
\|m(t)\|_{H^s} \leq \|m_0\|_{H^s} + C\int_0^t \|m\|_{L^\infty}^2\|m(\tau)\|_{H^s}d\tau,
$$

which combined with the Gronwall inequality yields that

$$
\|m(t)\|_{H^s} \leq \|m_0\|_{H^s}\exp\left\{C\int_0^t \|m(\tau)\|_{L^\infty}^2 d\tau\right\} \text{ for } \forall t\in[0,T). \qquad (4.9)
$$

Assume that $T<\infty$, it follows from (4.9) that

$$
\limsup_{t\uparrow T}\|m(t)\|_{H^s} \leq M_T \triangleq \|m_0\|_{H^s}\exp\left\{C\int_0^T \|m(\tau)\|_{L^\infty}^2 d\tau\right\} < \infty.
$$

By using a standard method used in [15], we can extends the solution m to $[0,T+\epsilon]$ for some $\epsilon>0$ small enough, which contradict to the fact that $T>0$ is the maximum existence time of the solution. This completes the proof of the theorem for $s\in(\frac{1}{2},1)$.

Step 2. For $1\leq s<2$, differentiating the Equ. (3.1) with respect to x, we get

$$
\begin{aligned}
&\partial_t(m_x) + (u+u_x)^2\partial_x(m_x) + 2(u+u_x)(u_x+u_{xx})m_x \\
&= 2mm_x(u+u_x) + m^2(u_x+u_{xx}) - m_x(u+u_x)^2 - 2m(u_x+u_{xx}). \qquad (4.10)
\end{aligned}
$$

By virtue of the Lemma 4.2 with $s-1\in[0,1)$, we obtain

$$
\begin{aligned}
\|\partial_x m(t)\|_{H^{s-1}} &\leq \|\partial_x m_0\|_{H^{s-1}} + C\int_0^t \|(u+u_x)^2\|_{W^{1,\infty}}\|\partial_x m(\tau)\|_{H^{s-1}}d\tau \\
&\quad + \int_0^t \Big(\|2mm_x(u+u_x)\|_{H^{s-1}} + \|(m^2-2m)(u_x+u_{xx})\|_{H^{s-1}} \\
&\quad + \|m_x(u+u_x)^2\|_{H^{s-1}} + 2\|(u+u_x)(u_x+u_{xx})m_x\|_{H^{s-1}}\Big)d\tau. \qquad (4.11)
\end{aligned}
$$

Taking advantage of (2) of Lemma 4.1 and (4.4) − (4.5), we have

$$
\begin{aligned}
&\|2mm_x(u+u_x)\|_{H^{s-1}} + \|(m^2-2m)(u_x+u_{xx})\|_{H^{s-1}} \\
&\leq C(\|u+u_x\|_{H^s}\|m^2\|_{L^\infty} + \|u+u_x\|_{L^\infty}\|\partial_x(m^2)\|_{H^{s-1}}) \\
&\leq C(\|m\|_{H^s}\|m\|_{L^\infty}^2 + \|m\|_{L^\infty}\|m^2\|_{H^s}) \\
&\leq C\|m\|_{H^s}\|m\|_{L^\infty}^2, \qquad (4.12)
\end{aligned}
$$

$$
\begin{aligned}
&\|m_x(u+u_x)^2\|_{H^{s-1}} + 2\|(u+u_x)(u_x+u_{xx})m_x\|_{H^{s-1}} \\
&\leq \|(u+u_x)^2\|_{H^s}\|m\|_{L^\infty} + \|(u+u_x)^2\|_{L^\infty}\|m_x\|_{H^{s-1}}
\end{aligned}
$$

$$\begin{aligned} &\leq \|u+u_x\|_{L^\infty}\|m\|_{L^\infty}\|u+u_x\|_{H^s} + \|u+u_x\|_{L^\infty}^2\|m\|_{H^s} \\ &\leq C\|m\|_{L^\infty}^2\|m\|_{H^s}, \end{aligned} \tag{4.13}$$

where we used the fact that $\|\partial_x f\|_{H^{s-1}} \leq C\|f\|_{H^s}$. It then follows from (4.6), (4.11)-(4.13) that

$$\|\partial_x m(t)\|_{H^{s-1}} \leq \|m_0\|_{H^s} + C\int_0^t \|m\|_{L^\infty}^2\|m\|_{H^s}d\tau,$$

which together with (4.9) with $s-1$ instead of s and the Gronwall inequality ensures that, for $s\in(1,2)$, (4.9) holds. Taking the similar argument in Step 1, we prove can the theorem for $s\in(1,2)$.

