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DISTRIBUTIVE LATTICE POLYMORPHISMS ON REFLEXIVE GRAPHS

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Abstract. In this paper we give two characterisations of the class of reflexive graphs admitting *distributive lattice polymorphisms* and use these characterisations to address the problem of recognition: we find a polynomial time algorithm to decide if a given reflexive graph G, in which no two vertices have the same neighbourhood, admits a distributive lattice polymorphism.

1. Introduction

1.1. Motivation

It is well known (see for example [3]) that the Constraint Satisfaction Problem, a framework for many combinatorial problems, can be stated as a problem of finding a homomorphism between relational structures. Moreover ([9]) any such homomorphism problem can be reduced to a retraction problem for a *reflexive* graph, in which every vertex has a loop.

The problem $\mathrm{Ret}(G)$ of retraction to G is NP-complete for most reflexive graphs G and in the cases when the problem is known to be polynomial time solvable, the polynomial time algorithm is tied to the existence of a *polymorphism* on G (an edge preserving vertex map from G^d to G for some d) which satisfies some particular identity.

A reflexive graph G is a *lattice* graph if there exists a *compatible lattice*, a lattice on its vertex set such that the meet and join operations, $\wedge$ and $\vee$, are polymorphisms of G. It is a *distributive lattice graph* or DL graph if there exists a distributive such lattice L, in which case we call (G, L) a DL *pair*.

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It was shown in [4] that $\mathrm{Ret}(G)$ can be solved, for a relational structure G, by a linear monadic Datalog program with at most one extensional predicate per rule, if and only if G is a retract of a DL graph.

To give a bit more context, a d-ary polymorphism $f : G^d \to G$ is a *totally symmetric idempotent* (TSI) polymorphism if $f(v, v, \ldots, v) = v$ for all $v \in V(G)$ and if $f(v_1, \ldots, v_d) = f(u_1, \ldots, u_d)$ whenever $\{v_1, \ldots, v_d\} = \{u_1, \ldots, u_d\}$. The class TSI of reflexive graphs G admitting TSI polymorphisms of all arities is important, as this is the class for which $\mathrm{Ret}(G)$ can be solved by a monadic Datalog program with at most one extensional predicate per rule.

It is of interest to get a graph theoretic characterisation of the class TSI. The two main sources of TSI polymorphisms are near-unanimity (NU) polymorphisms and semilattice (SL) polymorphisms. While NU polymorphisms have been well studied, and the classes of graphs admitting them have several nice characterisations, no such study had been attempted for SL polymorphisms until [14].

A (meet) semilattice ordering on the vertices of a graph defines a 2-ary operation $\vee$ on the vertex set. If this operation is a polymorphism of the graph, then it is an SL *polymorphism* of the graph. The problem of characterising reflexive graphs admitting SL polymorphisms was difficult, and we restricted our attention to those graphs G that admit SL polymorphisms for which the Hasse diagram of the semilattice ordering both a tree and a subgraph of G. We showed that the class of such graphs extends the class of chordal graphs. We were unable to say much in the case when the semilattice is not a tree. The other extreme is when the ordering is a lattice. In this case, the Hasse diagram is not a tree except when the lattice is a chain. This leads us to consider lattice polymorphisms.

1.2. Results

In this paper we give two explicit characterisations of the class of reflexive DL graphs, and use these characterisations to address the problem of recognition.

For our first characterisation, we recall a well known result of Birkhoff [2]. For a poset P, a subset D is a *downset* if $b \in D$ and $a \leq b$ imply $a \in D$. The family $\mathcal{D}(P)$ of all downsets of P is a distributive lattice under the ordering $\subseteq$. The meet and join operations are $\cap$ and $\cup$, respectively. Birkhoff showed that any distributive lattice L is isomorphic to $\mathcal{D}(J_L)$ for the poset J_L of join irreducible elements of L.

Viewing a comparability $a \leq b$ as an arc (a, b), a poset P is just a transitive acyclic (except for loops) reflexive digraph. So we can talk of a sub-digraph A of P. In doing so, we will always use $a \leq b$ to mean $(a, b) \in P$, and $a \to b$ to mean $(a, b) \in A$.

Definition 1.1 ($G(P, A)$)**.** For a poset P and a sub-digraph A of P, let $G(P, A)$ be the graph on $\mathcal{D}(P)$, in which two downsets $D, D' \in \mathcal{D}(P)$ are adjacent if A

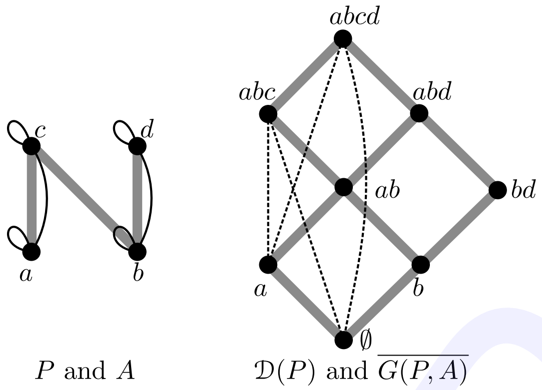


FIGURE 1. Poset P and lattice $\mathcal{D}(P)$ in thick light edges. Digraph A and (the complement of) graph $G(P,A)$ in dark.

contains all arcs (x,y) of P for which both x and y are in $D-D'$, or both are in $D'-D$.

See Figure 1 for an example. The left side shows a poset P represented by its Hasse diagram (defined in Section 2) in thick light edges, and a sub-digraph A in dark edges, missing only the arc (b,c). On the right is the downset lattice $\mathcal{D}(P)$ again represented by its Hasse diagram, and the graph $G(P,A)$ missing only edges between vertices one of which contains b and c and the other of which contains neither of them.

Our first main result is the following characterisation of reflexive DL graphs.

Theorem 1.2. *A reflexive graph G is a* DL *graph if and only if there is a poset P with a sub-digraph A such that $G \cong G(P,A)$.*

We prove this in Section 4 using a somewhat technical theorem, Theorem 4.3, which says that for any DL pair (G,L), G is isomorphic to $G(J_L,A)$ for some sub-digraph A of J_L, and using Lemma 4.4, which says that $G(P,A)$ is always a DL graph.

We frequently use the universal algebraic notion of an identity. An *identity* is a statement using variables, and such relations as graph adjacency, poset comparability, equality, and logical implication. A graph (or poset, or both) is said to *satisfy* the identity if the statement is true for all assignments of vertices of the graph (elements of the poset) to the variables. Many of our identities will involve both a graph and a poset on the same set of vertices; in this case we will say that the graph satisfies the identity under, or with respect to, the poset.

In [11] it was shown that a graph (without loops) is a *proper interval graph* if and only if there is a total ordering $\leq$ of its vertices under which the graph satisfies the so-called *min-max* identity:

$$(u' \leq u \leq v \leq v' \text{ and } u' \sim v') \Rightarrow u \sim v. \tag{1}$$

We take this as our definition of a proper interval graph in the reflexive context, and say that a proper interval graph is in *min-max form* if its vertices are labeled $\{0, 1, \ldots, n\}$ for some n so that it satisfies (1) under the natural induced ordering.

Simple arguments (see Fact 2.1) show that any reflexive graph that is compatible with a chain lattice is a proper interval graph. As any distributive lattice is embeddable in a product of chains, it follows that any DL graph is a subgraph of a categorical product (defined in Section 2) of proper interval graphs. In fact Dilworth [6] showed, and we recall this in more detail in Section 5, that any chain decomposition of P yields an embedding of $\mathcal{D}(P)$ into a product of chains. The embedding shown in Figure 1 comes from the decomposition of P into the chains $a \prec c$ and $b \prec d$. Figure 2 shows the embedding corresponding to the decomposition of P into the three chains a, d, and $b \prec c$. For the embedding in Figure 2 the graph $G(P, A)$ is an induced subgraph of proper interval graphs, in this case paths, on the chain factors. It turns out that this happens when certain edges in the complement of A in P are contained in the chain decomposition of P.

For any DL pair (G, L), we get, in Theorem 5.1, an embedding of L into a product of chains such that G is an induced subgraph of a corresponding product of proper interval graphs. Further, it is an induced subgraph of quite a particular form.

The vertex set of a product $\mathcal{G} = \prod_{i=1}^{d} G_i$ of proper interval graphs G_i is a set of d-tuples $x = (x_1, \ldots, x_d) \in \prod_{i=1}^{d}\{0, 1, \ldots, n_i\}$ for some $n_1, \ldots, n_d$. A *vertex interval* of $\mathcal{G}$ is a set of the form

$$[\alpha_{[i]}, \beta^{[j]}] = \{x \in V(\mathcal{G}) \mid \alpha \leq x_i \text{ and } x_j \leq \beta\}$$

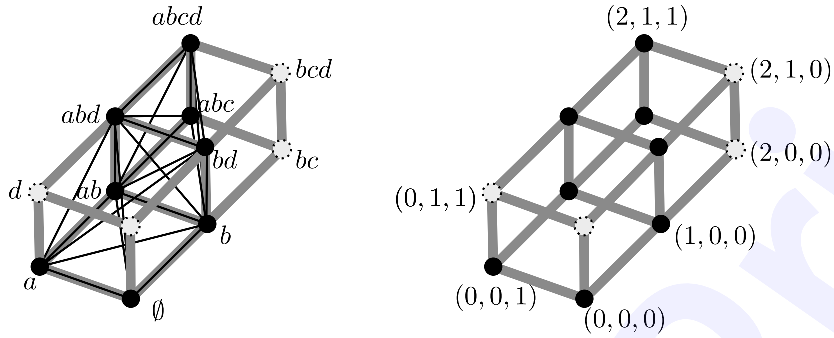
for some $i, j \in [d]$, $\alpha \leq n_i$ and $\beta \leq n_j$. (See right side of Figure 2.)

