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FOURIER SERIES OF HIGHER-ORDER EULER FUNCTIONS AND THEIR APPLICATIONS

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Abstract. In this paper, we give some identities for the higher-order Euler functions arising from the Fourier series of them. In addition, we investigate some formulae related to Bernoulli functions which are derived from our identities.

1. Introduction

As is well known, the Euler polynomials are defined by the generating function

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\quad \text{(see [1-20])}. \tag{1.1}$$

When $x=0$, $E_n=E_n(0)$ are called Euler numbers. For any real x, we define

$$\langle x\rangle = x-[x]\in[0,1). \tag{1.2}$$

Note that $\langle x\rangle$ is the fractional part of x. Then $E_m(\langle x\rangle)$ are functions defined on $(-\infty,\infty)$ and periodic with period 1, which are called Euler functions. For $m\in\mathbb{N}$, the Fourier series of $E_m(\langle x\rangle)$ is given by

$$E_m(\langle x\rangle)=\sum_{n=-\infty}^{\infty}a_n^{(m)}e^{(2n+1)\pi ix},\quad (a_n^{(m)}\in\mathbb{C}),\quad \text{(see [9, 10, 13])}. \tag{1.3}$$

where

$$a_n^{(m)}=\int_0^1E_m(x)e^{-(2n+1)\pi ix}dx,\quad (i=\sqrt{-1}). \tag{1.4}$$

From (1.4), we note that

(1.5)

$$a_n^{(m)}=\frac{m}{(2n+1)\pi i}a_n^{(m-1)}=\frac{m(m-1)}{\big((2n+1)\pi i\big)^2}a_n^{(m-2)}=\cdots=\frac{m!a_n^{(1)}}{\big((2n+1)\pi i\big)^{m-1}}.$$

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Thus, by (1.5), we get

$$a_n^{(m)} = 2\frac{m!}{\big((2n+1)\pi i\big)^{m+1}}, \ (m \in \mathbb{N}), \quad (\text{see } [9, 10]). \tag{1.6}$$

So, from (1.3) and (1.6), we have

$$E_m(\langle x \rangle) = 2m! \sum_{n=-\infty}^{\infty} \frac{e^{(2n+1)\pi i x}}{\big((2n+1)\pi i\big)^{m+1}}, \quad (\text{see } [9, 10]). \tag{1.7}$$

by (1.7), we get

$$E_m = 2m! \sum_{n=-\infty}^{\infty} \frac{1}{\big((2n+1)\pi i\big)^{m+1}}, \quad (m \in \mathbb{N} \cup \{0\}). \tag{1.8}$$

Thus, from (1.8), we have

$$E_{2m+1} = (-1)^{m+1} 2 \cdot \frac{(2m+1)!}{\pi^{2m+2}} \sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)^{2m+2}}, \quad (\text{see } [9, 10, 13]). \tag{1.9}$$

By (1.9), we get

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2m+2}} = (-1)^{m+1} \frac{E_{2m+1}}{4(2m+1)!} \pi^{2m+2}, \quad (\text{see } [9, 10]). \tag{1.10}$$

For $r \in \mathbb{N}$, the higher-order Euler polynomials are defined by the generating function

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see } [9, 10, 13, 14, 17, 19]). \tag{1.11}$$

When $x = 0$, $E_n^{(r)} = E_n^{(r)}(0)$ are called the higher-order Euler numbers (see [17, 19]). For any real number x, $E_m^{(r)}(\langle x \rangle)$ are functions defined on $(-\infty, \infty)$ and periodic with period 1, which are called Euler functions of order r. In this paper, we give some new identities of the higher-order Euler functions which are derived from the Fourier series of $E_n^{(r)}(\langle x \rangle)$. In addition, we investigate some formulae related to Bernoulli functions.