Step 3. In the case of $s\geq 2$, by applying $\Lambda^s = (1-\partial_x^2)^{\frac{s}{2}}$ to the first equation of (3.1), we get

$$\begin{aligned} &\partial_t(\Lambda^s m) + (u+u_x)^2\partial_x(\Lambda^s m) \\ &= (u+u_x)^2\Lambda^s m_x - \Lambda^s[(u+u_x)^2 m_x] + \Lambda^s[m^2(u+u_x) - m(u+u_x)^2]. \end{aligned} \tag{4.14}$$

Multiplying both sides of (4.14) by $\Lambda^s m$ and integrating with respect to x on $\mathbb{R}$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(\Lambda^s m)^2dx \\ =&\ -\frac{1}{2}\int_{\mathbb{R}}(u+u_x)^2\partial_x(\Lambda^s m)^2dx \\ &+\int_{\mathbb{R}}\Big((u+u_x)^2\Lambda^s m_x - \Lambda^s[(u+u_x)^2m_x]\Big)\Lambda^s m dx \\ &+\int_{\mathbb{R}}\Lambda^s m\Lambda^s[m^2(u+u_x) - m(u+u_x)^2]dx. \end{aligned}$$

By integrating by parts and using the Hölder inequality, we obtain

$$\begin{aligned} \frac{d}{dt}\|\Lambda^s m(t)\|_{L^2} \leq&\ \|(u+u_x)(u_x+u_{xx})\|_{L^\infty}\|\Lambda^s m(t)\|_{L^2} \\ &+\|\Lambda^s[m^2(u+u_x) - m(u+u_x)^2]\|_{L^2} \\ &+\|(u+u_x)^2\Lambda^s m_x - \Lambda^s[(u+u_x)^2 m_x]\|_{L^2}. \end{aligned} \tag{4.15}$$

Integrating both sides of the inequality (4.15) from 0 to t, we have

$$\begin{aligned} \|m(t)\|_{H^s} \leq&\ \|m_0\|_{H^s} + \int_0^t \|(u+u_x)(u_x+u_{xx})\|_{L^\infty}\|m(\tau)\|_{H^s}d\tau \\ &+\int_0^t \|(u+u_x)^2\Lambda^s m_x - \Lambda^s[(u+u_x)^2m_x]\|_{L^2}d\tau \\ &+\int_0^t (\|m^2(u+u_x)\|_{H^s} + \|m(u+u_x)^2\|_{H^s})d\tau. \end{aligned} \tag{4.16}$$

By means of the commutator estimate (see (2) of Lemma 4.1) and (4.4)-(4.5), we have

(4.17)
$$\begin{aligned}
&\|(u+u_x)^2\Lambda^s m_x - \Lambda^s[(u+u_x)^2 m_x]\|_{L^2}\\
&\le C(\|\Lambda^s(u+u_x)^2\|_{L^2}\|m_x\|_{L^\infty} + \|\partial_x(u+u_x)^2\|_{L^\infty}\|\Lambda^{s-1}m_x\|_{L^2})\\
&\le C(\|u+u_x\|_{L^\infty}\|(u+u_x)\|_{H^s}\|m_x\|_{L^\infty} + \|(u+u_x)(u_x+u_{xx})\|_{L^\infty}\|m\|_{H^s})\\
&\le C(\|m\|_{L^\infty}\|m\|_{H^s}\|m\|_{H^{\frac{3}{2}+\epsilon}} + \|m\|_{L^\infty}^2\|m\|_{H^s}),
\end{aligned}$$

where we used the Sobolev embedding $H^{\frac{3}{2}+\epsilon} \hookrightarrow W^{1,\infty}$ for $0<\epsilon<\frac{1}{2}$. Moreover, by virtue of (1) of Lemma 4.1, we can estimate that

$$\begin{aligned}
(4.18)\quad &\|m^2(u+u_x)\|_{H^s} + \|m(u+u_x)^2\|_{H^s}\\
&\le C(\|m\|_{L^\infty}^2\|u+u_x\|_{H^s} + \|m^2\|_{H^s}\|u+u_x\|_{L^\infty} + \|m\|_{L^\infty}\|(u+u_x)^2\|_{H^s}\\
&\quad + \|m\|_{H^s}\|u+u_x\|_{L^\infty}^2)\\
&\le C(\|m\|_{L^\infty}^2\|m\|_{H^s} + \|m\|_{L^\infty}\|u+u_x\|_{L^\infty}\|u+u_x\|_{H^s})\\
&\le C\|m\|_{L^\infty}^2\|m\|_{H^s}.
\end{aligned}$$