In Section 5 we show that the following, our second characterisation of reflexive DL graphs, follows almost immediately from Theorem 5.1.

Theorem 1.3. *A reflexive graph G is a* DL *graph if and only if it is the induced subgraph of a product $\mathcal{G} = \prod_{i=1}^{d} G_i$ of proper interval graphs G_i in min-max form, that we get by removing vertex intervals.*

A reflexive graph G is *R-thin* if no two vertices have the same neighbourhood. For questions of $\text{Ret}(G)$, one may always assume that G is R-thin as there are simple linear time reductions between $\text{Ret}(G)$ and $\text{Ret}(G^R)$ where G^R, defined formally in Section 6.4, is the R-thin graph we get from G by removing all but one vertex from every set of vertices sharing the same neighbourhood. In Section 6 we prove the following.

Theorem 1.4. *There is a polynomial time algorithm to decide whether or not a given R-thin reflexive graph is a* DL *graph.*



$\mathcal{D}(P)$ and $G(P, A)$ $\quad$ $\mathcal{P} - [1_{[2]}, 0^{[1]}] - [2_{[1]}, 0^{[3]}]$

Figure 2. Left: The lattice $\mathcal{D}(P)$ from Figure 1 embedded in a product of three chains, and the graph $G(P, A)$ from Figure 1 embedded as an induced subgraph of the product of paths on those chains. Right: The usual labelling on the product of chains showing $\mathcal{D}(P)$ as $\mathcal{P} - [1_{[2]}, 0^{[1]}] - [2_{[1]}, 0^{[3]}]$.

While for questions about $\mathrm{Ret}(G)$ one may assume that G is R-thin, distributive lattice polymorphisms are unusual in the fact that the existence of a compatible distributive lattice for G^R does not imply the existence of one for G. We finish off Section 6 with some notes on deciding if a non R-thin graph is a DL graph.

2. Definitions, notation, and basic observations

For any element u of an ordering $\leq$ of a set X, the downset $\langle u] = \{x \in X \mid x \leq u\}$ is the set of elements below u, and $[u\rangle = \{x \in X \mid x \geq u\}$ is the set of elements above it. We write $a \prec b$, and say that $a \prec b$ is a *cover*, if b *covers* a; that is, if $a < b$ and there is no x such that $a < x < b$. It is standard to depict a poset by its Hasse diagram– its sub-digraph of covers– and to depict direction of the covers simply by assuming that the greater element is higher on the page. As we draw a lattice and a graph on the same set of vertices, the edges of our Hasse diagram are the thicker lighter edges, and the graph edges are thin and dark.

Recall that a *lattice* is a pair $(L, \leq)$ where $\leq$ is a partial order on L such that the greatest lower bound and least upper bound are uniquely defined for any pair of elements. These define the meet, $\wedge$, and join, $\vee$, operations respectively. It is a basic fact that the operations $\wedge$ and $\vee$, and the lattice, define each other by the identities

$$u \leq v \iff (u \wedge v) = u \text{ and } u \leq v \iff (u \vee v) = v.$$

The lattice is *distributive* if the meet and join distribute. As our lattices are finite, the meet and join operations are well defined for any set of elements, and there is a maximum element, or *unit*, denoted **1**, and a minimum element or *zero* denoted **0**. As is customary, we denote a lattice $(L, \leq)$ by L when there is no risk of confusion.

A lattice L on the vertices of a graph G was defined to be compatible if its meet and join operations $\wedge$ and $\vee$ are polymorphisms. Explicitly, G is compatible with L whenever it satisfies the following identity under L, where '$\sim$' denotes adjacency in G:

$$(u \sim u' \text{ and } v \sim v') \Rightarrow (u \wedge v \sim u' \wedge v' \text{ and } u \vee v \sim u' \vee v'). \tag{2}$$

Along with (1), we consider another useful identity for vertex orderings of graphs.

$$u \sim v \sim w \text{ and } (u \leq v \geq w \text{ or } u \geq v \leq w) \Rightarrow u \sim w. \tag{3}$$

Fact 2.1. *Under a compatible lattice ordering L of a reflexive graph G, G satisfies identities* (1) *and* (3)*.*

Proof. For (1), as $v \sim v$ we get $u = u \wedge v \sim w \wedge v = v$ and $v = u \vee v \sim w \vee v = w$, as needed. For (3), assuming $u \leq v \geq w$, we get $u = (u \wedge v) \sim (v \wedge w) = w$. □

The product $(L_1 \times L_2, \leq)$ of two lattices $(L_1, \leq_1)$ and $(L_2, \leq_2)$ is defined by

$$(a_1, a_2) \leq (b_1, b_2) \text{ if } a_i \leq_i b_i \text{ for } i = 1, 2.$$

The operations $\vee$ and $\wedge$ of the product are defined componentwise from the corresponding operations of the factors. Thus the product of distributive lattices is a distributive lattice. The (*categorical*) *product* $G_1 \times G_2$ of two graphs G_1 and G_2, is the graph with vertex set $V(G_1) \times V(G_2)$ and edgeset

$$\{(u_1, u_2)(v_1, v_2) \mid u_i v_i \in G_i \text{ for } i = 1, 2\}.$$

The following is standard.

Lemma 2.2. *If for $i = 1, 2$, G_i is a reflexive graph compatible with a lattice L_i, then $G_1 \times G_2$ is compatible with $L_1 \times L_2$.*

Proof. Let $(u_1, u_2) \sim (v_1, v_2)$ and $(u_1', u_2') \sim (v_1', v_2')$ in $G_1 \times G_2$. Then

$$(u_1, u_2) \wedge (u_1', u_2') = (u_1 \wedge u_1', u_2 \wedge u_2') \sim (v_1 \wedge v_1', v_2 \wedge v_2') = (v_1, v_2) \wedge (v_1', v_2'),$$

and similarly $(u_1, u_2) \vee (u_1', u_2') = (v_1, v_2) \vee (v_1', v_2')$. □

A sublattice L' of a lattice L is any subset that is closed under the meet and join operations. The following is clear from the definition of compatibility.

Fact 2.3. *If a graph G is compatible with a lattice L, and L' is a sublattice of L, then the subgraph G' of G induced by L' is compatible with L'.*

A *conservative set* (or subalgebra) in a reflexive graph G is a subset of $V(G)$ that is closed under any polymorphism. It is a basic fact, (see [3]), that sets of the form $\{x \in V(G) \mid d(x, x_0) \leq d\}$ for some vertex x_0 and integer d are conservative, as are the intersections of such sets. In particular, components and maximal cliques of G are examples of conservative sets. We use this to prove the following, which allows us to restrict our attention to connected graphs.

Lemma 2.4. *A graph is a (distributive) lattice graph if and only if each component is.*

Proof. If a graph G is disconnected, and each of its components has a compatible lattice L_i, then let L be the *simple join* of the component lattices; that is, let L be the lattice on the set $\bigcup_{i=1}^{d} L_i$ with the ordering $\leq_L$ defined by $x \leq_L y$ if $x \leq y$ in some L_i or if $x \in L_i$ and $y \in L_j$ for $i < j$. It is easy to check that this lattice is compatible with G, and that it is distributive if the component lattices are.

On the other hand, if a disconnected graph has a compatible lattice, then as each component is a subalgebra, and subalgebras are closed under polymorphisms, each component is closed under the lattice operations. Thus each component induces a sublattice, so is compatible with the component by Fact 2.3. If a lattice is distributive, then so is any sublattice. □

The following, which does not hold for semilattices, is a huge simplification.

Proposition 2.5. *For a connected reflexive graph G with a compatible lattice L, the Hasse diagram of L is a subgraph of G.*

Proof. It is enough to show for any cover $v \prec u$, that uv is an edge of G.

Observe first that the upset $[v\rangle$ is a connected subgraph of G. Indeed as G is connected, for u_0 and u_p in $[v\rangle$, there is a path $u_0 \sim u_1 \sim \cdots \sim u_p$ between them in G. So $(v \vee u_0) \sim (v \vee u_1) \sim \cdots \sim (v \vee u_p)$ is a walk between them in $[v\rangle$.

The same proof in connected $[v\rangle$ then shows that the downset $\langle u]$ in $[v\rangle$ is connected. But it contains only u and v, so uv is an edge of G. □

3. Some examples

As all but the minimum and maximum vertices of a lattice must have at least one cover and be covered by one other vertex, the following is immediate from Proposition 2.5. The *degree* of a vertex in a reflexive graph the number of neighbours it has, distinct from itself.

Example 3.1. For a connected reflexive graph G with a degree one vertex v, v must be the minimum or maximum vertex of any compatible lattice L. In particular, the only reflexive trees with compatible lattices are reflexive paths.

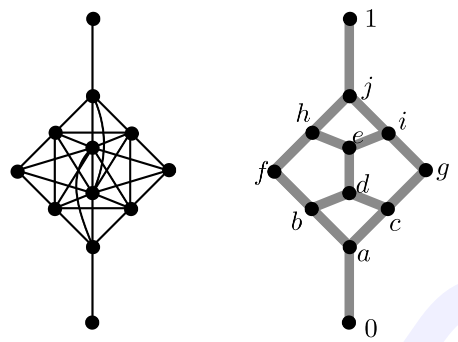


FIGURE 3. Graph (left) with compatible lattice (right) but no compatible distributive lattice.

