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PARTIALLY ABELIAN REPRESENTATIONS OF KNOT GROUPS

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Abstract. A knot complement admits a pseudo-hyperbolic structure by solving Thurston's gluing equations for an octahedral decomposition. It is known that a solution to these equations can be described in terms of region variables, also called w-variables. In this paper, we consider the case when pinched octahedra appear as a boundary parabolic solution in this decomposition. The w-solution with pinched octahedra induces a solution for a new knot obtained by changing the crossing or inserting a tangle at the pinched place. We discuss this phenomenon with corresponding holonomy representations and give some examples including ones obtained from connected sum.

1. Introduction

For a knot diagram D of a knot K in S^3, D. Thurston [6] introduced a way to decompose $M = S^3 \setminus (K \cup \{\text{two points}\})$ into ideal octahedra by placing an octahedron at each crossing and then identifying their faces appropriately along the knot diagram. One can obtain an ideal triangulation $\mathcal{T}_D$ of M by dividing each octahedron into ideal tetrahedra. Then one can give a "pseudo-hyperbolic structure" on M through this ideal triangulation by solving Thurston's gluing equations for $\mathcal{T}_D$ requiring the product of cross-ratios (or shape parameters) around each edge of $\mathcal{T}_D$ to be 1. Since the cross-ratios determine the shapes of each ideal hyperbolic octahedron and vice versa, these hyperbolic octahedra, giving a pseudo-hyperbolic structure on M, will be called a solution. Even though the gluing equations only guarantee that the sum of dihedral angles around each edge is a multiple of 2π, not 2π, one still can consider a (pseudo-) developing map of M and its holonomy representation as W. Thurston did in [7], whenever a solution to the gluing equations is given.

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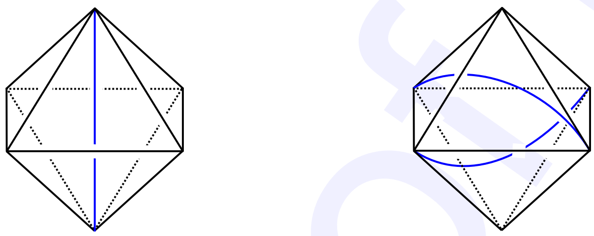
An octahedral decomposition has been used by several authors very successfully in conjunction with the volume conjecture. Yokota [8] used a 4-term triangulation $\mathcal{T}_{4D}$ of M motivated by the optimistic limit of the Kashaev invariant presenting the gluing equations as derivatives of a potential function. In a similar manner, Cho and Murakami [3] suggested a 5-term triangulation $\mathcal{T}_{5D}$ of M applying the optimistic limit to the colored Jones polynomial formulation of the state sum of quantum invariant. They present the gluing equations for $\mathcal{T}_{5D}$ in terms of region variables, also called w-variables, which are non zero complex valued variables assigned to each region of a diagram D. The 5-term triangulation $\mathcal{T}_{5D}$ has a nice property that any non trivial boundary parabolic representation of a knot group can be derived from a solution to the gluing equations for $\mathcal{T}_{5D}$ as a holonomy [2]. On the other hand, the 4-term triangulation $\mathcal{T}_{4D}$ does not have such property since the octahedron at a crossing in $\mathcal{T}_{4D}$ can not be pinched, i.e., the top and bottom vertices of the octahedron can not coincide, while the octahedron in $\mathcal{T}_{5D}$ can.



(a) 4-term triangulation (b) 5-term triangulation

Each solution of the gluing equations gives rise to a holonomy representation of a knot group and among them only boundary parabolic ones will be considered in this paper. We observed that some interesting phenomena arise when pinched octahedra appear in a solution. We first suggest the notion of R-related diagrams as follows. Let a solution to the gluing equations for $\mathcal{T}_{5D}$ have pinched octahedra. Then it also satisfies the gluing equations for $\mathcal{T}_{5D'}$ where D' is a new diagram obtained from D by changing a crossing at which a pinched octahedron is assigned (Theorem 3.1). We say two such diagrams D and D', having a "common w-solution", are R-related. Here 'R' stands for 'representation' meaning that both knots K and K', represented by D and D' respectively, have representations of knot groups with the same image group in $\mathrm{PSL}(2,\mathbb{C})$. These representations are called partially abelian representations where meaning of "partially abelian" will be explained in the following section. We also show that whenever a pinched solution arises, we can replace the crossing, where the pinch occurs, by rational tangles with a relatively easy change of w-solutions (Theorem 3.4). This shows that we can construct lots of "bigger" knots having the same representations and the complex volume as the one we started with. In the last section, we describe how we can find examples of R-related diagrams through the connected sum.