Therefore, we deduce from (4.16)-(4.18) that

$$\begin{aligned}
\|m(t)\|_{H^s} \le \|m_0\|_{H^s} + \int_0^t \|m\|_{L^\infty}^2\|m(\tau)\|_{H^s}d\tau\\
+ C\int_0^t \|m\|_{L^\infty}\|m\|_{H^s}\|m\|_{H^{\frac{3}{2}+\epsilon}}d\tau.
\end{aligned}$$

Taking the Gronwall inequality, we obtain

$$\|m(t)\|_{H^s} \le \|m_0\|_{H^s}\exp\left\{C\int_0^t(\|m\|_{L^\infty}^2 + \|m\|_{H^{\frac{3}{2}+\epsilon}}^2)d\tau\right\}. \tag{4.19}$$

Assume that $T<\infty$ and $\int_0^T \|m(t)\|_{L^\infty}^2 dt < \infty$, as a conclusion of the Step 2 for $\frac{3}{2}+\epsilon \in (1,2)$, we see that $\sup_{t\in[0,T)}\|m(t)\|_{H^{\frac{3}{2}+\epsilon}}$ is uniformly bounded. Hence we deduce from (4.19) that

$$\limsup_{t\to T}\|m(t)\|_{H^s} < \infty,$$

which contradict to the fact that $T>0$ is the maximum existence time of the solution.

This completes the proof of the Theorem 4.3. □

Remark 4.4. The maximum existence time T can be chosen independent of the regularity index s. Indeed, let $m_0 \in H^s(s>\frac{1}{2})$ and consider some $s' \in (\frac{1}{2}, s)$. The Theorem 3.2 ensures the existence of unique solution u_s (resp.,$m_{s'}$) in H^s (resp.,$H^{s'}$) with the maximum existence time T_s (resp.,$T_{s'}$). Since $H^s \hookrightarrow H^{s'}$, it follows from the uniqueness that $T_s \le T_{s'}$, and also $m_s \equiv m_{s'}$ on $[0,T_s)$. On the other hand, if $T_s < T_{s'}$, then $m_{s'} \in C([0,T_s]; H^{s'}) \hookrightarrow L^2(0,T_s;L^\infty)$, which contradicts to the Theorem 4.3. Hence we have $T_s = T_{s'}$.

Corollary 4.5. *Let $m_0 \in H^s$ with $s > \frac{1}{2}$ and $T > 0$ be the maximum existence time of the solution to the Equ.* (3.1)*. Then the solution will blow up in finite time T if and only if*

$$\limsup_{t\to T} \|m(\cdot,t)\|_{L^\infty} = \infty.$$

Proof. By using the Sobolev embedding theorem, the Corollary 4.5 is a direct conclusion of the Theorem 4.3, we omit the details here. □

Consider the following ordinary differential equation

$$\begin{cases} \frac{dq(x,t)}{dt} = (u+u_x)^2(q(x,t),t), \ x\in\mathbb{R}, \ t\in[0,T), \\ q(x,0) = x, \ x\in\mathbb{R}, \end{cases} \tag{4.20}$$

for the flow generated by $(u+u_x)^2$.

Lemma 4.6. *Let $m_0 \in H^s$ with $s > \frac{1}{2}$, and $T > 0$ is the maximum existence time of solution to the Equ.* (3.1)*. Then the initial value problem* (4.20) *admits a unique solution $q \in C^1(\mathbb{R}\times[0,T);\mathbb{R})$ which is an increasing diffeomorphism of $\mathbb{R}$ with respect to x, and*

$$q_x(x,t) = \exp\left\{\int_0^t [2(u+u_x)(u_x+u_{xx})](t,q(x,t))d\tau\right\} > 0 \tag{4.21}$$

for all $x\in\mathbb{R}$ and $t\in[0,T)$.