2. Fourier series of higher-order Euler functions

From (1.11), we note that

$$E_m^{(r)}(x+1) + E_m^{(r)}(x) = 2E_m^{(r-1)}(x), \quad (m \geq 0). \tag{2.1}$$

Indeed

$$\begin{aligned}\sum_{m=0}^{\infty} E_m^{(r)}(x+1)\frac{t^m}{m!} &= \left(\frac{2}{e^t+1}\right)^r e^{(x+1)t} = \left(\frac{2}{e^t+1}\right)^r (e^t+1-1)e^{xt}\\ &= 2\left(\frac{2}{e^t+1}\right)^{r-1} e^{xt} - \left(\frac{2}{e^t+1}\right)^r e^{xt}\\ &= \sum_{m=0}^{\infty}\left(2E_m^{(r-1)}(x) - E_m^{(r)}(x)\right)\frac{t^m}{m!}.\end{aligned} \tag{2.2}$$

For $x = 0$ in (2.1), we have

$$E_m^{(r)}(1) = 2E_m^{(r-1)}(0) - E_m^{(r)}(0), \quad (m \geq 0). \tag{2.3}$$

Thus, by (2.3), we get

$$E_m^{(r)}(0) = E_m^{(r)}(1) \Leftrightarrow E_m^{(r)}(0) = E_m^{(r-1)}(0). \tag{2.4}$$

Assume that $m \geq 1$ and $r \geq 1$. Then $E_m^{(r)}(\langle x\rangle)$ is piecewise C^∞. In addition, $E_m^{(r)}(\langle x\rangle)$ is continuous for those (r, m) with $E_m^{(r)}(0) = E_m^{(r-1)}(0)$, and discontinuous with jump discontinuities at integers for those (r, m) with $E_m^{(r)}(0) \neq E_m^{(r-1)}(0)$. The Fourier series of $E_m^{(r)}(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(r,m)} e^{2\pi inx}, \tag{2.5}$$

where

$$\begin{aligned}C_n^{(r,m)} &= \int_0^1 E_m^{(r)}(\langle x\rangle)e^{-2\pi inx}dx = \int_0^1 E_m^{(r)}(x)e^{-2\pi inx}dx\\ &= \frac{1}{m+1}\left[E_{m+1}^{(r)}(x)e^{-2\pi inx}\right]_0^1 + \frac{2\pi in}{m+1}\int_0^1 E_{m+1}^{(r)}(x)e^{-2\pi inx}dx\\ &= \frac{1}{m+1}\left[E_{m+1}^{(r)}(1) - E_{m+1}^{(r)}(0)\right] + \frac{2\pi in}{m+1}C_n^{(r,m+1)}\\ &= \frac{2}{m+1}\left(E_{m+1}^{(r-1)}(0) - E_{m+1}^{(r)}(0)\right) + \frac{2\pi in}{m+1}C_n^{(r,m+1)}.\end{aligned} \tag{2.6}$$

Replacing m by $m-1$ in (2.6), we have

$$\frac{2\pi in}{m}C_n^{(r,m)} = C_n^{(r,m-1)} + \frac{2}{m}\left(E_m^{(r)}(0) - E_m^{(r-1)}(0)\right). \tag{2.7}$$

Case 1. Let $n \neq 0$. Then we have

$$\begin{aligned}C_n^{(r,m)} &= \frac{m}{2\pi in}C_n^{(r,m-1)} + \frac{1}{\pi in}\left(E_m^{(r)}(0) - E_m^{(r-1)}(0)\right)\\ &= \frac{m}{2\pi in}\left(\frac{m-1}{2\pi in}C_n^{(r,m-2)} + \frac{1}{\pi in}\left(E_{m-1}^{(r)}(0) - E_{m-1}^{(r-1)}(0)\right)\right)\\ &\quad + \frac{1}{\pi in}\left(E_m^{(r)}(0) - E_m^{(r-1)}(0)\right)\end{aligned} \tag{2.8}$$