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2. Region variables and pinched octahedra

2.1. Region variables

Let D be a knot diagram of a knot K with N crossings. We denote the crossings of D by $c_1, \ldots, c_N$ and the regions of D by $r_1, \ldots, r_{N+2}$. Let $\mathcal{O}_D$ be Thurston's octahedral decomposition of $M = S^3 \setminus (K \cup \{\text{two points}\})$ with respect to D. We denote the ideal octahedron of $\mathcal{O}_D$ at a crossing c_k by o_k. We divide each octahedron o_k into five tetrahedra by adding two edges as in Figure 1 and call the resulting ideal triangulation of M *the five-term triangulation* $\mathcal{T}_{5D}$. Considering the octahedra $o_1, \ldots, o_N$ to be hyperbolic, Cho and Murakami [3] suggested region variables as a way to describe the shape of the hyperbolic octahedra. A region variable w_j is a non zero complex valued variable assigned to each region r_j of D where the ratio of adjacent region variables around c_k becomes the shape parameter of a tetrahedron in o_k as in Figure 1. It turns out

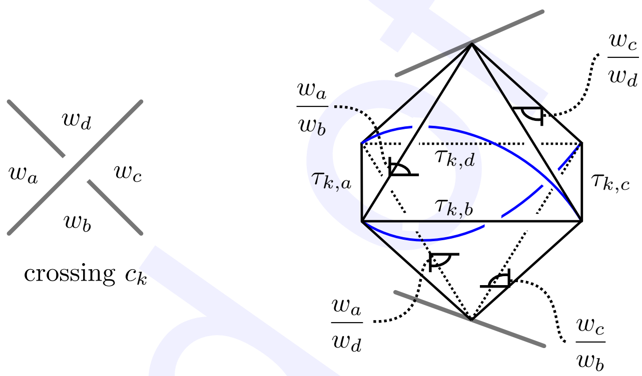


FIGURE 1. The 5-term triangulation and region variables.

that the hyperbolic ideal octahedra $o_1, \ldots, o_N$ whose shapes are determined by w-variables as above automatically satisfy the gluing equation for every edge of $\mathcal{T}_{5D}$ except for the edges corresponding to the regions of D. See Section 4.3 of [4] for details. The gluing equation corresponding to r_j is

$$\prod_{\text{corner crossing } c_k \text{ of } r_j} \tau_{k,j} = 1,$$

where the product is over all corner crossings of r_j and $\tau_{k,j}$ is the cross-ratio at the side edge of o_k corresponding to r_j (see Figure 1). Since the ratios of w-variables and τ's are cross-ratios at the edges of o_k, from the general relation of these cross-ratios of an octahedron, one can compute $\tau_{k,*}$'s in terms of region variables as follows:

$$(1)\quad \begin{cases} \tau_{k,a} = \dfrac{w_b w_d - w_a w_c}{(w_a - w_b)(w_a - w_d)} \\ \tau_{k,b} = \dfrac{(w_b - w_c)(w_b - w_a)}{w_a w_c - w_b w_d} \\ \tau_{k,c} = \dfrac{w_b w_d - w_a w_c}{(w_c - w_d)(w_c - w_b)} \\ \tau_{k,d} = \dfrac{(w_d - w_a)(w_d - w_c)}{w_a w_c - w_b w_d} \end{cases} \quad \text{for a crossing } c_k \text{ as in Figure 1}$$

Definition 2.1. A *region variable* w_j is a non-zero complex valued variable assigned to each region r_j of a diagram D. A $(N+2)$-tuple of region variables $w = (w_1, \ldots, w_{N+2})$ is a *boundary parabolic solution* (to Thurston's gluing equations for $\mathcal{T}_{5D}$) if it satisfies

(a) (gluing equation)

$$(2)\quad \prod_{\text{corner crossing } c_k \text{ of } r_j} \tau_{k,j} = 1$$

for every region r_j of D

(b) (non-degeneracy condition) $w_a w_c - w_b w_d \neq 0$ at each crossing as in Figure 1. We also require that every pair of adjacent region variables is distinct.

