Bull. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/BKMS.b160905
pISSN: 1015-8634 / eISSN: 2234-3016

RIQUIER AND DIRICHLET BOUNDARY VALUE PROBLEMS FOR SLICE DIRAC OPERATORS

Hongfen Yuan

Abstract. In recent years, the study of slice Dirac operators has attracted more and more attention in the literature. In this paper, Almansi-type decompositions for null solutions to the iterated slice Dirac operator and the generalized slice Dirac operator are obtained without a star-like domain centered at the origin. As applications, we investigate Riquier type problems and Dirichlet type problems in the theory of slice monogenic functions.

1. Introduction

A lot of attention has been paid to developing a new theory of slice monogenic functions (i.e., slice Clifford analysis) (see [5, 6]). It is a generalization of the theory of complex analysis to higher dimensions, where a real Clifford algebra takes over the role played by the complex numbers while still preserving the essential features of complex analysis. Analytic results in the flavor of complex analysis has been established in the theory of slice monogenic functions (see [7, 10, 11]). In other approaches to hypercomplex analysis, an important role is played by an underlying algebraic structure, namely the Lie superalgebra $osp(1|2)$, which allows us to find a representation theoretic interpretation of various function space decompositions (see [8]). In 2015, based on the Lie superalgebra structure, Cnudde, De Bie and Ren [4] defined a slice Dirac operator as follows.

Received November 12, 2016; Revised June 10, 2017; Accepted July 3, 2017.
2010 *Mathematics Subject Classification.* 30G35, 30E25, 35G15.
Key words and phrases. slice Dirac operator, Euler operator, generalized slice Dirac operator, Riquier problem, Dirichlet problem.
This work was supported by the National Natural Science Foundation of China (Grant No. 11426082), and the Science Foundation of Hebei Province (Grant No. A2016402034), Project of Handan Municipal Science and Technology Bureau under Grant No. 1534201097-10, and Foundation of Hebei University of Engineering under Grant Nos. 16121002014, 00070348, 00717.

Let Ω be a bounded subdomain of R^{m+1}. Then the slice Dirac operator is defined as

$$D_0 = e_0\partial_{x_0} + \frac{\underline{x}}{|\underline{x}|^2}\sum_{i=1}^{m} x_i\partial_{x_i},$$

where $\underline{x} = \sum_{i=1}^m x_i e_i$ and $|\underline{x}|^2 = \sum_{i=1}^m x_i^2 = -\underline{x}^2$. Here the vectors $e_i, i = 0,\ldots,m$ satisfy the equation $e_ie_j + e_je_i = -2\delta_{i,j},\ i,j = 0,\ldots,m$, which are basis vectors of the real Clifford algebra Cl_{m+1}. Null solutions to the slice Dirac operator are called slice monogenic functions. Also, 1-vector $x \in Cl_{m+1}$ is defined as

$$x = x_0e_0 + \sum_{i=1}^{m} x_ie_i = x_0e_0 + \underline{x}.$$

Based on their work, we study the commutativity between D_0, x and the shifted Euler operator $\mathbb{E}_s$. Furthermore, using commutativity, we construct Almansi-type decompositions for slice polymonogenic functions (i.e., null solutions to the iterated slice Dirac operator D_0^k).

The following assertion stated by E. Almansi [1] is well known: if $f(x)$ is a semiharmonic function of order m in a star domain Ω_0 centered at the origin of coordinates, then there exist unique functions $f_1(x),\ldots,f_m(x)$, each harmonic in Ω_0 such that $f(x) = f_0(x) + |x|^2 f_1(x) + \cdots + |x|^{2(m-1)} f_{m-1}(x)$. The classical decomposition theorem is called the Almansi decomposition, which is related to the Fischer decomposition of polynomials. The results in the case of complex analysis, Clifford analysis and Clifford analysis in superspace have been developed in [2, 15, 17, 18]. More recently, the result is useful in the study of boundary value problems. Riemann-Hilbert problems for null-solutions to iterated generalized Cauchy-Riemann equations in axially symmetric domains are considered based on Almansi decompositions for generalized Cauchy-Riemann operators (see [12]). Riquier problems in harmonic and Clifford analysis (see [3]) are studied by Almansi decompositions (see [13, 14, 19]). However, Almansi type decompositions for slice polymonogenic functions and their applications are still not considered. In this paper, we try to fill part of this gap. We investigate Riquier type problems and Dirichlet type problems in the theory of slice monogenic functions by Almansi type decompositions for the slice Dirac operator. This is a starting point for further research, in particular on boundary value problems for slice monogenic functions which are studied with the help of Almansi type decompositions and for which corresponding but quite different results will be published in a subsequent paper.

2. Almansi type decompositions for slice Dirac operators

In this section we show Almansi type decompositions for slice Dirac operators by four different forms.

2.1. Almansi type decomposition I

In [2, 17], the authors investigated Almansi decompositions in star-like domain with center 0. In this section, we give an Almansi decomposition for the iterated slice Dirac operator D_0^k in star-like domain with center a.

We begin with the following definitions:

Definition 2.1. Let $a = \sum_{i=0}^{m} e_i a_i \in R^{m+1}$ and $|\underline{a}| \neq 0$. Let Ω_a be a star-like subdomain of R^{m+1} with center a. The shifted Euler operator defined on the space $C^1(\Omega_a, Cl_{m+1})$ is given by

$$\mathbb{E}_s = s\mathbf{I}_{\Omega_a} + \sum_{i=0}^{m} (x_i - a_i)\partial_{x_i},$$

where $s > 0$ and $\mathbf{I}_{\Omega_a}$ denotes the identity operator on the space $C^1(\Omega_a, Cl_{m+1})$.