Proposition 3.2. *Neither the class of graphs admitting compatible lattices, nor the class admitting compatible distributive lattices, is closed under retraction.*

Proof. It is easy to see that the reflexive biclique $K_{1,4}$ is a retract of the product P_2^2 of two reflexive paths. P_2^2 has a distributive lattice by Lemma 2.2, but $K_{1,4}$ does not, by Example 3.1. □

Proposition 3.3. *There are graphs that have compatible lattices but have no compatible distributive lattices.*

Proof. Let G be the graph on the left of Figure 3. It is easy but tedious to verify, using (2) that the non-distributive lattice shown on the right is compatible. We show that there is no distributive lattice that is compatible with G.

Assume, towards contradiction, that G has a compatible distributive lattice. By Proposition 2.5, $\mathbf{0}$ and $\mathbf{1}$ must be the vertices labelled 0 and 1 in the figure. Further $\mathbf{0}$ must have unique cover a and $\mathbf{1}$ must cover j. So $\langle j] \cap [a\rangle$ is a distributive sublattice with zero a and unit j.

As the set $\{d, e\}$ is the intersection of maximal cliques, it is a conservative set, so induces a sublattice. The only two element lattice is a chain, so we may assume, without loss of generality, that $d \leq e$.

The set $\{d, e, h\}$ is also an intersection of maximal cliques, so induces a sublattice of three elements, so must also be a chain. If $h \leq e$, then by (3) a and h are adjacent, so $h \geq e$. Similarly $b \leq d$.

The set $\{b, d, e, f, h\}$ is a maximal clique, so induces a lattice. As f is not adjacent to a or j, it follows from (3) that it can neither be above nor below d or e, so it is incomparable with them. Thus the sublattice induced on $\{b, d, e, f, h\}$ is as shown in the figure. It is well known that no lattice with this lattice as a sublattice is distributive. □

4. Downset construction

Our main result of this section is Theorem 4.3; after proving it, we will use it to prove Theorem 1.2. Before we prove Theorem 4.3, we make some easy observations about the construction $G(P,A)$ of Definition 1.1. Recall that A is a sub-digraph of a poset P; as only the arcset of A is important in the construction $G(P,A)$, *we will always assume that A has the same vertex set as P*. Recall also that we write $x \leq y$ if (x,y) is in P, and $x \to y$ if it is in A.

An arc (x',y') of P is *useless* for a sub-digraph A if there are x and y with $x \not\to y$, but $x' \leq x \leq y \leq y'$ and either $x' \neq x$ or $y' \neq y$. Removing all useless arcs from A, A clearly satisfies the following directed version of the identity (1) under P:

$$(u' \leq u \leq v \leq v' \text{ and } u' \to v') \Rightarrow u \to v. \tag{4}$$

Lemma 4.1. *The graph $G(P,A)$ is unchanged by adding or removing useless arcs from A. Thus in proofs about $G(P,A)$, A may be assumed to satisfy identity* (4) *under the poset P.*

Proof. Let $x' \leq x \leq y \leq y'$, and $x \not\to y$ in A. For any two downsets D and D' of P with x', y' in $D - D'$, we clearly have that x, y are in $D - D'$ as well, and so $D \not\sim D'$ in $G(P,A)$ whether $x' \to y'$ or not. □

For a sub-digraph A of a poset P, let A^{c} be the sub-digraph of P with arc set

$$\{(x,y) \mid (x,y) \in P - A\}.$$

The following is a useful alternate definition of adjacency in $G(P,A)$.

Lemma 4.2. *If A is a sub-digraph of a poset P, and D and D' are in $\mathcal{D}(P)$, then D and D' are adjacent in $G(P,A)$ if and only if the following both hold for all vertices x and y.*

- *If $x \in D$ and $(y,x) \in A^{\mathrm{c}}$, then $y \in D'$.*
- *If $x \in D'$ and $(y,x) \in A^{\mathrm{c}}$, then $y \in D$.*

Proof. The definition of adjacency of D and D' is clearly equivalent to the statement that neither $D - D'$ nor $D' - D$ induces an edge of A^{c}. That $D - D'$ induces no edge in A^{c} is equivalent to the statement that for all (y,x) in A^{c} with x, y in D, either $y \notin D'$ or $x \notin D'$. As D and D' are downsets, this reduces to the statement that for all (y,x) in A^{c} with $x \in D$, $y \notin D'$. □

Theorem 4.3. *For any reflexive graph G compatible with a distributive lattice L, G is isomorphic to $G(J_L, A)$ for a unique sub-digraph A of J_L such that A and J_L satisfy* (4).

Proof. Let G be compatible with a distributive lattice L. By [2] we have that $L \cong \mathcal{D}(J_L)$, so we denote vertices of G by downsets of the poset J_L.

We define a sub-digraph $A = A(G, J_L)$ of J_L as follows. For a vertex p of J_L, let

$$C_p = \bigcup\{X \in \mathcal{D}(J_L) \mid p \notin X\}$$

be the maximum downset not containing p. Let any arc (y, x) of J_L be in A if $\langle x] \cap C_y$ and $\langle x]$ are adjacent in G. We show $G \cong G(J_L, A)$.

Let D and D' in $\mathcal{D}(J_L)$ be adjacent in G. To show they are adjacent in $G(J_L, A)$, it is enough to show, without loss of generality, that any arc $(y, x) \in J_L$ for $x, y \in D - D'$, is in A. So we show that $\langle x] \cap C_y$ and $\langle x]$ are adjacent, which gives us $y \to x$ in A by the definition of A. As $D' \sim D$ and $\langle x] \sim \langle x]$ in G we have

$$(\langle x] \cap D') \sim (\langle x] \cap D) = \langle x].$$

But as $y \notin D'$ we have $D' \leq C_y$, and so we also have

$$\langle x] \cap D' \leq \langle x] \cap C_y \leq \langle x].$$

Thus by (1) we get $\langle x] \sim (\langle x] \cap C_y)$, as needed.

On the other hand, let D and D' be non-adjacent downsets of J_L. Then we must show that there is some arc $(y, x) \in J_L$ for $x, y \in D - D'$ or $x, y \in D' - D$ that is not in A. Assume, towards contradiction, that all such arcs with $x, y \in D - D'$ are in A. Then for each, we saw above that $\langle x] \sim (\langle x] \cap C_y)$. Fixing x and taking the intersection over all $y \leq x$ in $D - D'$, we get

$$\langle x] = \bigcap \langle x] \sim \bigcap (\langle x] \cap C_y) = \bigcap (\langle x] - [y\rangle) =: T_x,$$

where $T_x = \langle x] - \bigcup [y\rangle$ is contained in $\langle x] - D'$ as the union is over all $y \leq x$ that are in $D - D'$. Now taking the union over all $x \in D - D'$ we get that

$$D = \bigcup \langle x] \sim \bigcup T_x \subseteq \bigcup (\langle x] - D') \subseteq D - D'.$$

By (1) we get $D \sim D - D'$. Similarly we get $D' \sim D \cap D'$. But as $D \cap D' \leq D, D'$ we get $D \sim D'$ from (3). This is a contradiction.

Now, Lemma 4.1 allows us to assume that A satisfies (4) with respect to J_L. The uniqueness of A then follows by observing that $G(J_L, A')$ would be different for any other sub-digraph A' of J_L satisfying (4): simply let (y, x) be an arc of A' but not A. Then the edge $\langle x] \sim \langle y]$ is in $G(J_L, A')$ but not in $G(J_L, A)$. □

Theorem 1.2 is immediate from Theorem 4.3 and the following lemma.

Lemma 4.4. *If P is a poset and $A \subseteq P$, then $\mathcal{D}(P)$ is compatible with $G(P, A)$.*

Proof. We use Lemma 4.2 for the definition of adjacency in $G = G(P, A)$. Now, assuming that $D \sim D'$ and $E \sim E'$, we must show both $D \cup E \sim D' \cup E'$ and $D \cap E \sim D' \cap E'$.

For the former, let $x \in D \cup E$ and (x, y) be in A^c. Then x is in D or in E, so as $D \sim D'$ and $E \sim E'$, we have that y is in D' or in E'. Thus $y \in D' \cup E'$. That $x \in D' \cup E'$ implies $y \in D \cup E$ is the same, so $D \cup E \sim D' \cup E'$.

The proof of the latter is similar. □

Now, consider $G(C, A)$ where C is a chain. All downsets, except $\emptyset$, are of the form $\langle c]$ for some $c \in C$. We write $\langle -1]$ in place of $\emptyset$ to allow the following observation, which follows from the fact that A satisfies (4) under C: two downsets $\langle x]$ and $\langle y]$, for $y \leq x$ are adjacent if and only if $(y+1, x)$ is an arc of A. The following is then clear, and is the starting point of our next characterisation of reflexive DL graphs.

Fact 4.5. *For a chain C and a sub-digraph A, $G(C, A)$ satisfies* (4) *with respect to the ordering C, so is a proper interval graph.*

Proof. Assume that $\langle u] \subsetneq \langle v] \subsetneq \langle w]$ and $\langle u] \sim \langle w]$ in $G(C, A)$. So $u < v < w$ and $u+1 \to w$ in A. As we may assume that A satisfies (4) we have $u+1 \to v$ and $v+1 \to w$ in A, and so $\langle u] \sim \langle v]$ and $\langle v] \sim \langle w]$. □

5. Reflexive DL graphs as subgraphs of products of proper interval graphs

In this section we prove Theorem 5.1 and give some related results. In particular, we use it to prove Theorem 1.3. An *embedding* $e : (G, L) \to (G', L')$ of DL pairs is a map $e : V(G) \to V(G')$ that is simultaneously a graph homomorphism $G \to G'$ and a lattice embedding $L \to L'$. If the image $e(G)$ is an induced subgraph of $\mathcal{G}$ we call e an *induced embedding.* A $\mathcal{GP}$*-embedding* of a DL pair (G, L) is an embedding into a pair $(\mathcal{G}, \mathcal{P})$ such that $\mathcal{G} = \prod_{i=1}^{d} G_i$, $\mathcal{P} = \prod_{i=1}^{d} P_i$, and for each i, (G_i, P_i) is a DL pair with a chain P_i, so G_i is a proper interval graph.