$$
\begin{aligned}
&= \frac{m(m-1)}{(2\pi in)^2} C_n^{(r,m-2)} + \frac{m}{2}\frac{1}{(\pi in)^2}\left(E_{m-1}^{(r)}(0) - E_{m-1}^{(r-1)}(0)\right) \\
&\quad + \frac{1}{\pi in}\left(E_m^{(r)}(0) - E_m^{(r-1)}(0)\right) \\
&= \frac{m(m-1)}{(2\pi in)^2}\left(\frac{m-2}{2\pi in} C_n^{(r,m-3)} + \frac{1}{\pi in}\left(E_{m-2}^{(r)}(0) - E_{m-2}^{(r-1)}(0)\right)\right) \\
&\quad + \frac{m}{2}\frac{1}{(\pi in)^2}\left(E_{m-1}^{(r)}(0) - E_{m-1}^{(r-1)}(0)\right) + \frac{1}{\pi in}\left(E_m^{(r)}(0) - E_m^{(r-1)}(0)\right) \\
&= \frac{m(m-1)(m-2)}{(2\pi in)^3} C_n^{(r,m-3)} \\
&\quad + \frac{m(m-1)}{2^2}\frac{1}{(\pi in)^3}\left(E_{m-2}^{(r)}(0) - E_{m-2}^{(r-1)}(0)\right) \\
&\quad + \frac{m}{2}\frac{1}{(\pi in)^2}\left(E_{m-1}^{(r)}(0) - E_{m-1}^{(r-1)}(0)\right) + \frac{1}{\pi in}\left(E_m^{(r)}(0) - E_m^{(r-1)}(0)\right) \\
&= \cdots \\
&= \frac{m!}{(2\pi in)^{m-1}} C_n^{(r,1)} + \sum_{k=1}^{m-1} \frac{(m)_{k-1}}{2^{k-1}}\frac{1}{(\pi in)^k}\left(E_{m-k+1}^{(r)}(0) - E_{m-k+1}^{(r-1)}(0)\right),
\end{aligned}
$$

where $(x)_n = x(x-1)\cdots(x-n+1)$, for $n \geq 1$, and $(x)_0 = 1$. Here we note that

$$
\begin{aligned}
C_n^{(r,1)} &= \int_0^1 E_1^{(r)}(x) e^{-2\pi inx} dx = \int_0^1 \left(x + E_1^{(r)}\right) e^{-2\pi inx} dx \\
&= \int_0^1 x e^{-2\pi inx} dx + E_1^{(r)} \int_0^1 e^{-2\pi inx} dx \\
&= -\frac{1}{2\pi in}\left[x e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi inx}\int_0^1 e^{-2\pi inx} dx \\
&= -\frac{1}{2\pi in}.
\end{aligned} \tag{2.9}
$$

Thus, by (2.8) and (2.9), we get

$$
\begin{aligned}
C_n^{(r,n)} &= \frac{-m!}{(2\pi in)^m} + \sum_{k=1}^{m-1} \frac{2(m)_{k-1}}{(2\pi in)^k}\left(E_{m-k+1}^{(r)}(0) - E_{m-k+1}^{(r-1)}(0)\right) \\
&= \sum_{k=1}^{m} \frac{2(m)_{k-1}}{(2\pi in)^k}\left(E_{m-k+1}^{(r)}(0) - E_{m-k+1}^{(r-1)}(0)\right).
\end{aligned} \tag{2.10}
$$

Here we used the fact that $E_n^{(r)}(0) = \sum_{l_1+\cdots+l_r=n} \binom{n}{l_1,\cdots,l_r} E_{l_1}(0)\cdots E_{l_r}(0)$.

Case 2. Let $n = 0$. Then, we note that

$$
C_0^{(r,m)} = \int_0^1 E_m^{(r)}(x) dx = \frac{1}{m+1}\left[E_{m+1}^{(r)}(x)\right]_0^1 \tag{2.11}
$$

$$= \frac{1}{m+1}\left(E_{m+1}^{(r)}(1) - E_{m+1}^{(r)}(0)\right)$$
$$= \frac{2}{m+1}\left(E_{m+1}^{(r-1)}(0) - E_{m+1}^{(r)}(0)\right).$$

Assume first that $E_m^{(r)}(0) = E_m^{(r-1)}(0)$. Then $E_m^{(r)}(1) = E_m^{(r)}(0)$ and $m \geq 2$. Note that $E_m^{(r)}(\langle x\rangle)$ is piecewise C^∞ and continuous. Hence the Fourier series of $E_m^{(r)}(\langle x\rangle)$ converges uniformly to $E_m^{(r)}(\langle x\rangle)$, and