Definition 2.2. Let Ω_a be given as above. The operator $\mathbb{J}_s : C(\Omega_a, Cl_{m+1}) \to C(\Omega_a, Cl_{m+1})$ is defined by

$$\mathbb{J}_s f = \int_0^1 f[a + t(x - a)]t^{s-1}dt,$$

where $s > 0$.

Then we provide several lemmas with respect to the shifted Euler operator as follows.

Lemma 2.3. *Let Ω_a be as stated before. If $f(x) \in C^2(\Omega_a, Cl_{m+1})$, then*

$$D_0\mathbb{E}_s f(x) = \mathbb{E}_{s+1} D_0 f(x), \tag{1}$$

where $s > 0$.

Proof. With Definition 2.1 and the definition of the slice Dirac operator, we have

$$\begin{aligned}
& D_0\mathbb{E}_s f(x) \\
= & \left[e_0\partial_{x_0} + \frac{\underline{x}}{|\underline{x}|^2}\sum_{i=1}^{m} x_i\partial_{x_i}\right]\left[s\mathbf{I}_{\Omega_a} + \sum_{i=0}^{m}(x_i - a_i)\partial_{x_i}\right] f(x) \\
= & \, se_0\partial_{x_0} f(x) + e_0\partial_{x_0}\sum_{i=0}^{m}(x_i - a_i)\partial_{x_i} f(x) \\
& + s\frac{\underline{x}}{|\underline{x}|^2}\sum_{i=1}^{m} x_i\partial_{x_i} f(x) + \frac{\underline{x}}{|\underline{x}|^2}\sum_{i=1}^{m} x_i\partial_{x_i}\sum_{j=0}^{m}(x_j - a_j)\partial_{x_j} f(x) \\
= & \left[(s+1)e_0\partial_{x_0} + (s+1)\frac{\underline{x}}{|\underline{x}|^2}\sum_{i=1}^{m} x_i\partial_{x_i}\right] f(x) \\
& + \left[\sum_{i=0}^{m}(x_i - a_i)\partial_{x_i}(e_0\partial_{x_0}) + \sum_{j=0}^{m}(x_j - a_j)\partial_{x_j}\frac{\underline{x}}{|\underline{x}|^2}\sum_{i=1}^{m} x_i\partial_{x_i}\right] f(x)
\end{aligned}$$

$$= \mathbb{E}_{s+1} D_0 f(x). \qquad \square$$

Lemma 2.4. *Suppose the operators* $\mathbb{E}_s$ *and* $\mathbb{J}_s$ *are given as above. Then on the space* $C^1(\Omega_a, Cl_{m+1})$ *we have*

$$\mathbb{E}_s \mathbb{J}_s = \mathbb{J}_s \mathbb{E}_s = \mathbf{I}_{\Omega_a}, \tag{2}$$

$$\mathbb{E}_s[(x-a)f] = (x-a)[\mathbb{E}_{s+1} f], \tag{3}$$

where $\mathbf{I}_{\Omega_a}$ *denotes the identity operator on the space* $C^1(\Omega_a, Cl_{m+1})$.

Proof. Let $f \in C^1(\Omega_a, Cl_{m+1})$. Then we calculate

$$\begin{aligned} f(x) &= \int_0^1 \frac{d}{dt}[f(a+t(x-a))t^s]dt \\ &= \int_0^1 \left\{ t^s \sum_{i=0}^m (x_i - a_i) \frac{\partial f}{\partial x_i}[a+t(x-a)] + st^{s-1} f[a+t(x-a)] \right\} dt \\ &= \int_0^1 \left\{ t^{s-1} \left[\sum_{i=0}^m (x_i - a_i) \frac{\partial f}{\partial x_i} \right] [a+t(x-a)] + st^{s-1} f[a+t(x-a)] \right\} dt \\ &= \int_0^1 \left\{ \left[\sum_{i=0}^m (x_i - a_i) \frac{\partial f}{\partial x_i} \right] [a+t(x-a)] + sf[a+t(x-a)] \right\} t^{s-1} dt, \end{aligned}$$

which implies that $\mathbb{J}_s \mathbb{E}_s f = f$. A similar calculation gives

$$\mathbb{E}_s \mathbb{J}_s f = f.$$

Using the definition of the shifted Euler operator, we have

$$\begin{aligned} &\mathbb{E}_s(x-a)f \\ &= \left[s\mathbf{I} + \sum_{i=0}^m (x_i - a_i)\partial_{x_i} \right] \sum_{j=0}^m (x_j - a_j) e_j f \\ &= s\sum_{j=0}^m (x_j - a_j) e_j f + \sum_{j=0}^m (x_j - a_j) e_j f + \sum_{i,j=0}^m (x_j - a_j) e_j (x_i - a_i) \partial_{x_i} f \\ &= (s+1) f + (x-a) \sum_{j=0}^m \partial_{x_j} (x_j - a_j) f \\ &= (x-a)\mathbb{E}_{s+1} f, \end{aligned}$$

which completes the proof. $\square$

Lemma 2.5. *Let* Ω_a *be given as in Definition 2.1. If* $f \in C^1(\Omega_a, Cl_{m+1})$ *is slice monogenic, then* $\mathbb{E}_s f(x)$ *and* $\mathbb{J}_s f(x)$ *are slice monogenic in* Ω_a.