Theorem 5.1. *For any* DL *pair (G, L), there is a $\mathcal{GP}$-embedding $e : (G, L) \to (\mathcal{G}, \mathcal{P})$ such that the image $e(L)$ is equal to $\mathcal{P} - \mathcal{V}$ for some union $\mathcal{V}$ of intervals of the form $[\alpha_{[i]}, \beta^{[j]}]$. Moreover, there is an induced such embedding.*

Before proving this, we observe that it implies Theorem 1.3.

Proof of Theorem 1.3. If G is a DL graph, then it belongs to some DL pair (G, L), and Theorem 5.1 gives us the necessary embedding of G into a product of proper interval graphs.

On the other hand, assume we get G from a product $\mathcal{G} = \prod G_i$ of proper interval graphs G_i in min-max form by removing vertex intervals $[\alpha_{[i]}, \beta^{[j]}]$. The ordering on the G_i is a chain lattice P_i, so induces on $V(\mathcal{G})$ a lattice $\mathcal{P} = \prod P_i$. By a result in [17], the subset induced by removing sets of the form $[\alpha_{[i]}, \beta^{[j]}]$ is a sublattice. By Fact 2.3, it is therefore compatible with the subgraph G of $\mathcal{G}$ that it induces. □

The main steps of the proof of Theorem 5.1 are as follows. In Subsection 5.1, we give necessary definitions, and recall a result from [18] that gives a correspondence, for a given distributive lattice L, between the chain covers $\mathcal{C}$ of J_L, and the embeddings $e_{\mathcal{C}}$ of L into products of chains. In Subsection 5.2 we observe that any embedding of L into a product $\mathcal{P}$ of chains induces an embedding of G into a corresponding product $\mathcal{G}$ of proper interval graphs, and so get that $e_{\mathcal{C}}$ is a $\mathcal{G}\mathcal{P}$-embedding of (G, L). We then prove two technical lemmas. In the first, Lemma 5.3, we describe exactly which vertices and edges we must remove from $\mathcal{G}$ to get G. In the second, Lemma 5.4 we find a property of the chain cover $\mathcal{C}$ that is necessary and sufficient for $e_{\mathcal{C}}$ to embed G as an induced subgraph of $\mathcal{G}$. Finally in Subsection 5.3 we formally prove Theorem 5.1 by observing that J_L has a chain cover with the property given in Lemma 5.4.

5.1. Preliminary definitions and results

A *chain cover* of a poset P is a family $\mathcal{C} = \{C_1, \ldots, C_d\}$ of subchains of P such that every element of P is in at least one chain. Given $\mathcal{C}$, label the elements of P so that the subchain C_i is $1^{(i)} \prec \cdots \prec {n_i}^{(i)}$ for some n_i; if an element is in more than one chain, it gets more than one label. It is clear that a downset D of P is uniquely defined by the tuple $e_{\mathcal{C}}(D) = (x_1, \ldots, x_d)$ where $x_i = |D \cap C_i|$. (Note that D is thus the downset generated by the set $\{{x_i}^{(i)} \mid i \in [d]\}$.)

We observed in [18] that $e_{\mathcal{C}}$ is in fact a lattice embedding of $\mathcal{D}(P)$ into the product $\mathcal{P}_{\mathcal{C}} = \prod_{i=1}^{d} P_i$ where P_i is the chain $0 \prec 1 \prec \cdots \prec n_i$ with one more element than C_i. As $L \cong \mathcal{D}(J_L)$, every chain cover of J_L gives an embedding $e_{\mathcal{C}}$ of L as a sublattice of a product $\mathcal{P}_{\mathcal{C}}$ of chains. In Corollary 6.6 of [18] we made this into a one-to-one correspondence by showing any embedding of L into a product of chains is equal to $e_{\mathcal{C}}$ for some chain cover $\mathcal{C}$ of J_L.

Further, we showed how the chain cover $\mathcal{C}$ can be used to describe $e_{\mathcal{C}}(L)$ explicitly as a sublattice of $\mathcal{P}_{\mathcal{C}}$. This requires the following notation, which explains the notation $[\alpha_{[i]}, \beta^{[j]}]$. Given the product $\mathcal{P}_{\mathcal{C}} = \prod_{i=1}^{d} P_i$ of chains, let

$$\alpha_{[i]} = (\underbrace{0, \ldots, 0, \alpha}_{i}, 0, \ldots, 0) \quad \text{and} \quad \beta^{[j]} = (n_1, \ldots, n_{j-1}, \beta, n_{j+1}, \ldots, n_d)$$

for all $i, j \in [d]$ and α, β with $0 \leq \alpha \leq n_i$ and $0 \leq \beta \leq n_j$.

Proposition 5.2 ([18]). *Let $\mathcal{V}$ be the union of the intervals*

$$[\alpha_{[i]}, \beta^{[j]}] = \{x \in \mathcal{P}_{\mathcal{C}} \mid \alpha \leq x_i \text{ and } x_j \leq \beta\},$$

over all comparable pairs $(\beta + 1)^{(j)} \leq \alpha^{(i)}$ in J_L. Then the image of the embedding $e_{\mathcal{C}} : L \to \mathcal{P}_{\mathcal{C}}$ is equal to $\mathcal{P}_{\mathcal{C}} - \mathcal{V}$.

5.2. Technical lemmas

Our first task is to show that for a DL pair (G, L), any embedding e of L into a product of chains induces an embedding of G into a corresponding product of proper interval graphs.

By Theorem 4.3, $G = G(J_L, A)$ for some sub-digraph A of J_L, and from [18] we have that $e = e_{\mathcal{C}}$ for some chain cover $\mathcal{C}$ of J_L. For each chain $C_i \in \mathcal{C}$ let A_i be the subgraph of A induced by the vertices of C_i. By Fact 4.5 we have that

$G_i = G(C_i, A_i)$ satisfies (4) under the chain P_i, so is a proper interval graph. Let $\mathcal{G} = \prod G_i$ be the product of these proper interval graphs. The embedding $e_{\mathcal{C}} : \mathcal{D}(J_L) \to \mathcal{P}_{\mathcal{C}}$ embeds $V(G)$ as a subset of $V(\mathcal{G})$. The following shows that this is in fact a graph embedding.

For each $i, j \in [d]$ and $0 \leq \alpha \leq n_i$ and $0 \leq \beta \leq n_j$, define the following set of possible edges in a graph on the vertices of G:

$$[\alpha_{[i]}\rangle \times \langle\beta^{[j]}] := \{\{x, y\} \mid x, y \in V(G), \alpha \leq x_i, \text{ and } y_j \leq \beta\}\,.$$

For a set $\mathcal{V} \subset V(\mathcal{G})$ and a set $\mathcal{E}$ of edges over $V(\mathcal{G})$, we write $\mathcal{G} - \mathcal{V} - \mathcal{E}$ for the graph we get from $\mathcal{G}$ by removing vertices of $\mathcal{V}$ and edges of $E(\mathcal{G}) \cap \mathcal{E}$.

Lemma 5.3. *Where $\mathcal{C}$ is a chain cover of J_L, $G = G(J_L, A)$ is a subgraph of the product $\mathcal{G} = \prod G_i$ of proper interval graphs $G_i = (C_i, A_i)$. In fact where*

$$\mathcal{V} = \bigcup\{[\alpha_{[i]}, \beta^{[j]}] \mid (\beta+1)^{(j)} \leq \alpha^{(i)} \in J_L\} \textit{ and}$$
$$\mathcal{E} = \bigcup\{[\alpha_{[i]}\rangle \times \langle\beta^{[j]}] \mid (\beta+1)^{(j)} \to \alpha^{(i)} \in A^{\mathrm{c}}\},$$

G is the subgraph $\mathcal{G} - \mathcal{V} - \mathcal{E}$ of $\mathcal{G}$.

Proof. Let D and D' be downsets of J_L and let $x = e_{\mathcal{C}}(D)$ and $y = e_{\mathcal{C}}(D')$. If $D \sim D'$ we have in particular that for each i, $(x_i+1)^{(i)} \to {y_i}^{(i)}$ if $x_i < y_i$ and $(y_i+1)^{(i)} \to {x_i}^{(i)}$ if $y_i < x_i$. So ${x_i}^{(i)} \sim {y_i}^{(i)}$ in $G_i(C_i, A_i)$. This shows that G is a subgraph of $\mathcal{G}$. By Proposition 5.2, $V(G) = V(\mathcal{G}) - \mathcal{V}$ so we are done when we prove the following claim.

Claim 1. *Vertices x and y of G are non-adjacent in G if and only if there is some arc $(\beta+1)^{(j)} \to \alpha^{(i)}$ in A^{c} such that $\{x, y\}$ is in the set $[\alpha_{[i]}\rangle \times \langle\beta^{[j]}]$ of edges of $\mathcal{G}$.*

Proof. Let D_x and D_y be the downsets of J_L for which $e_{\mathcal{C}}(D_x) = x$ and $e_{\mathcal{C}}(D_y) = y$. Then $D_x \not\sim D_y$ if and only if there is some $(\beta+1)^{(j)} \to \alpha^{(i)}$ in A^{c} with $\alpha^{(i)}, (\beta+1)^{(j)}$ in $D_x - D_y$ (or $D_y - D_x$, but wlog we assume the former). This is true if and only if

$$y_i < \alpha \leq x_i \text{ and } y_j \leq \beta < x_j. \tag{5}$$

But since $(\beta+1)^{(j)} \to \alpha^{(i)}$ in A^{c} we certainly have that $(\beta+1)^{(j)} \leq \alpha^{(i)}$ in J_L, so x and y are not in $[\alpha_{[i]}, \beta^{[j]}]$. This means that neither

$$(\alpha \leq x_i \text{ and } x_j \leq \beta) \text{ nor } (\alpha \leq y_i \text{ and } y_j \leq \beta) \tag{6}$$

holds. As the conjunction of (5) and the negation of (6) is logically equivalent to

$$\alpha \leq x_i \text{ and } y_j \leq \beta,$$

we get the claim. ◇

This completes the proof of the lemma. □

Now our goal is to embed G as an induced subgraph of $\mathcal{G}$, so we would like to decide when G can be expressed as $\mathcal{G}-\mathcal{V}$ in the above lemma. At first glance it would seem that this is exactly when $\mathcal{E}$ is empty, so when A^c is empty, but actually it is not necessary that A^c is empty, as many of the sets $[\alpha_{[i]}\rangle \times \langle\beta^{[j]}]$ may not actually contain edges of $\mathcal{G}$.