Proof. By Theorem 3.2, the solution $m \in C([0,T];H^s)\cap C^1([0,T];H^{s-1})$ with $s > \frac{1}{2}$, hence $u \in C^1([0,T];H^{s-1})$ with $s > \frac{5}{2}$. Noting that $H^{s-1}(\mathbb{R}) \hookrightarrow C^1(\mathbb{R})$ for $s > \frac{5}{2}$, both u and u_x are bounded, Lipschitz in the space variable x and of class in time. Therefore, it follows from the classical ODE theory that initial value problem (4.20) admits a unique solution $q \in C^1(\mathbb{R}\times[0,T);\mathbb{R})$.

Differentiating the equation (4.20) with respect to x to get

$$\begin{cases} \frac{d}{dt}q_x(x,t) = 2[(u+u_x)(u_x+u_{xx})](q(x,t),t)q_x(x,t), \ x\in\mathbb{R}, \ t\in[0,T), \\ q_x(x,0) = 1, \ x\in\mathbb{R}, \end{cases}$$

which leads to

(4.22)

$$q_x(x,t) = \exp\left\{\int_0^t [2(u+u_x)(u_x+u_{xx})](q(x,t),t)d\tau\right\}, \ x\in\mathbb{R}, \ t\in[0,T).$$

For all $\overline{T} < T$, it follows from the Sobolev embedding theorem that

$$\sup_{(x,t)\in\mathbb{R}\times[0,\overline{T})} \Big|[2(u+u_x)(u_x+u_{xx})](x,t)\Big| \le C\|m\|_{L^\infty}^2 \le C\|m\|_{H^s}^2 < \infty.$$

We infer from (4.22) that there exists a positive constant K such that $q_x(x,t) \geq e^{-Kt} > 0$, for $\forall (x,t) \in \mathbb{R} \times [0,T)$. which shows that the mapping $q(\cdot,t)$ is an increasing diffeomorphism of $\mathbb{R}$ before it blow-up. $\square$

Lemma 4.7. *Let $m_0 \in H^s$ with $s > \frac{1}{2}$, and $T > 0$ is the maximum existence time of the solution to the Equ. (3.1). Then the following property holds*:

(4.23)
$$m(q(x,t),t)q_x(x,t) = m_0(x) \exp\left\{ \int_0^t [(u+u_x)(u_x+u_{xx})](q(x,\tau),\tau)d\tau \right\}$$

for all $(x,t) \in \mathbb{R} \times [0,T)$. Moreover, If there exists a positive constant $C > 0$ such that $-[(u+u_x)(u_x+u_{xx})](x,t) \geq -C$ for all $(x,t) \in \mathbb{R} \times [0,T)$, then we have

$$\|m(\cdot,t)\|_{L^\infty} \leq Ce^{Ct}\|m_0\|_{L^\infty} \text{ for all } t \in [0,T). \tag{4.24}$$

Proof. Differentiating the left hand side of (4.23) with respect to t, and taking advantage of (4.22) and the Equ. (4.20), we obtain

(4.25)
$$\begin{aligned}
&\frac{d}{dt}[m(q(x,t),t)q_x(x,t)] \\
&= (m_t(q,t) + m_x(q,t)q_t(x,t))q_x(x,t) + m(q,t)q_{xt}(x,t) \\
&= [m_t + m_x(u+u_x)^2 + 2m(u+u_x)(u_x+u_{xx})](q,t)q_x(x,t) \\
&= [m(u+u_x) - (u+u_x)^2 + 2(u+u_x)(u_x+u_{xx})](q,t)m(q(x,t),t)q_x(x,t) \\
&= [(u+u_x)(u_x+u_{xx})](q,t)[m(q(x,t),t)q_x(x,t)].
\end{aligned}$$

Consider the unknown function $m(q(\cdot,t),t)q_x(\cdot,t)$ with respect to t, we can immediately deduce the Equ. (4.23) from (4.25). By Lemma 4.6, (4.23) and the assumption, we have

$$\begin{aligned}
\|m(\cdot,t)\|_{L^\infty} &= \|m(q(\cdot,t),t)\|_{L^\infty} \\
&= \left\| m_0 q_x^{-1}(\cdot,t) \exp\left\{ \int_0^t [(u+u_x)(u_x+u_{xx})](q(\cdot,\tau),\tau)d\tau \right\} \right\|_{L^\infty} \\
&= \left\| m_0 \exp\left\{ -\int_0^t [(u+u_x)(u_x+u_{xx})](q(\cdot,\tau),\tau)d\tau \right\} \right\|_{L^\infty} \\
&\leq \|m_0 e^{Ct}\|_{L^\infty} \leq Ce^{Ct}\|m_0\|_{L^\infty}.
\end{aligned} \tag{4.26}$$

This completes the proof of Lemma 4.7. $\square$

Remark 4.8. (1) By (4.21) and (4.23), if the initial data m_0 do not change sign, the solution $m(x,t)$ along the characteristics $q(x,t)$ will not change sign for any $t \in [0,T)$.