Proof. Assume $f \in C^1(\Omega_a, Cl_{m+1})$ is slice monogenic. Then $D_0 f(x) = 0$ and

$$D_0 \mathbb{J}_s f = \left(e_0 \partial_{x_0} + \frac{\underline{x}}{|\underline{x}|^2} \sum_{i=1}^m x_i \partial_{x_i} \right) \left[\int_0^1 f(a+t(x-a)) t^{s-1} dt \right]$$

$$= \int_0^1 \left[\left(e_0 \partial_{x_0} + \frac{\underline{x}}{|\underline{x}|^2} \sum_{i=1}^m x_i \partial_{x_i} \right) f \right] (a + t(x-a)) t^s dt = 0.$$

It is easy to see that $D_0 \mathbb{E}_s f = 0$ by the definition of the shifted Euler operator. □

Theorem 2.6. *If $f \in C^k(\Omega_a, Cl_{m+1})$ is slice polymonogenic, then there exist unique slice monogenic functions $f_1, \ldots, f_k$ in Ω_a such that*

$$f(x) = \sum_{i=0}^{k-1} (x-a)^i f_i(x), \tag{4}$$

where $f_1, \ldots, f_k$ are given by

$$\begin{cases} f_0(x) = [\mathbf{I}_{\Omega_a} - (x-a) G_1 D_0] \cdots \left[\mathbf{I}_{\Omega_a} - (x-a)^{k-1} G_{k-1} D_0^{k-1} \right] f(x), \\ f_1(x) = G_1 D_0 \\ \qquad \left[\mathbf{I}_{\Omega_a} - (x-a)^2 G_2 D_0^2 \right] \cdots \left[\mathbf{I}_{\Omega_a} - (x-a)^{k-1} G_{k-1} D_0^{k-1} \right] f(x), \\ \vdots \\ f_{k-2}(x) = G_{k-2} D_0^{k-2} \left[\mathbf{I}_{\Omega_a} - (x-a)^{k-1} G_{k-1} D_0^{k-1} \right] f(x), \\ f_{k-1}(x) = G_{k-1} D_0^{k-1} f(x), \end{cases} \tag{5}$$

with $G_k = \frac{1}{C_k} \mathbb{J}_1 \cdots \mathbb{J}_{1+\left[\frac{k-1}{2}\right]}$, $C_k = (-2)^k \left[\frac{k}{2}\right]!$.

Conversely, if functions $f_1, \ldots, f_k$ are slice monogenic, then the sum in (4) *is a slice polymonogenic function.*

Proof. First of all, we prove that if $q(x)$ is slice monogenic, then

$$D_0^k (x-a)^k q(x) = C_k \mathbb{E}_{1+\left[\frac{k-1}{2}\right]} \cdots \mathbb{E}_1 q(x), \tag{6}$$

where $C_k = (-2)^k \left[\frac{k}{2}\right]!$. We prove (6) by induction. For $k = 1$, inspired by Cnudde [4], we have

$$D_0 (x-a) q(x) = -[(x-a) D_0 + 2\mathbb{E} + 2] q(x) = (-2) \mathbb{E}_1 q(x).$$

Thus, we have

$$D_0 (x-a)^{2s} q(x) = -2s (x-a)^{2s-1} q(x) + (x-a)^{2s} D_0 q(x) = -2s (x-a)^{2s-1} q(x)$$

and

$$\begin{aligned} D_0 (x-a)^{2s+1} q(x) &= -2(x-a)^{2s} (s + \mathbb{E} + 1) q(x) - (x-a)^{2s+1} D_0 q(x) \\ &= -2(x-a)^{2s} \mathbb{E}_{s+1} q(x). \end{aligned}$$

Suppose that

$$D_0^l (x-a)^l q(x) = C_l \mathbb{E}_{1+\left[\frac{l-1}{2}\right]} \cdots \mathbb{E}_1 q(x)$$

for $k = l$. We prove (6) for $k = l+1$. If $l = 2s$, then

$$D_0^{2s+1} (x-a)^{2s+1} q(x) = D_0^{2s} D_0 x^{2s+1} q(x) = D_0^{2s} [-2(x-a)^{2s} \mathbb{E}_{s+1} q(x)],$$

and

$$D_0^{2s}[-2(x-a)^{2s}\mathbb{E}_{s+1}q(x)] = D_0^{2s-1}[4s(x-a)^{2s-1}\mathbb{E}_{s+1}q(x)].$$

Equation (6) follows directly from the above induced formulas and the assumption of induction.

Then we apply the operator D_0^{k-1} on both sides of the equation (4):

$$\begin{aligned} D_0^{k-1}f(x) &= D_0^{k-1}\left(\sum_{i=0}^{k-1}(x-a)^i f_i(x)\right) \\ &= D_0^{k-1}\left[(x-a)^{k-1}f_{k-1}(x)\right] \\ &= C_{k-1}\mathbb{E}_{1+\left[\frac{k-2}{2}\right]}\cdots\mathbb{E}_1 f_{k-1}(x), \end{aligned}$$

which implies that

$$f_k(x) = \frac{1}{C_{k-1}}\mathbb{J}_1\cdots\mathbb{J}_{1+\left[\frac{k-2}{2}\right]}D_0^{k-1}f(x).$$

Thus, (5) follows by induction.

Finally, it is easy to prove that the sum in (4) is a slice polymonogenic function if $f_1, \ldots, f_k$ are slice monogenic. □

From Theorem 2.6, we can also get the Fischer decomposition for polynomials. In fact, if f is a homogeneous polynomial of degree $k-1$, then $D_0^k f = 0$.

2.2. Almansi type decomposition II

In this section, we obtain the Almansi type decomposition for the operator D_0^{2k} by another way.