The non-degeneracy condition (b) holds if and only if every tetrahedron of $\mathcal{T}_{5D}$ is non-degenerate. (We refer Section 4.3 of [4] for details.)

2.2. Pinched octahedra

Let region variables w be a boundary parabolic solution and let o_k be the hyperbolic ideal octahedron of $\mathcal{O}_D$ at a crossing c_k whose cross-ratios are determined by w. One can construct a pseudo-developing map of $M = S^3 \setminus (K \cup \{\text{two points}\})$ by placing the octahedra $o_1, \ldots, o_N$ consecutively in $\mathbb{H}^3$ in the fashion arranged in the universal cover $\widetilde{M}$. Then one can obtain a holonomy representation $\rho : \pi_1(M) \to \mathrm{PSL}(2, \mathbb{C})$, which is boundary parabolic, of the knot group by the rigidity of a developing map.

In [4], they observed that an octahedron o_k may be *pinched*, i.e., the top and bottom vertices of o_k may coincide.

Proposition 2.2. *Let m_k and $\widehat{m}_k$ be Wirtinger generators winding the over-arc and the incoming under-arc of c_k, respectively. Then the following are equivalent.*

(a) *The hyperbolic octahedron o_k is pinched.*
(b) $w_a - w_b + w_c - w_d = 0$ *for Figure 1.*
(c) $\tau_{k,j} = 1$ *for some region r_j adjacent to c_k.*
(d) $\tau_{k,j} = 1$ *for every region r_j adjacent to c_k.*
(e) $\rho(m_k)$ *and* $\rho(\widehat{m}_k)$ *commute.*

Proof. One can easily check that conditions (b), (c), and (d) are equivalent to each others using equation (1). Moreover, a simple cross-ratio computation gives that $\tau_{k,j} = 1$ if and only if the top and bottom vertices of the octahedron o_k coincide (see Propositions 4.13 and 4.14 in [4]). For condition (e) let us

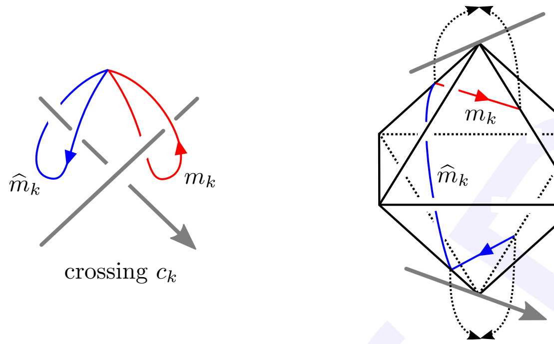


FIGURE 2. Wirtinger generators around a crossing c_k.

consider the Wirtinger generators m_k and $\widehat{m}_k$. Since m_k and $\widehat{m}_k$ wind the top and bottom vertices of o_k respectively as in Figure 2, one can see that $\rho(m_k)$ and $\rho(\widehat{m}_k)$ fix the top and bottom vertices of a developing image of o_k, respectively (see Remark 5.12 of [4] for details). Since both $\rho(m_k)$ and $\rho(\widehat{m}_k)$ are parabolic elements, $\rho(m_k)$ and $\rho(\widehat{m}_k)$ commute if and only if the top and bottom vertices coincide, i.e., o_k is pinched. □

We call the holonomy representation associated to a solution with pinched octahedra, or simply a pinched solution, a *partially abelian representation* with respect to the diagram. We stress that the notion of partially abelian representations depends on a diagram. Note that if every octahedron is pinched, then the solution gives an abelian representation. We also say "a solution is pinched at a crossing c_k" to refer "o_k is pinched". Note that condition (e) is also equivalent to (e′) the ρ-images of any two Wirtinger generators around c_k commute.