(2) Since the mapping $q(\cdot,t)$ is a increasing diffeomorphism of $\mathbb{R}$ with respect to x, if $v(\cdot,t) \in L^\infty$, then we have $\|v(\cdot,t)\|_{L^\infty} = \|v(q(\cdot,t),t)\|_{L^\infty}$ for all $t \in [0,T)$.

Corollary 4.9. *Under the conditions of the Theorem 4.7, if m_0 has compact support satisfying supp $m_0 \in [a,b]$, then the corresponding solution m has also compact support such that supp $m(\cdot,t) \in [q(a,t), q(b,t)]$ for all $t \in [0,T)$, where $q(x,t)$ is the solution of Equ.* (4.21).

Proof. By (4.23) and (4.22), we have

$$\begin{aligned} m(q(x,t),t) &= m_0(x) q_x(x,t)^{-1} \exp\left\{ \int_0^t [(u+u_x)(u_x+u_{xx})](q(x,\tau),\tau) d\tau \right\} \\ &= m_0(x) \exp\left\{ -\int_0^t [(u+u_x)(u_x+u_{xx})](q(x,\tau),\tau) d\tau \right\}. \end{aligned}$$

Noting that m_0 has compact support in $[a,b]$, it follows from the previous equality that $m(q(x,t),t) = 0$, for all $x \in (-\infty, a) \cup (b, \infty)$. Moreover, since the mapping $q(\cdot,t)$ is an increasing diffeomorphism of $\mathbb{R}$ with respect to variable x, we have

$$m(x,t) = 0 \text{ for all } x \in (-\infty, q(a,t)) \cup (q(b,t), \infty),$$

which implies that the solution $m(x,t)$ has compact support in $[q(a,t), q(b,t)]$. □

Based on Lemma 4.7, we have the following new blow-up criteria.

Theorem 4.10. *Let $m_0 \in H^s$ with $s > \frac{1}{2}$, and $T > 0$ is the maximum existence time of the solution to the Equ.* (3.1)*. Then the solution blows up in finite time if and only if*

$$\limsup_{t\to T} \sup_{x\in\mathbb{R}} \{[(u+u_x)(u_x+u_{xx})](x,t)\} = +\infty. \tag{4.27}$$

Proof. On the one hand, if (4.27) holds, then by virtue of the Sobolev embedding theorem, we have

$$\begin{aligned} &\limsup_{t\to T} \sup_{x\in\mathbb{R}} \{[(u+u_x)(u_x+u_{xx})](x,t)\} \\ &\le \limsup_{t\to T} \|(u+u_x)(\cdot,t)\|_{L^\infty} \|(u_x+u-m)(\cdot,t)\|_{L^\infty} \\ &\le \limsup_{t\to T} \|m(\cdot,t)\|_{L^\infty}^2 \le C \limsup_{t\to T} \|m(\cdot,t)\|_{H^s}^2, \end{aligned} \tag{4.28}$$

which shows that the solution $m(x,t)$ will blow up at the finite time T.

On the other hand, if the solution blows up in finite time, and there exists a positive constant C such that $\limsup_{t\to T} \sup_{x\in\mathbb{R}} \{[(u+u_x)(u_x+u_{xx})](x,t)\} \le C$, or equivalently,

$$-[(u+u_x)(u_x+u_{xx})](x,t) \ge -C, \text{ for all } (x,t) \in \mathbb{R} \times [0,T).$$

By means of Lemma 4.7, we have

$$\int_0^T \|m(\cdot,t)\|_{L^\infty}^2 dt \le \int_0^T C^2 e^{2Ct} \|m_0\|_{L^\infty}^2 dt = \frac{C}{2} \|m_0\|_{L^\infty}^2 (e^{2CT} - 1) < \infty. \tag{4.29}$$

By (4.29) and Theorem 4.3, the solution $m(x,t)$ globally exits, which contradicts to the fact that $T<\infty$ is the maximum existence time of the solution to the Equ. (3.1).