Theorem 2.7. *Let $f(x) \in C^{2k}(\Omega_a, Cl_{m+1})$. If $f(x)$ is a solution to the equation $D_0^{2k}f = 0$, then there exist unique functions $f_0, \ldots, f_{k-1}$ such that*

$$f(x) = \sum_{i=0}^{k-1}(x-a)^{2i}f_i(x), \quad x \in \Omega_a, \tag{7}$$

where each $f_i(x)$ satisfies the equation $D_0^2 f = 0$, and given by

$$\begin{cases} f_{k-1}(x) = \dfrac{\mathbb{J}_{k-1}\cdots\mathbb{J}_1 D_0^{2(k-1)}f(x)}{4^{k-1}(k-1)!}, \\ f_{k-2}(x) = \dfrac{\mathbb{J}_{k-2}\cdots\mathbb{J}_1 D_0^{2(k-2)}[f(x)-(x-a)^{2(k-1)}f_{k-1}(x)]}{4^{k-2}(k-2)!}, \\ f_{k-3}(x) = \dfrac{\mathbb{J}_{k-3}\cdots\mathbb{J}_1}{4^{k-3}(k-3)!} \\ \qquad D_0^{2(k-3)}[f(x)-(x-a)^{2(k-1)}f_{k-1}(x)-(x-a)^{2(k-2)}f_{k-2}(x)], \\ \vdots \\ f_1(x) = \dfrac{\mathbb{J}_1 D_0^2[f(x)-(x-a)^{2(k-1)}f_{k-1}(x)-\cdots-(x-a)^2 f_2(x)]}{4}, \\ f_0(x) = f(x)-(x-a)^{2(k-1)}f_{k-1}(x)-\cdots-(x-a)^2 f_1(x). \end{cases} \tag{8}$$

Conversely, if functions $f_0, \ldots, f_{k-1}$ *satisfy the equation* $D_0^2 f = 0$*, then the function* $f(x)$ *given by* (7) *is a solution to the equation* $D_0^{2k} f = 0$.

Proof. First of all, we prove that

$$D_0^{2s}[(x-a)^{2s} f(x)] = 4^s s! \mathbb{E}_1 \cdots \mathbb{E}_s f(x),$$

where $f(x) \in C^{2s}(\Omega_a, Cl_{m+1})$ is a solution to the equation $D_0^2 f = 0$. Direct calculation yields

$$\begin{aligned}
& D_0^{2s}[(x-a)^{2s} f(x)] \\
&= D_0^{(2s-1)} D_0[(x-a)^{2s} f(x)] \\
&= D_0^{(2s-1)}[-2s(x-a)^{(2s-1)} f(x) + (x-a)^{2s} D_0 f(x)] \\
&= D_0^{2(s-1)} D_0[-2s(x-a)^{(2s-1)} f(x) + (x-a)^{2s} D_0 f(x)] \\
&= D_0^{2(s-1)}[2^2 s(x-a)^{2(s-1)}(s+\mathbb{E}+1) f(x) + 2s(x-a)^{(2s-1)} D_0 f(x)] \\
&\quad + D_0^{2(s-1)}[-2s(x-a)^{(2s-1)} D_0 f(x) + (x-a)^{2s} D_0^2 f(x)] \\
&= D_0^{2(s-1)}[2^2 s(x-a)^{2(s-1)}(s+\mathbb{E}+1)] f(x) \\
&= D_0^{2(s-1)}[2^2 s(x-a)^{2(s-1)}(s+\mathbb{E})] f(x) + D_0^{2(s-1)}[2^2 s(x-a)^{2(s-1)} f(x)] \\
&= \cdots \\
&= 4^s s! \mathbb{E}_1 \cdots \mathbb{E}_s f(x).
\end{aligned}$$

Secondly, differentiating both sides of the equation (7), we have

$$\begin{aligned}
D_0^{2(k-1)} f(x) &= D_0^{2(k-1)} \left[\sum_{i=0}^{k-1} (x-a)^{2i} f_i(x) \right] \\
&= D_0^{2(k-1)} \left[(x-a)^{2(k-1)} f_{k-1}(x) \right] \\
&= 4^{k-1}(k-1)! \mathbb{E}_1 \cdots \mathbb{E}_{k-1} f_{k-1}(x).
\end{aligned}$$

Thus, we see that

$$f_{k-1}(x) = \frac{1}{4^{k-1}(k-1)!} \mathbb{J}_{k-1} \cdots \mathbb{J}_1 D_0^{2(k-1)} f(x)$$

by Lemma 2.4. Using the same steps, it can be shown that

$$f_{k-2}(x) = \frac{1}{4^{k-2}(k-2)!} \mathbb{J}_{k-2} \cdots \mathbb{J}_1 D_0^{2(k-2)} [f(x) - (x-a)^{2(k-1)} f_{k-1}(x)].$$

Thus, we have (7) by induction.

Conversely, if $f_1, \ldots, f_{k-1}$ satisfy the equation $D_0^2 f = 0$, then by Lemma 2.5, we find that

$$D_0^{2k} f(x) = D_0^{2k} \left[\sum_{i=0}^{k-1} (x-a)^{2i} f_i(x) \right]$$

$$= D_0^2 D_0^{2(k-1)} \left[\sum_{i=0}^{k-1} (x-a)^{2i} f_i(x) \right]$$
$$= 4^{k-1}(k-1)! D_0^2 \mathbb{E}_1 \cdots \mathbb{E}_{k-1} f_{k-1}(x) = 0,$$

which implies that the function $f(x)$ given by (7) is a solution to the equation $D_0^{2k} f = 0$. □

2.3. Almansi type decomposition III

In this section, in a similar way as in the proof of Theorem 2.7, we obtain the Almansi type decomposition for the iterated slice Dirac operator D_0^{2k} by another form as follows.