We have been using Lemma 4.1 to remove from A arcs of J_L that are useless for A. We may also use it to add to A those arcs of J_L that are useless for A. Doing so, the complement (in J_L) is a 'reduced' version of A^c: a graph $\mathrm{red}(A^c)$ that generates the usual A^c by composition with J_L.

Lemma 5.4. *For a chain cover $\mathcal{C}$ of J_L, $e_{\mathcal{C}}(G(J_L, A))$ is an induced subgraph of $\mathcal{G}$ if and only if $\mathrm{red}(A^c)$ is a sub-digraph of $\bigcup\mathcal{C} := \bigcup_{C_i\in\mathcal{C}} C_i$.*

Proof. Let $G = G(J_L, A)$. On the one hand, let $\mathrm{red}(A^c)$ be a subgraph of $\bigcup\mathcal{C}$. As $e_{\mathcal{C}}(G) = \mathcal{G}-\mathcal{V}-\mathcal{E}$ by Lemma 5.3, it is enough to show that no edge of $\mathcal{E}$, so no edge of $[\alpha_{[i]}\rangle \times \langle\beta^{[j]}]$ for any $(\beta+1)^{(j)} \to \alpha^{(i)}$ in $\mathrm{red}(A^c)$, is in $\mathcal{G}$. As $\mathrm{red}(A^c)$ is a subgraph of $\bigcup\mathcal{C}$ for any arc $(\beta+1)^{(j)} \to \alpha^{(i)}$, the endpoints have labels in a common chain k, so there are δ and γ with $\delta+1 \leq \gamma$ such that $(\beta+1)^{(j)} = (\delta+1)^{(k)}$ and $\alpha^{(i)} = \gamma^{(k)}$. Thus $\beta^{[j]} = \delta^{[k]}$ and $\alpha_{[i]} = \gamma_{[k]}$, which implies

$$[\alpha_{[i]}\rangle \times \langle\beta^{[j]}] = [\gamma_{[k]}\rangle \times \langle\delta^{[k]}].$$

So let $\{x, y\}$ be an edge of $[\alpha_{[i]}\rangle \times \langle\beta^{[j]}]$. It is thus an edge of of $[\gamma_{[k]}\rangle \times \langle\delta^{[k]}]$, and so $y_k \leq \delta$ and $\gamma \leq x_k$. As $(\beta+1)^{(j)} \to \alpha^{(i)}$ and so $(\delta+1)^{(k)} \to \gamma^{(k)}$ in $\mathrm{red}(A^c)$, $(\delta+1)^{(k)} \not\to \gamma^{(k)}$ in A, and so in A_k. Thus $\{x_k^{(k)}, y_k^{(k)}\}$ is not in G_k, and so $\{x, y\}$ is not in $\mathcal{G}$.

On the other hand, assume that $\mathrm{red}(A^c)$ is not a subgraph of $\bigcup\mathcal{C}$. Then there is some $(\beta+1)^{(j)} \to \alpha^{(i)}$ in $\mathrm{red}(A^c)$ such that $\alpha^{(i)}$ and $(\beta+1)^{(j)}$ are not both in C_k for any k. Now, as $(\beta+1)^{(j)} \to \alpha^{(i)}$ in $\mathrm{red}(A^c)$, we know that $\langle\alpha^{(i)}], \langle\beta^{(j)}]$ are not adjacent in G but we show that they are adjacent in $\mathcal{G}$ by showing that their projections onto any of the C_k induces an edge of G_k.

Indeed, $(\langle\alpha^{(i)}]-\alpha^{(i)}) \sim \langle\beta^{(j)}]$, or otherwise there is an arc in A^c in $(\langle\alpha^{(i)}]-\alpha^{(i)}) - \langle\beta^{(j)}]$ which would contradict the existence of $(\beta+1)^{(j)} \to \alpha^{(i)}$ in the reduced $\mathrm{red}(A^c)$. For any k such that $\alpha^{(i)} \notin C_k$, edge $\langle\alpha^{(i)}] - \alpha^{(i)}$ and $\langle\alpha^{(i)}]$ project to the same image in G_k. As the image, under the projection, of the edge $(\langle\alpha^{(i)}]-\alpha^{(i)}) \sim \langle\beta^{(j)}]$ is an edge of G_k, the projection of $\langle\alpha^{(i)}]$ and $\langle\beta^{(j)}]$ therefore induces an edge in G_k, as needed.

Similarly $\langle\alpha^{(i)}] \sim \langle\beta^{(j)}]\cup\{(\beta+1)^{(j)}\}$ is an arc showing that the projection of $\langle\alpha^{(i)}]$ and $\langle\beta^{(j)}]$ onto G_k induces an edge of G_k, for any k such that $(\beta+1)^{(j)} \notin C_k$. □

5.3. Proof of Theorem 5.1

We give now the formal proof of Theorem 5.1.

Let (G, L) be a DL pair. Theorem 4.3 provides us a sub-digraph A of J_L such that $G \cong G(J_L, A)$. By Lemma 5.3 every chain cover $\mathcal{C}$ of J_L yields a $\mathcal{GP}$-embedding $e_{\mathcal{C}} : L \cong \mathcal{P}_{\mathcal{C}} - \mathcal{V}$ into a product of chains. By Lemma 5.4, such $e_{\mathcal{C}}$ is induced if and only if $\mathrm{red}(A^c)$ is a subgraph of $\bigcup \mathcal{C}$. We can assure this by taking every arc of $\mathrm{red}(A^c)$ as a two element chain in $\mathcal{C}$ and then covering then rest of J_L with one element chains.

This completes the proof of Theorem 5.1.

5.4. Tight embeddings

Theorem 5.1 tells us that every DL pair (G, L) admits an induced $\mathcal{GP}$-embedding into some pair $(\mathcal{G}, \mathcal{P})$.

A lattice embedding is *tight* if every cover of L is a cover of $\mathcal{P}$. Classical results of Birkhoff and Dilworth give a correspondence between tight embeddings of L into products of chains, and *chain decompositions* of J_L: chain covers $\mathcal{C}$ consisting of disjoint chains. By Lemma 5.3 any DL pair (G, L) has a tight $\mathcal{GP}$-embedding, but if $\mathrm{red}(A^c)$ has any vertices with in-degree or out-degree greater than 2, then by Lemma 5.4, it is not induced. In fact, we will see at the end of the next section that there are DL graphs G such that there are no compatible lattices L for which (G, L) has a tight induced $\mathcal{GP}$-embedding.

6. Recognition of R-thin DL graphs

Recall that a graph is *R-thin* if no two of its vertices have the same neighbourhood. As our graphs are reflexive, neighbourhoods and closed neighbourhoods are the same thing.

The factorization of a categorical product of graphs was shown to be unique (up to certain obviously necessary assumptions which include R-thinness) by Dörfler and Imrich [7]. Feigenbaum and Schäffer [10] showed that a categorical product can be factored in polynomial time. On the other hand, it was shown in [5] that proper interval graphs can be recognised in linear time. So products $\mathcal{G}$ of proper interval graphs can be recognised in polynomial time. However, as we have seen, products of proper interval graphs are not the only DL graphs. Indeed, Theorem 1.3 tells us that certain subgraphs of products of proper interval graphs are also DL graphs.

In this chapter we prove Theorem 1.4, which says that there is a polynomial time algorithm to decide whether or not an R-thin reflexive graph is a DL graph. We begin by outlining the two main steps of the algorithm.

In the first step, we find a special subgraph S of G using an algorithm from [12]. We define this graph S in Subsection 6.1, and then spend considerable effort to prove Lemma 6.6, which essentially says that for any distributive lattice L which is compatible with G, S is sandwiched between the symmetrisation of L and the symmetrisation of its Hasse diagram. This assures that if we can properly orient the edges of S (according to their orientation in L), then we can recover L by taking the transitive closure of S.

The second step, in Subsection 6.2, is to define an orientation $\vec{S}$ of the edges of S with respect to some choice of 0 and 1 in G. In Lemma 6.9 that if (G, L) is a DL pair such that 0 and 1 are the zero and unit of L, respectively, then the orientation $\vec{S}$ agrees with the orientation of the edges of S in L, so yields L by taking the transitive closure. Thus checking if the transitive closure of $\vec{S}$ is a distributive lattice, and checking if it is compatible with G, we determine if G is compatible with any distributive lattice with the given choice of extremal elements. If it is, then G is a DL graph; if not, we repeat with another choice of 0 and 1. Doing this for the n^2 choices of the vertices 0 and 1 in G, we decide if G is a DL graph.

These steps are put together in the formal proof of Theorem 1.4 in Subsection 6.3.

6.1. The subgraph S of non-dispensable edges

The following definition can be found in [12].

Definition 6.1. An edge xy of G *dispensable* if it satisfies the following conditions.