This completes the proof of Theorem 4.10. $\square$

References

[1] H. Bahouri, J. Y. Chemin, and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, in: Grundlehren der Mathematischen Wissenschaften, vol. 343, Springer, Heidelberg, 2011.

[2] A. Bressan and A. Constantin, *Global conservative solutions of the Camassa-Holm equation*, Arch. Ration. Mech. Anal. **183** (2007), no. 2, 215–239.

[3] ———, *Global dissipative solutions of the Camassa-Holm equation*, Anal. Appl. **5** (2007), no. 1, 1–27.

[4] R. Camassa and D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett. **71** (1993), no. 11, 1661–1664.

[5] J. Chemin, *Localization in Fourier space and Navier-Stokes system*, in: Phase Space Analysis of Partial Differential Equations, pp. 53–136, Proceedings, in: CRM Series, Pisa, 2000.

[6] G. M. Coclite and K. H. Karlsen, *On the well-posedness of the Degasperis-Procesi equation*, J. Funct. Anal. **233** (2006), no. 1, 60–91.

[7] A. Constantin, *Global existence of solutions and breaking waves for a shallow water equation: a geometric approach*, Ann. Inst. Fourier (Grenoble) **50** (2000), 321–362.

[8] ———, *On the scattering problem for the Camassa-Holm equation*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **457** (2001), no. 2008, 953–970.

[9] A. Constantin and J. Escher, *Global existence and blow-up for a shallow water equation*, Ann. Scuola Norm. Super. Pisa Cl. Sci. **26** (1998), no. 2, 303–328.

[10] ———, *Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation*, Comm. Pure Appl. Math. **51** (1998), no. 5, 475–504.

[11] ———, *Wave breaking for nonlinear nonlocal shallow water equations*, Acta Math. **181** (1998), no. 2, 229–243.

[12] A. Constantin and L. Molinet, *Global weak solutions for a shallow water equation*, Comm. Math. Phys. **211** (2000), no. 1, 45–61.

[13] A. Constantin and W. Strauss, *Stability of peakons*, Comm. Pure Appl. Math. **53** (2000), no. 5, 603–610.

[14] R. Danchin, *A few remarks on the Camassa-Holm equation*, Differential Integral Equations **14** (2001), no. 8, 953–988.

[15] ———, *Fourier analysis methods for PDEs*, Lecture Notes, 14, 2005.

[16] A. Degasperis, D. D. Holm, and A. N. W. Hone, *A new integral equation with peakon solutions*, Theoret. Math. Phys. **133** (2002), 1463–1474.

[17] H. R. Dullin, G. A. Gottwald, and D. D. Holm, *On asymptotically equivalent shallow water wave equations*, Phys. D **190** (2004), no. 1-2, 1–14.

[18] J. Escher, Y. Liu, and Z. Yin, *Global weak solutions and blow-up structure for the Degasperis-Procesi equation*, J. Funct. Anal. **241** (2006), no. 2, 457–485.

[19] ———, *Shock waves and blow-up phenomena for the periodic Degasperis-Procesi equation*, Indiana Univ. Math. J. **56** (2007), no. 1, 87–177.

[20] Y. Fu, G. Gui, Y. Liu, and C. Qu, *On the Cauchy problem for the integrable modified Camassa-Holm equation with cubic nonlinearity*, J. Differential Equations **255** (2013), no. 7, 1905–1938.

[21] B. Fuchssteiner, *Some tricks from the symmetry-toolbox for nonlinear equations: generalizations of the Camassa-Holm equation*, Phys. D **95** (1996), no. 3-4, 229–243.