Theorem 2.8. *If $f(x) \in C^{2k}(\Omega_a, Cl_{m+1})$ is a solution to the equation $D_0^{2k} f = 0$, then there exist unique functions $f_0, \ldots, f_{k-1}$ in Ω_a satisfying the equation $D_0^2 f = 0$ such that*

$$f(x) = \sum_{i=0}^{k-1} [1 + (x-a)^2]^i f_i(x), \tag{9}$$

where

$$\left\{ \begin{array}{l} f_{k-1}(x) = \frac{1}{4^{k-1}(k-1)!} \mathbb{J}_{k-1} \cdots \mathbb{J}_1 D_0^{2(k-1)} f(x), \\ f_{k-2}(x) = \frac{1}{4^{k-2}(k-2)!} \mathbb{J}_{k-2} \cdots \mathbb{J}_1 \\ \quad [f(x) - (1 + (x-a)^2)^{k-1} f_{k-1}(x)], \\ f_{k-3}(x) = \frac{1}{4^{k-3}(k-3)!} \mathbb{J}_{k-3} \cdots \mathbb{J}_1 \\ \quad [f(x) - (1 + (x-a)^2)^{k-1} f_{k-1}(x) - (1 + (x-a)^2)^{k-2} f_{k-2}(x)], \\ \quad \vdots \\ f_1(x) = \frac{1}{4} \mathbb{J}_1 D_0^2 \\ \quad [f(x) - (1 + (x-a)^2)^{k-1} f_{k-1}(x) - \cdots - (1 + (x-a)^2)^2 f_2(x)], \\ f_0(x) = f(x) \\ \quad -[1 + (x-a)^2]^{k-1} f_{k-1}(x) - \cdots - [1 + (x-a)^2] f_1(x). \end{array} \right. \tag{10}$$

Conversely, if functions $f_0, \ldots, f_{k-1}$ satisfy the equation $D_0^2 f = 0$, then the function $f(x)$ given by (9) *is a solution to the equation $D_0^{2k} f = 0$.*

2.4. Almansi type decomposition IV

In this section, we obtain the Almansi type decomposition for the generalized iterated slice Dirac operator $D_{0,\lambda}^k$ by the generalized Euler operator.

Definition 2.9. We define the generalized slice Dirac operator by

$$D_{0,\lambda} = D_0 - \lambda,$$

where D_0 is the slice Dirac operator and λ is a complex number.

Definition 2.10. Let $\Omega \subset R^{m+1}$. The generalized Euler operator defined on the space $C^1(\Omega, Cl_{m+1})$ is given by

$$\mathbf{U}_\lambda = \lambda \mathbf{I}_\Omega + \sum_{i=0}^{m} x_i \partial_{x_i},$$

where λ is a complex number and $\mathbf{I}_\Omega$ is the identity operator on the space $C^1(\Omega, Cl_{m+1})$.

Lemma 2.11. *If $f(x) \in C^1(\Omega, Cl_{m+1})$ is a solution to the equation $D_{0,\lambda} f = 0$, then*

$$C_k D_{0,\lambda}^k \mathbf{U}_\lambda^k f = f, \tag{11}$$

where $C_k = \frac{1}{k!\lambda^k}$ and $k \in \mathbf{N}$.

Proof. Assume that f is a solution to the equation $D_{0,\lambda} f = 0$. Then for $k = 1$, it follows that

$$D_{0,\lambda}\mathbf{U}_\lambda f = (D_0 - \lambda)\mathbf{U}_\lambda f = D_0 \mathbf{U}_\lambda f - \lambda \mathbf{U}_\lambda f = \mathbf{U}_{\lambda+1} D_0 f - \lambda \mathbf{U}_\lambda f = \lambda f.$$

Suppose that for $k = l$, $C_l D_{0,\lambda}^l \mathbf{U}_\lambda^l f = f$, where $C_l = \frac{1}{l!\lambda^l}$. For $k = l+1$,

$$D_{0,\lambda}^{l+1}\mathbf{U}_\lambda^l f = D_{0,\lambda} D_{0,\lambda}^l \mathbf{U}_\lambda^l f = \frac{1}{C_l} D_{0,\lambda} f = 0.$$

Then it follows that

$$\begin{aligned}
D_{0,\lambda}^{l+1}\mathbf{U}_\lambda^{l+1} f &= D_{0,\lambda}^l D_{0,\lambda} \mathbf{U}_\lambda \mathbf{U}_\lambda^l f \\
&= D_{0,\lambda}^l (\mathbf{U}_{\lambda+1} D_{0,\lambda} + \lambda) \mathbf{U}_\lambda^l f \\
&= D_{0,\lambda}^{l-1} D_{0,\lambda} \mathbf{U}_{\lambda+1} D_{0,\lambda} \mathbf{U}_\lambda^l f + \frac{\lambda}{C_l} f \\
&= D_{0,\lambda}^{l-1} \mathbf{U}_{\lambda+2} D_{0,\lambda}^2 \mathbf{U}_\lambda^l f + \frac{2\lambda}{C_l} f \\
&= \cdots \\
&= \mathbf{U}_{\lambda+l+1} D_{0,\lambda}^{l+1} \mathbf{U}_\lambda^l f + \frac{(l+1)\lambda}{C_l} f = \frac{1}{C_{l+1}} f.
\end{aligned}$$

Therefore, we have (11) by induction. □

Theorem 2.12. *If $f(x) \in C^k(\Omega, Cl_{m+1})$ satisfies the equation $D_{0,\lambda}^k f = 0$, then there exist unique functions $f_0, \ldots, f_{k-1}$ such that*

$$f(x) = \sum_{i=0}^{k-1} \mathbf{U}_\lambda^i f_i(x), \quad x \in \Omega, \tag{12}$$

where $f_0, \dots, f_{k-1}$ *are solutions to the equation* $D_{0,\lambda} f = 0$ *and given by*

$$(13)\quad \begin{cases} f_0(x) = (\mathbf{I}_\Omega - C_1 \mathbf{U}_\lambda D_{0,\lambda}) \cdots \left(\mathbf{I}_\Omega - C_{k-1} \mathbf{U}_\lambda^{k-1} D_{0,\lambda}^{k-1}\right) f(x), \\ f_1(x) = C_1 D_{0,\lambda} \left(\mathbf{I}_\Omega - C_2 \mathbf{U}_\lambda^2 D_{0,\lambda}^2\right) \cdots \left(\mathbf{I}_\Omega - C_{k-1} \mathbf{U}_\lambda^{k-1} D_{0,\lambda}^{k-1}\right) f(x), \\ \quad \vdots \\ f_{k-2}(x) = C_{k-2} D_{0,\lambda}^{k-2} (\mathbf{I}_\Omega - C_{k-1} \mathbf{U}_\lambda^{k-1} D_{0,\lambda}^{k-1}) f(x), \\ f_{k-1}(x) = C_{k-1} D_{0,\lambda}^{k-1} f(x), \end{cases}$$

with $C_k = \frac{1}{k!\lambda^k}$.