(i) $\exists z$ such that $N(x) \subsetneq N(z) \subsetneq N(y)$, or
(ii) $\exists z$ such that $N(y) \subsetneq N(z) \subsetneq N(x)$, or
(iii) $\exists z$ such that $N(x) \cap N(y) \subsetneq N(x) \cap N(z)$ and $N(x) \cap N(y) \subsetneq N(y) \cap N(z)$.

The *skeleton* $S(G)$ of G is the graph S that we get from G by removing loops and dispensable edges. Observe that when G is R-thin, we can replace the $\subsetneq$ in the first two conditions with $\subset$; they are equivalent.

The following is clear; for more detailed discussion of the complexity, see [12].

Fact 6.2. *Given a reflexive graph G, one can find $S(G)$ in time polynomial in the size of G.*

The rest of this subsection is dedicated to proving Lemma 6.6. This would be easy if for a $\mathcal{GP}$-embedding of (G, L) into $(\mathcal{G}, \mathcal{P})$, the R-thinness of G implied the R-thinness of the factors of $\mathcal{G}$. As it does not, however, we first prove Lemma 6.5.

As we mentioned in Subsection 5.4, not all DL pairs (G, L) have tight induced $\mathcal{GP}$-embeddings. Given an induced $\mathcal{GP}$-embedding e of (G, L), a *non-tight cover* of e is a cover $x \prec y$ of G such that the number $\mathrm{nt}(x \prec y) = |\{i \in d \mid e(x_i) \neq e(y_i)\}|$ of coordinates in which the image vertices differ, is greater than one. An induced embedding $e : (G, L) \to (\mathcal{G}, \mathcal{P})$ is *tightest* if it minimises, over all induced $\mathcal{GP}$-embeddings of (G, L), the sum of $\mathrm{nt}(x \prec y)$ over all non-tight covers $x \prec y$ of G.

Claim 6.3. *If $x \prec y$ is a non-tight cover in a tightest induced $\mathcal{GP}$-embedding of a* DL *pair (G, L), then for every vertex z of G either $z \leq x$ or $y \leq z$.*

Proof. Indeed, $x < z < y$ is impossible as $x \prec y$. If $x < z$ but $z \parallel y$ (i.e., z and y are incomparable), then $z_i > y_i$ for some i and so $x_i < (z \wedge y)_i$, giving us $x < (z \wedge y) < y$, which is again impossible as $x \prec y$. Similarly $x \parallel z$ and $x < y$ is impossible. Finally, if $x \parallel z$ and $y \parallel z$, then taking $z' = x \vee z$ we get that $x < z'$ and $z' \parallel y$, which we have already seen is impossible. □

Claim 6.4. *If $x \prec y$ is a non-tight cover in a tightest induced $\mathcal{GP}$-embedding of a* DL *pair (G, L), then for all $i \in [d]$ for which $x_i \neq y_i$, x_i and y_i have different neighbourhoods in G_i.*

Proof. Assume $x_i < y_i$, but that x_i and y_i have the same neighbourhoods. We get a contradiction by exhibiting a $\mathcal{GP}$-embedding $e' : (G, L) \to (\mathcal{G}', \mathcal{P}')$ that is tighter than the original embedding $e : (G, L) \to (\mathcal{G}, \mathcal{P})$.

Let P'_i be the chain we get from P_i by removing the interval $[x_i + 1, y_i]$ and reducing the label on everything in $[y_i\rangle$ by $c = y_i - x_i$. Let G'_i be the graph we get from G_i by removing the same elements. As x_i and y_i have the same neighbourhoods in G_i, so does everything between them, and so and (G'_i, P'_i) is a DL pair. Construct $\mathcal{G}'$ and $\mathcal{P}'$ from $\mathcal{G}$ and $\mathcal{P}$ respectively by replacing the factors G_i and P_i with G'_i and P'_i respectively.

Consider the map $e' : (\mathcal{G}, \mathcal{P}) \to (\mathcal{G}', \mathcal{P}')$ which is the identity on every coordinate of every vertex except that for vertices v with $v_i \geq y_i$, it reduces the i^{th} coordinate by c. The map e' is a lattice embedding Indeed, it clearly induces an isomorphism of $\langle x]$ in $\mathcal{P}$ to $\langle x]$ in $\mathcal{P}'$ and an isomorphism of $[y\rangle$ in $\mathcal{P}$ to $[e'(y)\rangle$ in $\mathcal{P}'$. By Claim 6.3 the only cover of G not in $\langle x]$ or $[y\rangle$ is $x \prec y$. As $x \prec y$ is non-tight, x and y differ in at least two coordinates, and so even though e' reduces y_i to x_i, we have $e'(x) \lneq e'(y)$, so e' is an embedding. As $e'(y)_i = e'(x)_i$ we have that $\mathrm{nt}(e'(x) \prec e'(y)) < \mathrm{nt}(x \prec y)$, and so e' is a tighter lattice embedding than e, as needed. That e' is also a graph embedding is immediate, as x_i and y_i have the same neighbourhoods. □

Lemma 6.5. *If G is R-thin, then each G_i in a tightest induced $\mathcal{GP}$-embedding of (G, L) is R-thin.*

Proof. Towards contradiction, assume that some G_i contains vertices a and b with the same neighbourhoods. As G_i is a proper interval graph, we may assume $b = a + 1$. By Claim 6.4 no non-tight cover projects onto $a \prec a + 1$ so there is some $x \in \mathcal{G}$ with $x_i = a$ such that x and the vertex $x' \in \mathcal{G}$ that we get from x by increasing the i coordinate by 1, are both in G. But x and x' have the same neighbourhood in $\mathcal{G}$, and so as G is an induced subgraph of $\mathcal{G}$, they have the same neighbourhood in G, contradicting the fact that G is R-thin. □

With this proved, we are ready to prove Lemma 6.6. We will use the following notation. Assume an embedding of G into some product $\mathcal{G} = \prod G_i$ of proper interval graphs G_i. For a vertex v_i of G_i we let $v_i^+ = \max\{N_{G_i}(v_i)\}$ be the greatest neighbour of v_i in G_i and $v_i^- = \min\{N_{G_i}(v_i)\}$ be the least neighbour.

As G_i is proper interval, $v_i \leq u_i$ implies $v_i^+ \leq u_i^+$ and $v_i^- \leq u_i^-$. As G_i is R-thin, strict inequality $v_i < u_i$ implies strict inequality in at least one of $v_i^+ \leq u_i^+$ and $v_i^- \leq u_i^-$.

Lemma 6.6. *Let (G, L) be a* DL *pair, G be R-thin, and S be the skeleton $S(G)$ of G. The following hold.*

(a) *Every edge of S is between comparable vertices of L, and*
(b) *For every cover $x \prec y$ of L, xy is an edge of S.*

Proof. First we prove part (a), by showing that any edge xy between incomparable vertices x and y is dispensable. Indeed, as x and y are incomparable, we have that $x \wedge y$ and $x \vee y$ are distinct and different from x and y. Further as $\wedge$ and $\vee$ are polymorphisms, any common neighbour of x and y is a neighbour of both of $x \wedge y$ and $x \vee y$, so $N(x) \cap N(y) \subseteq N(x \wedge y), N(x \vee y)$. By R-thinness, $N(x \wedge y)$ and $N(x \vee y)$ are distinct, so one of them properly contains $N(x) \cap N(y)$. Thus xy is dispensable.

Now, part (b) is harder. Let $x \prec y$ be a cover of L; we show that it is not dispensable. Assume some tightest induced $\mathcal{GP}$-embedding $(G, L) \to (\mathcal{G}, \mathcal{P})$. We have two cases.

Case: $x \prec y$ is a non-tight cover. For any z in $L - \{x, y\}$, we may assume by Claim 6.3 that $x \prec y < z$. We show that item (i) of Definition 6.1 cannot hold. Any neighbour of y in $[y\rangle$ is a neighbour of z by (3) (of Section 2) and so any w in $N(y) - N(z)$ must be in $\langle x]$. But then by (1) it is adjacent to x, contradicting $N(x) \subset N(z)$. Items (ii) and (iii) cannot hold, as by (1) any common neighbour of x and z is also a neighbour of y.

Case: $x \prec y$ is a tight cover. By the R-thinness of G, we may assume without loss of generality that there is some $v \in N(y) - N(x)$. So immediately, condition (ii) of Definition 6.1 does not hold. We may assume, by permuting indices of the interval graphs G_i, and possibly reversing the ordering on the first one, that $y_1 = x_1 + 1$

Assume (i) holds, that is, there is some z with $N(x) \subsetneq N(z) \subsetneq N(y)$. As z has some neighbour in $N(y) - N(x)$ we get $x_1 < y_1 \leq z_1$, so in particular $y_1^+ < z_1^+$. As $x \sim y \sim z$ we have $z_1^- \leq y_1 \leq x_1^+$. There is some vertex w adjacent to y but neither z nor x. As it is in $N(y) - N(x)$ we have $x_1^+ < w_1 < y_1^+$. Putting these together we have

$$z_1^- \leq y_1 \leq x_1^+ < w_1 \leq y_1 < z_1^+$$

and so $w \not\sim z$ means we may assume $w_2 \not\sim z_2$, and so $z_2 < y_2 \leq z_2^+ < w_2$.

Now let w' be the vertex in $\mathcal{G}$ we get from x by replacing x_1 with x_1^+ and x_2 with $z_2^+ + 1$. So $w' \not\sim z$ in $\mathcal{G}$, while $w \sim x$ and $x_2 = y_2 < w_2' \leq w_2$ implies $x_2 \sim w_2'$, so $w' \sim x$ in $\mathcal{G}$. As $N(x) \subset N(z)$ in G, w' cannot be in G, so is in some vertex interval $[\alpha_{[i]}, \beta^{[j]}]$ removed from $\mathcal{G}$ to get G.