[22] K. Grayshan, *Peakon solutions of the Novikov equation and properties of the data-to-solution map*, J. Math. Anal. Appl. **397** (2013), no. 2, 515–521.
[23] G. Gui, Y. Liu, P. Olver, and C. Qu, *Wave-breaking and peakons for a modified Camassa-Holm equation*, Comm. Math. Phys. **319** (2013), no. 3, 731–759.
[24] Z. Guo, *Some properties of solutions to the weakly dissipative Degasperis-Procesi equation*, J. Differential Equations **246** (2009), no. 11, 4332–4344.
[25] A. Himonas and C. Holliman, *The Cauchy problem for the Novikov equation*, Nonlinearity **25** (2012), no. 2, 449–479.
[26] A. Himonas and D. Mantzavinos, *The Cauchy problem for the Fokas-Olver-Rosenau-Qiao equation*, Nonlinear Anal. **95** (2014), no. 1, 499–529.
[27] ———, *Hölder continuity for the Fokas-Olver-Rosenau-Qiao equation*, J. Nonlinear. Sci. **24** (2014), no. 6, 1105–1124.
[28] H. Holden and X. Raynaud, *Global conservative solutions of the Camassa-Holm equations – a Lagrangian point of view*, Comm. Partial Differential Equations **32** (2007), no. 10-12, 1511–1549.
[29] Z. Jiang and L. Ni, *Blow-up phenomenon for the integrable Novikov equation*, J. Math. Anal. Appl. **385** (2012), no. 1, 551–558.
[30] T. Kato and G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pure Appl. Math. **41** (1988), no. 7, 891–907.
[31] S. Y. Lai, *Global weak solutions to the Novikov equation*, J. Funct. Anal. **265(2013)** (2013), no. 4, 520–544.
[32] S. Y. Lai, N. Li, and Y. H. Wu, *The existence of global strong and weak solutions for the Novikov equation*, J. Math. Anal. Appl. **399** (2013), no. 2, 682–691.
[33] J. Lenells and M. Wunsch, *On the weakly dissipative Camassa-Holm, Degasperis-Procesi, and Novikov equations*, J. Differential Equations **255** (2013), no. 3, 441–448.
[34] Y. Liu and Z. Yin, *Global existence and blow-up phenomena for the Degasperis-Procesi equation*, Commun. Math. Phys. **267** (2006), no. 3, 801–820.
[35] H. Lundmark, *Formation and dynamics of shock waves in the Degasperis-Procesi equation*, J. Nonlinear Sci. **17** (2007), no. 3, 169–198.
[36] Y. Mi and C. Mu, *On the Cauchy problem for the modified Novikov equation with peakon solutions*, J. Differential Equations **254** (2013), no. 3, 961–982.
[37] L. Ni and Y. Zhou, *Well-posedness and persistence properties for the Novikov equation*, J. Differential Equations **250** (2011), no. 7, 3002–3021.
[38] V. Novikov, *Generalizations of the Camassa-Holm equation*, J. Phys. A **42** (2009), no. 34, 342002, 14 pp.
[39] P. J. Olver and P. Rosenau, *Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support*, Phys. Rev. E **53** (1996), no. 2, 1900–1906.
[40] G. Rodríguez-Blanco, *On the Cauchy problem for the Camassa-Holm equation*, Nonlinear Anal. **46** (2001), no. 3, 309–327.
[41] F. Tiglay, *The periodic Cauchy problem for Novikov's equation*, Int. Math. Res. Not. **2011** (2011), no. 20, 4633–4648.
[42] Z. Xin and P. Zhang, *On the weak solutions to a shallow water equation*, Comm. Pure Appl. Math. **53** (2000), no. 11, 1411–1433.
[43] Z. Yin, *Global weak solutions to a new periodic integrable equation with peakon solutions*, J. Funct. Anal. **212** (2004), 182–194.
[44] W. Yan, Y. Li, and Y. Zhang, *Global existence and blow-up phenomena for the weakly dissipative Novikov equation*, Nonlinear Anal. **75** (2012), no. 4, 2464–2473.
[45] ———, *The Cauchy problem for the integrable Novikov equation*, J. Differential Equations **253** (2012), no. 1, 298–318.
[46] L. Zhao and S. Zhou, *Symbolic analysis and exact travelling wave solutions to a new modified Novikov equation*, Appl. Math. Comput. **217** (2010), no. 2, 590–598.

[47] Q. Zhang, *Global wellposedness of cubic Camassa-Holm equations*, Nonlinear Anal. **133** (2016), 61–73.
[48] S. Zhou, M. Xie, and F. Zhang, *Persistence properties for the Fokas-Olver-Rosenau-Qiao equation in weighted L^p spaces*, Bound. Value. Probl. **2015** (2015), 1–11.
[49] Y. Zhou, *Blow-up phenomenon for the integrable Degasperis-Procesi equation*, Phys. Lett. A **328** (2004), no. 2, 157–162.

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