Conversely, if functions $f_0, \dots, f_{k-1}$ *satisfy the equation* $D_{0,\lambda} f = 0$*, then the function* $f(x)$ *given by* (12) *is a solution to the equation* $D_{0,\lambda}^k f = 0$.

Proof. If we let the operator $D_{0,\lambda}^{k-1}$ act on Eq. (12), then it follows by Lemma 2.11 that

$$D_{0,\lambda}^{k-1} f(x) = D_{0,\lambda}^{k-1} \left(f_0(x) + \sum_{i=1}^{k-1} (\mathbf{U}_\lambda)^i f_i(x)\right) = D_{0,\lambda}^{k-1} \mathbf{U}_\lambda^{k-1} f_{k-1}(x) = \frac{f_{k-1}(x)}{C_{k-1}},$$

which implies that

$$f_{k-1}(x) = C_{k-1} D_{0,\lambda}^{k-1} f(x).$$

Similarly, if we let the operator $D_{0,\lambda}^{k-2}$ act on $f(x) - \mathbf{U}_\lambda^{k-1} f_{k-1}(x)$, then we have

$$f_{k-2}(x) = C_{k-2} D_{0,\lambda}^{k-2} (\mathbf{I} - C_{k-1} \mathbf{U}_\lambda^{k-1} D_{0,\lambda}^{k-1}) f(x).$$

Thus, we have (13) by induction.

Conversely, suppose that the functions $f_0, \dots, f_{k-1}$ satisfy the equation $D_{0,\lambda} f = 0$. It follows by Lemma 2.11 that

$$D_{0,\lambda}^k f(x) = D_{0,\lambda}^k \left[f_0(x) + \sum_{i=1}^{k-1} (\mathbf{U}_\lambda)^i f_i(x)\right] = 0,$$

which completes the proof. □

3. Riquier type problems for slice Dirac operators

In [14], the Riquier problem for polyharmonic equations is established. In [13], Karachik obtained a solution of the Riquier problem in harmonic analysis by using the 0-normalized system of functions with respect to the Laplace operator. In [19], the authors studied the Riquier problem in superspace by the 0-normalized system of functions with respect to the super Laplace operator. In this section, we investigate Riquier type problems in the theory of slice monogenic functions by another method.

3.1. Riquier type problem for slice Dirac operators

In this section, using Theorem 2.6, we consider the Riquier type problem for the iterated slice Dirac operator D_0^k as follows.

Given $\phi_i(y) \in C(\partial\Omega_a, Cl_{m+1})$, find a function f such that

$$D_0^i f \in C(\overline{\Omega_a}, Cl_{m+1})$$

for $i = 0, \ldots, k-1$, and

$$\begin{cases} D_0^k f = 0, \quad f \in C^k(\Omega_a, Cl_{m+1}), \\ D_0^i f|_{\partial\Omega} = \phi_i(y). \end{cases} \tag{14}$$

Theorem 3.1. *Suppose that $f_i(x),\ i = 0, \ldots, k-1$, satisfy the following equations*

$$\begin{cases} D_0 f_i(x) = 0, \quad f_i(x) \in C(\Omega_a, Cl_{m+1}), \\ f_i(x)|_{\partial\Omega_a} = \frac{1}{C_i}\mathbb{J}_1 \cdots \mathbb{J}_{\left[\frac{i+1}{2}\right]}\left(\phi_i(y) - \sum_{j=i+1}^{k-1} D_0^i[(x-a)^j f_j(x)]\right), \\ f_i(x) \in C^i(\overline{\Omega_a}, Cl_{m+1}),\ \mathbb{E}_0 = \mathbb{J}_0 = \mathbf{I}_{\Omega_a},\ C_i = (-2)^i \left[\frac{i}{2}\right]!. \end{cases} \tag{15}$$

Then the function $f(x) \in C^k(\Omega_a, Cl_{m+1})$ given by

$$f(x) = \sum_{i=0}^{k-1} (x-a)^i f_i(x) \tag{16}$$

is a solution of the problem (14).

Proof. Let $f(x) \in C^k(\Omega_a, Cl_{m+1})$. Because the functions $f_i(x)$ satisfy the equations $D_0 f_i(x) = 0$, it follows by Theorem 2.6 that

$$D_0^k f(x) = 0,$$

where $f(x)$ is given in (16).

For $i = 0$, using (15), we have

$$\begin{aligned} f(x)|_{\partial\Omega} &= f_0(x)|_{\partial\Omega} + \sum_{i=1}^{k-1}(x-a)^i f_i(x)|_{\partial\Omega} \\ &= \phi_0(y) - \sum_{i=1}^{k-1}(x-a)^i f_i(x)|_{\partial\Omega} + \sum_{i=1}^{k-1}(x-a)^i f_i(x)|_{\partial\Omega} = \phi_0(y). \end{aligned}$$

For $i = k-1$,

$$D_0^{k-1} f(x) = D_0^{k-1}\left(\sum_{i=0}^{k-1}(x-a)^i f_i(x)\right) = C_{k-1}\mathbb{E}_{\left[\frac{k}{2}\right]} \cdots \mathbb{E}_1 f_{k-1}(x). \tag{17}$$