Proposition 2.3. *Suppose that a region of D has n corner crossings. If a solution w is pinched at $n-1$ octahedra among them, then it is also pinched at the last crossing.*

Proof. The proof directly follows from condition (d) of Proposition 2.2 and the gluing equation (2) for the region. (Alternatively, one may use Proposition 2.2(e).) □

3. *R*-related diagrams

3.1. Crossing change and diagram change

In this section, we propose the notion of R-relatedness of knot diagrams by the following property: If two diagrams D and D' are R-related, then the knots K and K', given by D and D' respectively, have boundary parabolic representations with the same image group in $\mathrm{PSL}(2,\mathbb{C})$. To exclude the trivial case we assume that representations in this section are not abelian, equivalently solutions are not pinched at every crossing.

Theorem 3.1. *Let region variables w be a boundary parabolic solution for a diagram D. Suppose that w is pinched at crossings $\{c_k \mid k \in J\}$ for some index set J. Then w is also a boundary parabolic solution for a diagram D^J, which is obtained from D by changing the crossings $\{c_k \mid k \in J\}$.*

Proof. Let $\tau_{*,*}$ (resp., $\tau^J_{*,*}$) be the τ-values in equation (1) for the region variables w with respect to the diagram D (resp., D^J). It is clear from equation (1) that $\tau_{k,*} = \tau^J_{k,*}$ for $k \notin J$. Also, conditions (a) and (c) of Proposition 2.2 tell us that $\tau_{k,*} = \tau^J_{k,*} = 1$ for $k \in J$. Therefore, the solution w also satisfies the gluing equations for every region of D^J. □

We say such two diagrams D and D^J in Theorem 3.1 are *R-related*. Let K (resp., K^J) be a knot represented by D (resp., D^J). The solution w induces a representation of both the knot groups of K and K^J, and we denote them by ρ and ρ^J, respectively. One can describe ρ^J by ρ as follows. Let m_i, m_j, and m_l (resp., m^J_i, m^J_j, and m^J_l) be Wirtinger generators around a crossing $c_k (k \in J)$ of D (resp., D^J) as in Figure 3. Then $\rho^J(m^J_i) = \rho(m_i) = \rho(m_l)$, $\rho^J(m^J_j) = \rho(m_j)$, and $\rho^J(m^J_l) = \rho(m_j)$. Note also that we have $\rho(m_i) = \rho(m_l)$ and $\rho^J(m^J_j) = \rho^J(m^J_l)$.

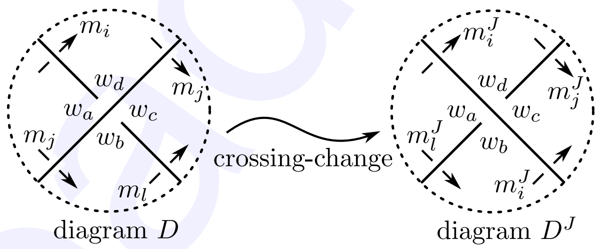


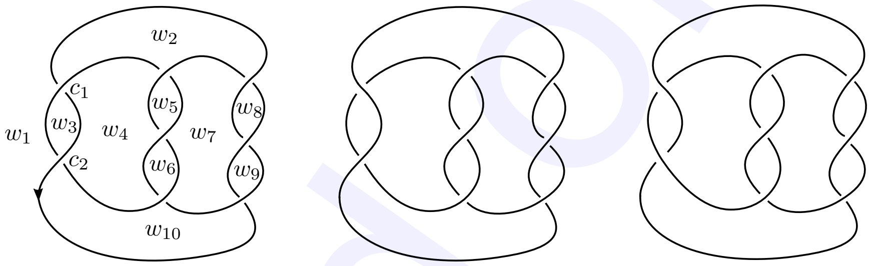
FIGURE 3. A crossing-change and Wirtinger generators.

Then it is clear that the image of ρ^J is the same as that of ρ. In particular, the complex volumes of K and K^J with respect to these representations are the same.