(2.12)
$$E_m^{(r)}(\langle x\rangle) = \frac{2}{m+1}\left(E_{m+1}^{(r-1)}(0) - E_{m+1}^{(r)}(0)\right) + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(\sum_{k=1}^{m}\frac{2(m)_{k-1}}{(2\pi i n)^k}\left(E_{m-k+1}^{(r)}(0) - E_{m-k+1}^{(r-1)}(0)\right)\right)e^{2\pi i n x}.$$

Before proceedings further, we recall the following facts about Bernoulli functions $B_n(\langle x\rangle)$:

$$B_m(\langle x\rangle) = -m!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^m},\ (m\geq 2),\quad (\text{see } [1,\ 18]), \tag{2.13}$$

and

$$-\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} B_1(\langle x\rangle) & \text{for } x \notin \mathbb{Z}\\ 0 & \text{for } x\in\mathbb{Z},\end{cases}\quad (\text{see } [1,\ 18]). \tag{2.14}$$

The series in (2.13) converges uniformly, but that in (2.3) converges only pointwise. We note that (2.12) can be rewritten as

$$\begin{aligned}
E_m^{(r)}(\langle x\rangle) &= \frac{2}{m+1}\left(E_{m+1}^{(r-1)}(0) - E_{m+1}^{(r)}(0)\right)\\
&\quad + \sum_{k=1}^{m}\frac{2(m)_{k-1}}{k!}\left(E_{m-k+1}^{(r-1)}(0) - E_{m-k+1}^{(r)}(0)\right)\cdot\left(-k!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^k}\right)\\
&= \frac{2}{m+1}\left(E_{m+1}^{(r-1)}(0) - E_{m+1}^{(r)}(0)\right)\\
&\quad + \sum_{k=2}^{m}\frac{2(m)_{k-1}}{k!}\left(E_{m-k+1}^{(r-1)}(0) - E_{m-k+1}^{(r)}(0)\right)B_k(\langle x\rangle) \qquad (2.15)\\
&\quad + 2\left(E_m^{(r-1)}(0) - E_m^{(r)}(0)\right)\times\begin{cases} B_1(\langle x\rangle) & \text{for } x\notin\mathbb{Z}\\ 0 & \text{for } x\in\mathbb{Z}.\end{cases}
\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 2.1. *Let $m \geq 2$, $r \geq 1$. Assume that $E_m^{(r)}(0) = E_m^{(r-1)}(0)$,*

(a) *$E_m^{(r)}(\langle x \rangle)$ has the Fourier series expansion*

$$E_m^{(r)}(\langle x \rangle) = \frac{2}{m+1}\left(E_{m+1}^{(r-1)}(0) - E_{m+1}^{(r)}(0)\right) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty}\left(\sum_{k=1}^{m}\frac{2(m)_{k-1}}{(2\pi in)^k}\left(E_{m-k+1}^{(r)}(0) - E_{m-k+1}^{(r-1)}(0)\right)\right)e^{2\pi inx}$$

for all $x \in (-\infty, \infty)$, where the convergence is uniform.

(b) $E_n^{(r)}(\langle x \rangle) = \dfrac{2}{m+1}\left(E_{m+1}^{(r-1)}(0) - E_{m+1}^{(r)}(0)\right)$

$$+ \sum_{k=2}^{m}\frac{2(m)_{k-1}}{k!}\left(E_{m-k++1}^{(r-1)}(0) - E_{m-k+1}^{(r)}(0)\right)B_k(\langle x \rangle)$$

for all $x \in (-\infty, \infty)$, where $B_k(\langle x \rangle)$ is the Bernoulli function.