As w is not in $[\alpha_{[i]}, \beta^{[j]}]$ we have $j = 1$ and $x_1^+ \leq \beta < w_1$. Also, as x is not in $[\alpha_{[i]}, \beta^{[j]}]$, we have $i = 1$ or $i = 2$. But a tightest embedding cannot have a vertex interval of the form $[\alpha_{[i]}, \beta^{[i]}]$ removed, so $i = 2$ and $x_2 < \alpha \leq w'_2 = z_2^+ + 1$. So $z_2 < x_2 < \alpha \leq z_2^+ + 1$.

Now we claim that the vertex x' which we get from x by reducing x_2 to z_2 has the same neighbourhood as x in G, a contradiction. Indeed $N(x')$ contains $N(x) \cap N(z)$, so contains $N(x)$, and all vertices v of $\mathcal{G}$ that are adjacent to x but not to x' have $\alpha \leq z_2^+ + 1 \leq v_2$ and $v_1 \leq x_1^+ \leq \beta$ so are in $[\alpha_{[2]}, \beta^{[1]}]$ which has been removed. Thus we have our contradiction, so (i) cannot hold.

The argument that (ii) cannot hold is essentially the same. Finally, assume (iii) holds. Clearly this implies that both $N(x) - N(y)$ and $N(y) - N(x)$ are non-empty, so $x_1^+ < y_1^+$ and $x_1^- < y_1^-$. Moreover, z has a neighbour $a \in N(Y) - N(X)$, so having $a_1 > x_1^+$, and similarly another neighbour b having $b_1 < y_1^-$. But then there is no viable value for z_1.

This completes the proof of (b) and so of the lemma. □

Compare Lemma 6.6 to similar statements in [12, Chap 8], where they show that S is closely related to what they call the *Cartesian skeleton* of a product graph G. Our proof is complicated by the fact that G is not a product, but a subgraph of a product.

6.2. Orienting edges of S

In this subsection, we take a graph G, and its skeleton $S = S(G)$ and we try to orient the arcs of S so that they are consistent with their orientation in L, which we do not know. Now this is impossible without some other knowledge of L: indeed, if L is compatible with G, then so is the lattice L^{-1} we get by reversing it, and the orientation of any arc of S is different depending on which lattice we consider. Knowing the extremal vertices of L is enough.

Given $\mathbf{0}$ and $\mathbf{1}$ in G, the first step is to observe that edges of S containing $\mathbf{1}$ are oriented towards it. And more generally, for edges xy in which x is closer to $\mathbf{1}$ than y is, we must have $y \to x$. Determining the orientation of edges of S whose endpoints have the same distance from $\mathbf{1}$ is a little more involved.

Definition 6.7. Given a graph G, its skeleton $S = S(G)$, and vertices $\mathbf{0}$ and $\mathbf{1}$ of G let $N_j = \{v \in G \mid \mathrm{dist}_G(\mathbf{1}, v) = j\}$ be the j^{th} neighbourhood of $\mathbf{1}$ in G, for each $j = 0, \ldots, \mathrm{dist}_G(\mathbf{0}, \mathbf{1})$, and let S_j be the subgraph of S induced by $\cup_{\alpha=0}^{j} N_j$.

Define a (partial) orientation $\vec{S} = \vec{S}(\mathbf{0}, \mathbf{1})$ of S as follows.

(i) For every edge uv of S, let $v \to u$ in $\vec{S}$ if $v \in N_j$ and $u \in N_{j-1}$ for some $j \in \{1, \ldots, \mathrm{dist}(\mathbf{0}, \mathbf{1})\}$.

(ii) For every edge uv of S_1 with both u and v in N_1 let $v \to u$ if $N_G(v) - N_G(u)$ contains an element of $N_1 \cup N_2$.

(iii) For $j \in \{2, \ldots, \mathrm{dist}(\mathbf{0}, \mathbf{1})\}$, do the following. Write $x \rightsquigarrow y$ if there is a directed path in S_{j-1} from x to y. For any edge uv of S_j with both u and v in N_j let $v \to u$ if any of the following hold.

(a) There exists $u' \in N_{j-1} \cap (N_G(u) - N_G(v))$ such that $v' \rightsquigarrow u'$ for all $v' \in N_{j-1} \cap N_G(v)$.
(b) There exists $v' \in N_{j-1} \cap (N_G(v) - N_G(u))$ such that $v' \rightsquigarrow u$ for all $u' \in N_{j-1} \cap N_G(u)$.
(c) $(N_G(v) - N_G(u))$ is non-empty but $N_{j-1} \cap (N_G(v) - N_G(u))$ is empty.

The following is clear.

Fact 6.8. *The construction of $\vec{S}(\mathbf{0}, \mathbf{1})$ from given G, $\mathbf{0}$ and $\mathbf{1}$ is polynomial in the size of G.*

What is not clear is that $\vec{S}(\mathbf{0}, \mathbf{1})$ properly orients every edge of S or that it is even a well defined orientation of S. This we prove now.

Lemma 6.9. *Assume that G is an R-thin graph and L is a compatible distributive lattice with zero $\mathbf{0}$ and unit $\mathbf{1}$. Let $S = S(G)$. Then $\vec{S}(\mathbf{0}, \mathbf{1})$ is an orientation of S; moreover if $x \to y$ in $\vec{S}$, then $x \leq y$ in L.*

Proof. We show for any edge uv of S with $u \leq v$ in L, that $u \to v$ in $\vec{S}$, **and** that $v \not\to u$ in $\vec{S}$. By Lemma 6.6(a), this implies that $\vec{S}$ is an orientation of S, and then explicitly yields the 'moreover' consequence of the lemma.

Case: u and v are not both in N_j for any j. By the definition of N_j we may assume that one of u and v is in N_j and the other is in N_{j-1}. We show that it is v that is in N_{j-1}, thus $u \to v$ by item (i) of Definition 6.7 and $v \to u$ is not included in $\vec{S}$ by any item of Definition 6.7.

Indeed, if it is u that is in N_{j-1}, then there is a path $u = x_{j-1} \sim x_{j-2} \sim \cdots \sim x_0 = \mathbf{1}$ of length $j-1$ from u to $\mathbf{1}$ in G. But then there is also a length $j-1$ path $v = v \vee x_{j-1} \sim v \vee x_{j-2} \sim \cdots \sim v \vee x_0 = \mathbf{1}$ from v to $\mathbf{1}$ in G, contradicting the fact that $v \in N_j$.

Case: u and v are both in N_1. We show that $N_G(u) - N_G(v)$ contains an element of $N_1 \cup N_2$, but that $N_G(v) - N_G(u)$ does not.

To see that $N_G(u) - N_G(v)$ does, observe that because $u \leq v$, we have $u_i \leq v_i$ for all $i \in [d]$. As $u_i^+ = n_i = v_i^+$ for all i, we have by R-thinness that $N_G(\mathbf{1}) \subsetneq N_G(v) \subsetneq N_G(u)$. Any vertex in $N_G(u) - N_G(v)$ is thus in N_2 as needed.

To see that $N_G(v) - N_G(u)$ does not contain an element of $N_1 \cup N_2$, observe that it is, in fact, empty. Indeed, for $w \in N_G(v)$, we have $w = w \wedge \mathbf{1} \sim v \wedge u = u$, so $w \in N_G(u)$.

Case: u and v are both in N_j for some $j \geq 2$.

Only item (iii) of Definition 6.7 can apply to include either $v \to u$ or $u \to v$ in $\vec{S}$. We show that as $u \leq v$, u and v satisfy no part of (iii) so $v \not\to u$, but on reversing the roles of u and v, they satisfy some part of (iii), so $u \to v$.

First, we verify $v \not\to u$ in $\vec{S}$. To see that part (a) does not hold, consider some $u' \in N_{j-1} \cap N_G(u)$ such that $v' \rightsquigarrow u'$ for all $v' \in N_G(v) \cap N_{j-1}$. As there must be such a v' we get $u' = u' \wedge v' \sim u \wedge v = v$, so $u \notin N_{j-1} \cap (N_G(u) - N_G(v))$.

The proof that part (b) does not hold is similar. The proof that part (c) of (iii) is not satisfied is given as the following claim which we will also use later.

Claim 6.10. *If $N_G(v) - N_G(u)$ is non-empty for u and v in N_j with $u \leq v$, then the maximum neighbour v' of v is in $N_{j-1} \cap (N_G(v) - N_G(u))$.*

Proof. Let w be in $N_G(v) - N_G(u)$.

As $N_G(v)$ is conservative, (recall the definition of conservative sets preceding Lemma 2.4) it induces a sublattice L, so has a maximum element v'. This element must also be in $N_G(v) - N_G(u)$; as if we had $v' \sim u$, then $w = w \wedge v' \sim v \wedge u = u$, contradicting the fact that $w \notin N_G(u)$.

We now show that v' is in N_{j-1}. Indeed, as $u \in N_j$, some neighbour x of u must be in N_{j-1}. Let $x = x_{j-1} \sim x_{j-2} \sim \cdots \sim x_0 = \mathbf{1}$ be a length $j-1$ walk from x to $\mathbf{1}$. Taking the join of each element in the walk with v' we get a length $j-1$ walk $v' = v' \vee x_{i-1} \sim v' \vee x_{i-2} \cdots \sim v' \vee \mathbf{1} = \mathbf{1}$ from v' to $\mathbf{1}$. This shows that v' is in N_i for some $i \leq j-1$, but being a neighbour of u, it must be in N_{j-1}. ◇

Having proved $v \not\to u$ in $\vec{S}$, we now show $u \to v$ in $\vec{S}$. Or rather, as it makes reading item (iii) of Definition 6.7 easier, we assume $v \leq u$ and show $v \to u$. In our proof for j, we may assume that it has already been proved for $j-1$, and so all edges of S_{j-1} are properly oriented. Thus we can interpret $x \rightsquigarrow y$ in S_{j-1} as $x \leq y$.