Example 3.2 (The knot 8_5)**.** In Section 7.2 of [4], they presented a pinched solution $w = (w_1, \ldots, w_{10})$ for a diagram D of the knot 8_5 as in Figure 4(a), and argued that there is no others:

$$\begin{aligned}(w_1, \ldots, w_{10}) = \Big(& -\frac{1}{p+q-pqr} + \frac{1}{p} + \frac{1}{q},\ -\frac{1}{p+q-pqr} + \frac{1}{p} + r, \\ & -\frac{1}{-pqr+p+q} + \frac{1}{p} + \frac{2}{q} - r,\ -\frac{1}{p+q-pqr} + \frac{1}{p} + \frac{1}{q}, \\ & -\frac{1}{p+q-pqr} + \frac{1}{q} + r,\ \frac{1}{p} + \frac{1}{q},\ r,\ \frac{1}{p} + \frac{1}{q}, \\ & -\frac{1}{p+q-pqr} + \frac{1}{q} + r,\ -\frac{1}{p+q-pqr} + \frac{1}{p} + r \Big).\end{aligned}$$

One can check that w is pinched at the crossings c_1 and c_2 using condition (b) of Proposition 2.2. Then, by Theorem 3.1, it also satisfies the gluing equations for $D^{\{1\}}$ and $D^{\{1,2\}}$, which are diagrams of the granny knot and the $T(3,4)$ torus knot, respectively. In particular, the complex volume $(0+3.28987i)$ of the knot 8_5 with respect to the solution w is the same as that of the granny knot which is twice the complex volume $(0 + 1.64493i)$ of the irreducible representation of the trefoil knot. See also [1]. (Note that the trefoil knot has the unique irreducible boundary parabolic representation.)



(a) The knot 8_5 (b) The granny knot (c) The $T(3,4)$ torus knot

FIGURE 4. The 8_5 knot diagram and R-related diagrams.

Example 3.3 (The knot 8_{18})**.** Let us consider a diagram D of the 8_{18} knot and assign region variables $w_1, \ldots, w_{10}$ to D as in Figure 5. We first investigate the possibilities of crossings to be pinched. Suppose there is a solution w pinched at the crossing c_6. By Theorem 3.1, w is a solution for $D^{\{6\}}$, which is a diagram of the trefoil knot with a kink at the crossing c_8. Since Wirtinger generators around c_8 commute, w should be also pinched at c_8 by Proposition 2.2(e). Under this condition one can compute a representation ρ using the Wirtinger presentation. (We use Mathematica for the actual computation.) Also, one can obtain the solution w from ρ through [2]:

$$w = \big(p - qr + q, p - qr + q, pr + p - qr, pr + p - qr, pr - qr + q,$$

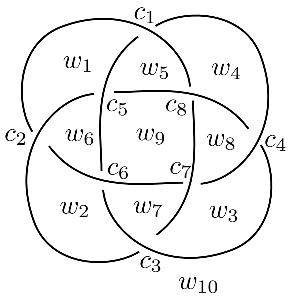


FIGURE 5. The 8_{18} knot diagram.

$$p-qr+2q, pr+p+q, 2pr+p-qr, pr+q, p-qr\big).$$

We note that w is pinched only at the crossings c_6 and c_8. Using the similar argument, we obtain a solution w' which is pinched only at the crossings c_2 and c_4:

$$w' = \big(p-qr+q, pr-2qr+q, 4pr-p-3qr+q, pr+p-qr, pr-qr+q,$$
$$pr-qr+q, 3pr-p-2qr+q, 3pr-p-2qr+q, 2pr-p-qr+q, p-qr\big).$$

One can check through Proposition 2.3 that other possibilities result in a solution pinched at every crossing or a solution in symmetry with either w or w'. Hence w and w' are the only pinched solutions for D.

Since both $D^{\{6,8\}}$ and $D^{\{2,4\}}$ represent the trefoil knot, we conclude that the knot 8_{18} has a boundary parabolic representation whose image is the modular group $\mathrm{PSL}(2,\mathbb{Z})$, which is the image of the irreducible representation of the trefoil knot.

Theorem 3.4. *Let w be a boundary parabolic solution for a diagram D. Suppose that w is pinched at a crossing c_k. Let D' be a diagram obtained from D by replacing c_k by the standard diagram of a rational tangle $[2n_1,\dots,2n_{k-1},2n_k+1]$, $n_i\in\mathbb{Z}$. Then there is a boundary parabolic solution w' for a diagram D' such that w' is pinched at every crossing in the tangle and coincide with w on the outside of the tangle.*

Proof. Let us denote the region variables around the crossing c_k by w_a, w_b, w_c, and w_d as in Figure 6. Then we have $w_a-w_b+w_c-w_d=0$ from Proposition 2.2(a). We prove the theorem by induction on k. For the case $k=1$, we replace c_k by a rational tangle $[2n_1+1]$. Compare with the diagram D, there are even number of new regions of D'. We define region variable w' by assigning w_c and w_a alternately to these new regions and leave w for other unchanged regions. See Figure 6. It is clear that w' is pinched at every crossing in the tangle, since we have $w_a-w_b+w_c-w_d=0$ at each crossing. Also, one can check that w' satisfies the gluing equation for every region of D' by Proposition 2.2(d).