Assume next that $E_m^{(r)}(0) \neq E_m^{(r-1)}(0)$. Then we note that $E_m^{(r)}(1) \neq E_m^{(r)}(0)$, and hence $E_m^{(r)}(\langle x \rangle)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. Thus the Fourier series of $E_m^{(r)}(\langle x \rangle)$ converges pointwise to $E_m^{(r)}(\langle x \rangle)$, for $x \notin \mathbb{Z}$, and converges to $\frac{1}{2}\left(E_m^{(r)}(0) + E_m^{(r)}(1)\right) = E_m^{(r-1)}(0)$, for $x \in \mathbb{Z}$. Thus, we obtain the following theorem.

Theorem 2.2. *Let $m \geq 1$, $r \geq 1$. Assume that $E_m^{(r)}(1) \neq E_m^{(r-1)}(0)$.*

(a)
$$\frac{2}{m+1}\left(E_{m+1}^{(r-1)}(0) - E_{m+1}^{(r)}(0)\right) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty}\left(\sum_{k=1}^{m}\frac{2(m)_{k-1}}{(2\pi in)^k}\left(E_{m-k+1}^{(r)}(0) - E_{m-k+1}^{(r-1)}(0)\right)\right)e^{2\pi inx}$$
$$= \begin{cases} E_m^{(r)}(\langle x \rangle) & \text{for } x \notin \mathbb{Z} \\ E_m^{(r-1)}(0) & \text{for } x \in \mathbb{Z}. \end{cases}$$

Here the convergence is pointwise.

(b)
$$\frac{2}{m+1}\left(E_{m+1}^{(r-1)}(0) - E_{m+1}^{(r)}(0)\right) + \sum_{k=1}^{m}\frac{2(m)_{k-1}}{k!}\left(E_{m-k+1}^{(r-1)}(0) - E_{m-k+1}^{(r)}(0)\right)B_k(\langle x \rangle)$$
$$= E_m^{(r)}(\langle x \rangle) \quad \text{for } x \notin \mathbb{Z},$$

and

$$\frac{2}{m+1}\left(E_{m+1}^{(r-1)}(0) - E_{m+1}^{(r)}(0)\right) + \sum_{k=2}^{m}\frac{2(m)_{k-1}}{k!}\left(E_{m-k+1}^{(r-1)}(0) - E_{m-k+1}^{(r)}(0)\right)B_k(\langle x \rangle)$$

$$= E_m^{(r-1)}(0) \quad \textit{for } x \in \mathbb{Z}.$$

Here $B_k(\langle x \rangle)$ is the Bernoulli function.

Remark. Note that

$$\begin{aligned}
\frac{1}{1+e^{-x}} &= \sum_{n=0}^{\infty} e^{-nx}(-1)^n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}x^k\right)^n(-1)^n \\
&= \sum_{n=0}^{\infty}(-1)^n\left(\sum_{a_1+a_2+\cdots=n}\frac{n!}{a_1!a_2!\cdots}\cdot\frac{(-1)^{a_1+2a_2+\cdots}}{(1!)^{a_1}(2!)^{a_2}\cdots}\right)x^{a_1+2a_2+\cdots}
\end{aligned}$$

Let $P(i,j) : a_1+2a_2+\cdots = i,\ a_1+a_2+\cdots = j$.Then

$$\begin{aligned}
\frac{1}{1+e^{-x}} &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}(-1)^n \sum_{P(m,n)}\frac{n!(-1)^m x^m}{a_1!a_2!\cdots a_m!(1!)^{a_1}\cdots(m!)^{a_m}}\right) \\
&= \sum_{m=0}^{\infty}(-1)^m\sum_{n=0}^{m}n!(-1)^n S_2(m,n)\frac{x^m}{m!}, \quad \text{(see [13, 9])},
\end{aligned}$$

where $S_2(m,n)$ is the stirling number of the second kind. By the definition of Euler number, we get

$$\frac{1}{1+e^{-x}} = \frac{1}{2}\left(\frac{2}{1+e^{-x}}\right) = \frac{1}{2}\sum_{m=0}^{\infty}(-1)^m E_m\frac{x^m}{m!}.$$

Thus, we see

$$E_m = 2\sum_{n=0}^{m}(-1)^n n! S_2(m,n), \quad \text{(see [9, 13])}.$$

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