For $0 < i < k-1$,

$$D_0^i f(x) = D_0^i\left[\sum_{j=0}^{k-1}(x-a)^j f_j(x)\right] \tag{18}$$
$$= C_i \mathbb{E}_{\left[\frac{i+1}{2}\right]}\cdots\mathbb{E}_1 f_i(x) + \sum_{j=i+1}^{k-1} D_0^i[(x-a)^j f_j(x)].$$

Letting $x \to \partial\Omega$, and using the second equality in (15), we have

$$D_0^i f|_{\partial\Omega} = \phi_i(y),\ i = 1, \dots, k-1.$$

Thus, we have the conclusion. □

3.2. Riquier type problem for generalized slice Dirac operators

In this section, we investigate the Riquier type problem for the generalized slice Dirac operator $D_{0,\lambda}^k$ by Theorem 2.12, as follows:

Assume $\varphi_i(y) \in C(\partial\Omega, Cl_{m+1})$. Find a function f such that

$$D_{0,\lambda}^i f \in C(\overline{\Omega}, Cl_{m+1})$$

for $i = 0, \dots, k-1$, and

$$\begin{cases} D_{0,\lambda}^k f = 0, \quad f \in C^k(\Omega, Cl_{m+1}), \\ D_{0,\lambda}^i f|_{\partial\Omega} = \varphi_i(y). \end{cases} \tag{19}$$

Theorem 3.2. *Suppose that $f_i(x),\ i = 0, \dots, k-1$, satisfy the following equations*

$$\begin{cases} D_{0,\lambda} f_i(x) = 0, \quad f_i(x) \in C^1(\Omega, Cl_{m+1}), \\ f_i(x)|_{\partial\Omega} = \dfrac{1}{i!\lambda^i}\left[\varphi_i(y) - \displaystyle\sum_{j=i+1}^{k-1} D_{0,\lambda}^i \mathbf{U}_\lambda^j f_j(x)|_{\partial\Omega}\right],\ i = 0, \dots, k-2, \\ f_i(x) \in C(\overline{\Omega}, Cl_{m+1}), \quad D_{0,\lambda}^i \mathbf{U}_\lambda^j f_j(x) \in C(\overline{\Omega}, Cl_{m+1}), \\ f_{k-1}(x)|_{\partial\Omega} = \dfrac{1}{(k-1)!\lambda^{k-1}}\varphi_{k-1}(y), \quad f_{k-1}(x) \in C(\overline{\Omega}, Cl_{m+1}). \end{cases} \tag{20}$$

Then the function $f(x)$ given by

$$f(x) = \sum_{i=0}^{k-1} \mathbf{U}_\lambda^i f_i(x) \tag{21}$$

is a solution of the problem (19).

Proof. The proof is similar to that of Theorem 3.1. We have the result by Theorem 2.12. □

4. Dirichlet type problems for null solutions to the iterated slice Dirac operator

In mathematics, a Dirichlet problem for Laplace's equation can be stated as follows: Given a function f that has values everywhere on the boundary of a region in $\mathbb{R}^n$, there is a unique function u twice continuously differentiable in the interior and continuous on the boundary, such that u is harmonic in the interior and $u = f$ on the boundary (see [9]). The Dirichlet problem is named after Peter Gustav Lejeune Dirichlet, which can be investigated for many PDEs, although originally it was posed for Laplace's equation. In [16], the author investigated Dirichlet type problems for Dunkl-Poisson equations. In this section, we study Dirichlet type problems for iterated slice Dirac equations.

4.1. Dirichlet type problem I

In this section, we consider the homogeneous boundary value problem for the inhomogeneous iterated slice Dirac equation in the ball as follows: Let B be the ball with center a and radius 1. Suppose that $f(x)$ is a solution to the equation $D_0^2 f = 0$. Find a function $u(x)$ satisfying

$$\left\{ \begin{array}{l} D_0^{2k} u(x) = 0, \quad x \in B, \\ u(x) \mid_{\partial B} = 0, \quad x \in \partial B, \end{array} \right. \tag{22}$$

where $\partial B = \{x \| x - a|^2 = 1,\ x \in R^{m+1}\}$.

Theorem 4.1. *Suppose that $f(x)$ is a solution to the equation $D_0^2 f = 0$. Then Problem* (22) *has a solution.*

Proof. Let

$$u(x) = [1 + (x - a)^2]^{k-1} f(x),$$

where $f(x)$ is a solution to the equation $D_0^2 f = 0$. Then

$$D_0^{2k}[1 + (x - a)^2]^{k-1} f(x) = 0$$

by Theorem 2.8. It is easy to see that $[1 + (x - a)^2]^{k-1} f(x) \mid_{\partial B} = 0$, which implies that $u(x)$ is a solution of Problem (22). □

4.2. Dirichlet type problem II

In this section, we derive a solution of the following homogeneous boundary value problem for the inhomogeneous iterated slice Dirac equation in the ball: Let B be the ball with center a and radius 1. Suppose that $f(x)$ is a solution to the equation $D_0^2 f = 0$. Find a function $u(x)$ satisfying

$$\left\{ \begin{array}{l} D_0^{2k} u(x) = f, \quad x \in B, \\ u(x) \mid_{\partial B} = 0, \quad x \in \partial B. \end{array} \right. \tag{23}$$

Theorem 4.2. *Suppose that $f(x)$ is a solution to the equation $D_0^2 f(x) = 0$. Then Problem* (23) *has a solution*

$$u(x) = \frac{1}{4^k k!}[1 + (x - a)^2]^k \mathbb{J}_k \cdots \mathbb{J}_1 f(x).$$

Proof. Let

$$u(x) = \frac{1}{4^k k!}[1 + (x - a)^2]^k \mathbb{J}_k \cdots \mathbb{J}_1 f(x).$$