For $k>1$, the number of regions of D' increases by an even number as k increases by 1. We define w' by assigning w_b and w_d (resp., w_c and w_a)

alternately to the newly created regions if k increase to an even (resp., odd) number as in Figure 6. Then one can check that w' is a desired solution.

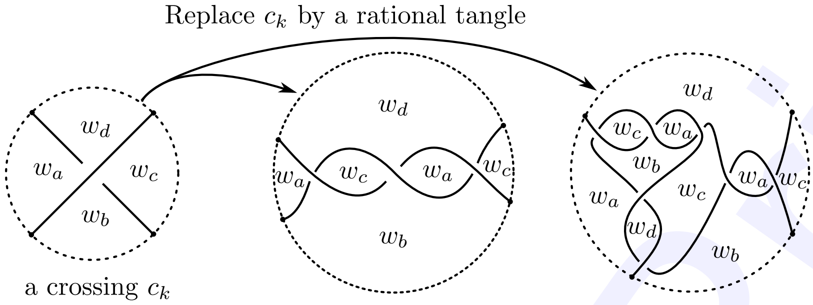


FIGURE 6. Rational tangles [3] and $[2, -2, 3]$.

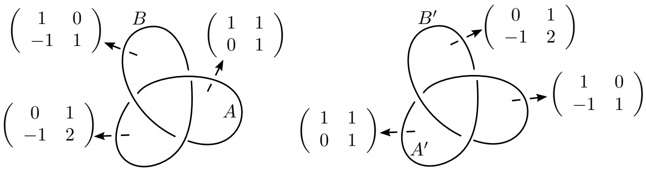
□

3.2. R-related diagrams from connected sum

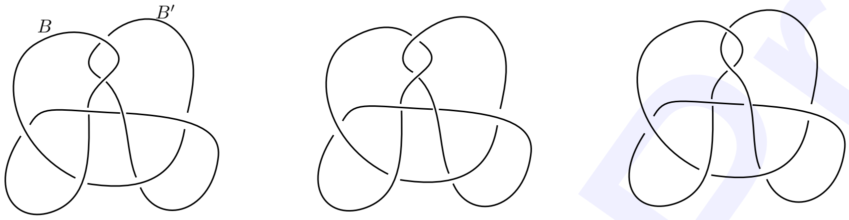
Now we give some examples of R-related diagrams using connected sum. Let D (resp., D') be a diagram of a knot K (resp., K') and let ρ (resp., ρ') be a boundary parabolic representation of the knot group of K (resp., K'). One can construct 1-parameter family of boundary parabolic representations for $K\#K'$ as follows. Let A and A' be arcs of D and D' respectively which are to be cut for the connected sum $D\#D'$. We may assume that both $\rho(m_A)$ and $\rho'(m_{A'})$ are $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ by conjugating ρ and ρ' appropriately where m_A (resp., $m_{A'}$) is the Wirtinger generator winding A (resp., A'). Then for any $r \in \mathbb{C}$ we define a representation $\rho\#_r\rho'$ of $D\#D'$ by assigning ρ to the Wirtinger generators of D and assigning $\left(\begin{smallmatrix}1&r\\0&1\end{smallmatrix}\right)\rho'\left(\begin{smallmatrix}1&-r\\0&1\end{smallmatrix}\right)$ to the Wirtinger generators of D'. (This construction is also described in [1].)