As $v \leq u$ we have $v_i \leq u_i$ for all i. By R-thinness there is a vertex w in either $N_G(u) - N_G(v)$ or in $N_G(v) - N_G(u)$. We show now that in the first case, part (a) of (iii) is satisfied, and then that in the second case, (b) or (c) are satisfied.

Indeed, for the first case, assume $w \in N_G(u) - N_G(v)$. By Claim 6.10 the maximum neighbour u' of u is in $N_{j-1} \cap (N_G(u) - N_G(v))$. To see that (a) is satisfied, we show $v' \leq u'$ for any neighbour v' of v in N_{j-1}. Indeed, $v' \sim v$ and $u' \sim u$ give $v' \vee u' \sim v \vee u = u$. As u' is the maximal neighbour of u this gives us $v' \vee u' \leq u'$. This implies $v' \vee u' = u'$, and so $v' \leq u'$, as needed.

Now, for the second case, assume $w \in N_G(v) - N_G(u)$. We further assume that item (iii) part (c) does not hold, and then show that part (b) must hold. Indeed, if (c) does not hold, then we may assume $w \in N_{j-1} \cap (N_G(v) - N_G(u))$. As $N_{j-1} \cap N_G(v)$ is a conservative set it induces a sublattice, so has a minimum element v'. But then for any neighbour u' of u in N_{j-1} we have from $v' \sim v$ and $u' \sim u$, that $v' \wedge u' \sim v \wedge u = v$. As S_{j-1} is conservative, $v' \wedge u'$ is in S_{j-1} so is in $S_{j-1} \cap N_G(v)$. Thus $v' \wedge u' \geq v'$ which implies $u' \geq v'$. As $v' \notin N_G(u)$ we have $u' > v'$, as needed.

This completes the proof of the lemma. □

As a consequence of this Lemma, we immediately get the following.

Fact 6.11. *Let G be an R-thin graph, (G, L) be a compatible pair, and $S = S(G)$. Making $\vec{S}(\mathbf{0}_L, \mathbf{1}_L)$ reflexive, and then closing transitively, yields the lattice L.*

6.3. Formal proof and corollaries

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. If follows from Lemma 6.6 and Fact 6.11 that the following algorithm correctly returns 'YES' if the input graph G is a DL graph and 'NO' otherwise. It follows from Facts 6.2 and 6.8 that it takes only polynomial time. This proves the theorem.
Input: An R-thin reflexive graph G.
Output: 'YES' if G is a DL graph, 'NO' otherwise.
Steps:

- (i) Construct $S(G)$ of Definition 6.1.
- (ii) For each choice of vertices $\mathbf{0}$ and $\mathbf{1}$ of G, do the following.
  - (a) Construct $\vec{S}(\mathbf{0}, \mathbf{1})$ of Definition 6.7.
  - (b) Check that all edges of S have exactly one direction in $\vec{S}(\mathbf{0}, \mathbf{1})$, and that $\vec{S}(\mathbf{0}, \mathbf{1})$ is acyclic. If this is true, continue with the next step; if not, move onto the next choice of $\mathbf{0}$ and $\mathbf{1}$.
  - (c) Add loops to $\vec{S}(\mathbf{0}, \mathbf{1})$, and let L be its transitive closure.
  - (d) Check that G and L are compatible. If they are, then return 'YES' and quit; if not, move onto the next choice of $\mathbf{0}$ and $\mathbf{1}$.
- (iii) All choices of $\mathbf{0}$ and $\mathbf{1}$ have been checked and we have not found a compatible lattice. Return 'NO'. □

Here is an unexpected consequence of our algorithm.

Corollary 6.12. *If G is R-thin, then for a given choice of vertices $\mathbf{0}$ and $\mathbf{1}$, there is at most one distributive lattice L (up to isomorphism), with minimum element $\mathbf{0}$ and maximum element $\mathbf{1}$ that is compatible with G.*

With this we get the following.

Proposition 6.13. *There are distributive lattice graphs that are not tight.*

Proof. Let (G, L) be the DL pair shown with a tight but non-induced $|PPI$-embedding in Figure 4. By Example 3.1, the shown $\mathbf{0}$ and $\mathbf{1}$ are the only possible extremal elements for a lattice compatible with G. As G is R-thin, we have by Corollary 6.12 that L is the only distributive lattice (up to isomorphism) compatible with G.

The poset J_L and sub-digraph $\mathrm{red}(A^c)$ are also shown. As $\mathrm{red}(A^c)$ contains a vertex with up-degree two, we have, following remarks in Subsection 5.4, that there is no tight induced embedding of (G, L). So G has no compatible distributive lattice with which it has a tight induced embedding. □

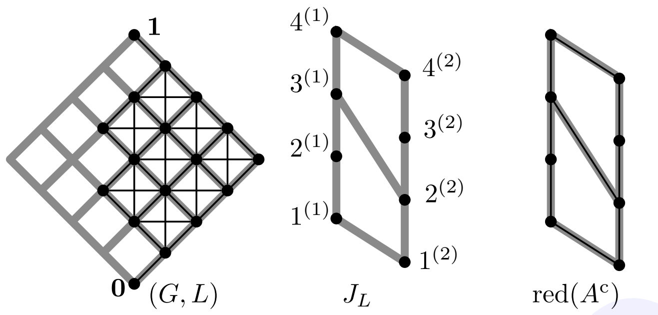


FIGURE 4. Compatible pair (G, L), poset J_L, and the graph $\text{red}(A^c)$

6.4. Non R-thin graphs

For a reflexive graph G we define a relation R on the vertex set by letting uRv if u and v have the same neighbourhood. Clearly this is an equivalence relation. The *R-thin reduction* of a graph G is the graph G^R whose vertices are the sets R and in which two sets are adjacent if there are any (and so all) edges between their member vertices.

The following shows our algorithm can be useful in showing that a non R-thin graph is not DL.

Lemma 6.14. *If a reflexive graph G is a* DL *graph, then its R-thin reduction is also* DL*.*

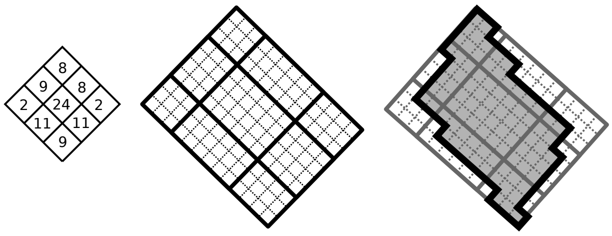
Proof. Assume that G is a reflexive DL graph that is not R-thin. We will find pairs of vertices that are identified in G^R and show that when we identify them, or reduce the number of coordinates in which they differ, we still have a DL graph. The fact that G^R is DL then follows by induction.

For some compatible L assume a tightest induced $\mathcal{GP}$-embedding $(G, L) \to (\mathcal{G}, \mathcal{P})$. Let x and y be vertices of G with the same neighbourhood. By Lemma 6.5 there is some G_j such that x_j and y_j have the same neighbourhood in G_j. We may assume that $x_j = y_j - 1$. For any vertex v in G with $v_j \geq y_j$, reduce v_j by 1. If under this reduction, two vertices now have the same co-ordinates, then identity them — they had the same neighbourhood so are identified in G^R. Clearly there is an embedding of this reduced graph into $\mathcal{G}' = \prod G_i'$ where $G_i' = G_i$ when $i \neq j$ and we get G_j' from G_j by identifying x_j and y_j. □

We conjecture the following.

Conjecture 6.15. *There is a polynomial time algorithm to decide whether or not a given reflexive graph is a* DL *graph.*

Notice that the graph G in Figure 3 is not R-thin. The vertices d and e have the same neighbourhoods. If we remove one of these vertices, then the



Diamond Tableau Divided Tableau Two walks

FIGURE 5. The Game of Conjecture 6.15

resulting lattice is distributive and is still compatible with the resulting R-thin reduction G^R. Thus the converse of Lemma 6.14 is unfortunately not true.

That said, one sees by reversing the operation in the proof of Lemma 6.14 that from an embedding of a DL pair, we can add a copy of every vertex that has the same value in some coordinate. Moreover one can argue that 'fattening' the lattice in a new dimension can be replicated in the existing dimensions. So resolving the conjecture comes down to solving a general version of the following game, which is described vaguely, but is clear from Figure 5.

Given a set of numbers in a diamond tableau, decide if one can

- divide the regions with square lines, and
- make two decreasing walks from the top to the bottom,

so that the number of divided regions in each of the original regions between the walks equals the number proscribed in the tableau. With some students [15], we show that this game has a polynomial time solution for tableaux of two dimensions.

7. A question

A partial characterisation of lattice graphs can be extracted from known literature. Indeed, it follows from [13] and [16] (see also [8]) that retracts of products of reflexive paths are exactly the reflexive graphs that admit majority, or 3-NU polymorphisms, that is, polymorphisms $f : V(G)^3 \to V(G)$ satisfying

$$f(x, y, z) = c \text{ if at least two of } x, y \text{ and } z \text{ are } c.$$

If a reflexive graph has a compatible lattice, then it also admits the following majority operation (seen, for example, in [1])

$$f(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (x \wedge z).$$

Thus all lattice graphs are retracts of products of paths. It would also be nice to see how our characterisations can be use to show this for DL graphs: that every DL graph is a retract of products of paths. In general, removing a vertex interval $[\alpha_{[i]}, \beta^{[j]}]$ is not a retraction.

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