Because $f(x)$ is a solution to the equation $D_0^2 f = 0$, it follows by Lemma 2.5 that the functions $\mathbb{J}_k \cdots \mathbb{J}_1 f(x)$ satisfy the equation $D_0^2 f = 0$. Applying Lemma 2.4, we find that

$$\begin{aligned} D_0^{2k} u(x) &= D_0^{2k} \left[\frac{1}{4^k k!}(1 + x^2)^k \mathbb{J}_k \cdots \mathbb{J}_1 f(x) \right] \\ &= \frac{1}{4^k k!} 4^k k! \mathbb{E}_1 \cdots \mathbb{E}_k \mathbb{J}_k \cdots \mathbb{J}_1 f(x) \\ &= f(x). \end{aligned}$$

Therefore, the function $u(x)$ is a solution of Problem (23). □

4.3. Dirichlet type problem III

In this section, we study the inhomogeneous boundary value problem for the inhomogeneous iterated slice Dirac equation in the ball as follows: Let B be the ball with center a and radius 1. Suppose that $f(x)$ is a solution to the equation $D_0^2 f = 0$. Find a function $u(x)$ satisfying

$$\left\{ \begin{array}{ll} D_0^{2k} u(x) = 0, & x \in B, \\ u(x) \mid_{\partial B} = P(x), & x \in \partial B, \end{array} \right. \tag{24}$$

where $\partial B = \{x \mid |x - a|^2 = 1,\ x \in R^{m+1}\}$ and $P(x)$ is a homogeneous polynomial of degree $2k - 1$.

Theorem 4.3. *Suppose that $f(x)$ is a solution to the equation $D_0^2 f(x) = 0$. Then Problem (24) has a solution*

$$u(x) = P(x) + [1 + (x - a)^2]^{k-1} f(x),$$

where $P(x)$ is a homogeneous polynomial of degree $2k - 1$.

Proof. Applying Theorems 2.7 and 4.2, we have the conclusion. □

5. Conclusions and future research

In this paper we have explicitly constructed Almansi-type decompositions in slice Clifford analysis. Furthermore, applying these decompositions, we investigate Riquier type problems and Dirichlet type problems. Therefore, Almansi decompositions are of great importance when studying boundary value problems in slice Clifford analysis. Based on Almansi-type decompositions, we will consider Riemann boundary value problems for null solutions to the iterated slice Dirac operator in a subsequent paper.

References

[1] E. Almansi, *Sullintegrazione dellequazione differenziale* $\Delta^{2m}u = 0$, Annali di Mat. **2** (1899), no. 3, 1–51.
[2] N. Aronszajn, T. M. Creese, and L. J. Lipkin, *Polyharmonic functions*, Oxford Mathematics Monographs, The Clarendon Press, Oxford University Press, New York, 1983.
[3] F. Brackx, R. Delanghe, and F. Sommen, *Clifford Analysis*, Pitman Research Notes in Mathematics Series, vol. 76. Pitman, Massachusetts, 1982.
[4] L. Cnudde, H. De Bie, and G. B. Ren, *Algebraic approach to Slice Monogenic functions*, Complex Anal. Oper. Theory **9** (2015), no. 5, 1065–1087.
[5] F. Colombo, I. Sabadini, and D. C. Struppa, *Slice monogenic functions*, Isr. J. Math. **171** (2009), 385–403.
[6] ———, *An extension theorem for slice monogenic functions and some of its consequences*, Israel J. Math. **177** (2010), 369–389.
[7] ———, *The Runge theorem for slice hyperholomorphic functions*, Proc. Amer. Math. Soc. **139** (2011), no. 5, 1787–1803.
[8] H. De Bie, B. Orsted, P. Somberg, and V. Soucek, *Dunkl operators and a family of realizations of osp*(1|2), Trans. Amer. Math. Soc. **364** (2012), no. 7, 3875–3902.
[9] P. G. L. Dirichlet, *Abh. Königlich*, Preuss. Akad. Wiss. (1850), 99–116.
[10] G. Gentili, I. Sabadini, M. Shapiro, F. Sommen, and D. C. Struppa, *Advances in Hypercomplex Analysis*, Springer INdAM Series, vol. 1. Springer, Milan, 2013.
[11] G. Gentili and D. C. Struppa, *A new theory of regular functions of a quaternionic variable*, Adv. Math. **216** (2007), no. 1, 279–301.
[12] F. L. He, M. Ku, U. Kähler, F. Sommen, and S. Bernstein, *Riemann-Hilbert problems for null-solutions to iterated generalized Cauchy-Riemann equations in axially symmetric domains*, Computers and Mathematics with Applications **71** (2016), no. 10, 1990–2000.
[13] V. V. Karachik, *Normalized system of functions with respect to the Laplace operator and its applications*, J. Math. Anal. Appl. **287** (2003), no. 2, 577–592.
[14] M. Nicolescu, *Les Fonctions Polyharmoniques*, Hermann, Paris, 1936.
[15] H. Malonek and G. B. Ren, *Almansi-type theorems in Clifford analysis*, Math. Methods Appl. Sci. **25** (2002), no. 16-18, 1541–1552.
[16] H. F. Yuan, *Dirichlet type problems for Dunkl-Poisson equations*, Bound. Value Probl. **2016** (2016), no. 222, 1–16.
[17] ———, *Solutions of the Poisson equation and related equations in super spinor space*, Comput. Methods Funct. Theory **16** (2016), no. 4, 699–715.
[18] H. F. Yuan and Y. Y. Qiao, *Solutions of the Dirac and related equations in superspace*, Complex Var. Elliptic Equ. **59** (2014), no. 9, 1315–1327.
[19] H. F. Yuan, Y. Y. Qiao, and H. J. Yang, *Normalized system for the super Laplace operator*, Adv. Appl. Clifford Algebras **22** (2012), no. 4, 1109–1128.

Hongfen Yuan
School of Mathematics and Physics
Hebei University of Engineering
Handan 056038, P. R. China
E-mail address: yhf0609@163.com