Now choose an arc B of D and an arc B' of D' such that they are parts of a common region in $D\#D'$. Suppose both $\rho(m_B)$ and $\rho'(m_{B'})$ do not fix ∞ where m_B (resp., $m_{B'}$) is the Wirtinger generator winding the arc B (resp., B'). Let us choose $r := \mathrm{Fix}(\rho(m_B)) - \mathrm{Fix}(\rho'(m_{B'}))$. Then the $\rho\#_r\rho'$ images of m_B and $m_{B'}$ commute since they are parabolic elements having a common fixed point. Therefore, applying Reidemeister second move for the arcs B and B' in the common region, we obtain two pinched crossings.

Example 3.5 (The granny knot)**.** Let D and D' be diagrams of the trefoil knot, and ρ and ρ' be representations described as in Figure 7(a). We choose arcs A, A', B, and B' as in Figure 7(a). Then we have $r = \mathrm{Fix}(\rho(m_B)) - \mathrm{Fix}(\rho'(m_{B'})) = -1-0 = -1$ and hence we obtain the irreducible representation $\rho\#_{-1}\rho'$ of $D\#D'$. Now apply Reidemeister second move for the arcs B and B' in $D\#D'$. Since the $\rho\#_{-1}\rho'$ images of m_B and $m_{B'}$ commute, $\rho\#_{-1}\rho'$ is also a representation for a diagram obtained by changing a crossing created by the Reidemeister move. This results in the knots 8_{21} and 8_{15} depending on the crossing-change as in Figure 7. Therefore each of the knots 8_{21} and 8_{15} has a



(a) Two trefoil knots to form the granny knot

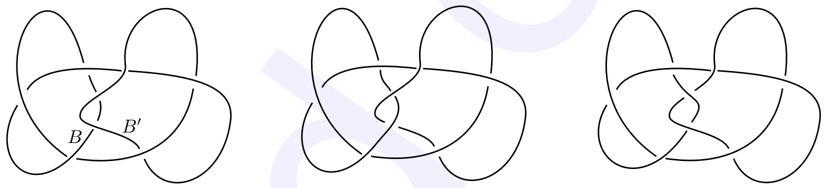


(a) the granny knot (b) the knot 8_{21} (c) the knot 8_{15}

FIGURE 7. R-related diagrams: the granny knot, 8_{21}, and 8_{15}.

representation whose image group is the same as the image of $\rho\#_{-1}\rho'$ of the granny knot.

One can choose arcs B and B' differently as in Figure 8 which results in the knots 8_{19} and 8_5. We note that other choices of B and B' does not give a new one.

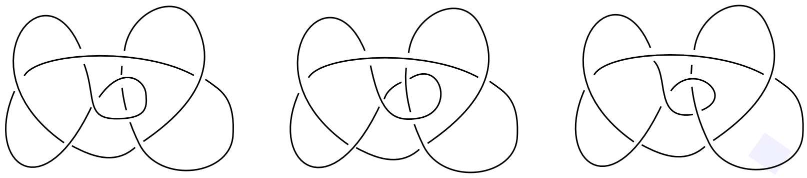


(a) the granny knot (b) the knot 8_{19} (c) the knot 8_5

FIGURE 8. R-related diagrams: the granny knot, 8_{19}, and 8_5.

Example 3.6 (The square knot)**.** We can apply the same argument to the square knot and obtain the knots 8_{20} and 8_{10} as in Figure 9. We check that these knots are only knots obtained from the square knot diagram. Again, we conclude that each of the knots 8_{20} and 8_{10} have a representation whose image is the same as that of the square knot.

The discussions in this section suggest implicitly a hierarchy on the set of knots. If two knots share a R-related diagram, in general one is "smaller" than the other in a certain sense as the discussions in this section indicate, i.e.,



(a) the square knot (b) the knot 8_{20} (c) the knot 8_{10}

FIGURE 9. R-related diagrams: the square knot, 8_{20}, and 8_{10}.

a representation of a smaller knot essentially appears as a pinched representation of the other knot. This may define a kind of order or a hierarchy on the set of knots. This also suggest a strong relation with the knot group epimorphism problem, even though this hierarchy looks weaker than the partial order defined by the knot group epimorphism [5]. As we saw in the examples of 8 crossing knots, all these knots obtained from granny and square knots by crossing changes are known to have an epimorphism to the trefoil knot, and in fact these are the only such knots with up to 8 crossings. We hope to investigate this "hierarchy" and the relationship with epimorphism problem more systematically in future papers.

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