Bull. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/BKMS.b160848
pISSN: 1015-8634 / eISSN: 2234-3016

CONTROLLABILITY FOR TRAJECTORIES OF SEMILINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Jin-Mun Jeong and Yong Han Kang

Abstract. In this paper, we first consider the existence and regularity of solutions of the semilinear control system under natural assumptions such as the local Lipschtiz continuity of nonlinear term. Thereafter, we will also establish the approximate controllability for the equation when the corresponding linear system is approximately controllable.

1. Introduction

In this paper, we are concerned with the global existence of solution and the approximate controllability for the semilinear control system:

$$\begin{cases} x^{'}(t) + Ax(t) = f(t, x(t)) + (Bu)(t), \quad t \in (0, T], \\ x(0) = x_0. \end{cases} \tag{1.1}$$

Let H and V be real Hilbert spaces such that V is a dense subspace in H. Let A be the operator associated with a sesquilinear form $a(\cdot, \cdot)$ defined on $V \times V$ satisfying Gårding's inequality:

$$(Au, v) = a(u, v), \quad u,\ v \in V$$

where V is a Hilbert space such that $V \subset H \subset V^*$. Then $-A$ generates an analytic semigroup in both H and V^*(see [18, Theorem 3.6.1]) and so the equation (1.1) may be considered as an equation in H as well as in V^*. The nonlinear operator f from $[0, T] \times V$ to H is assumed to be locally Lipschitz continuous with respect to the second variable. Let U be a Banach space of control variables and the controller operator B be a bounded linear operator from the Banach space $L^2(0, T; U)$ to $L^2(0, T; H)$. Let $x(t; f, u)$ be a solution of the equation (1.1) associated with a nonlinear term f and a control u. We will show the approximate controllability for the equation (1.1), namely that

Received October 21, 2016; Accepted February 24, 2017.
2010 *Mathematics Subject Classification.* Primary 35B37; Secondary 93C20.
Key words and phrases. approximate controllability, semilinear control system, local lipschtiz continuity, controller operator, reachable set.
This research was supported by Basic Science Research Program through the National research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2015R1D1A1A09059030).

the reachable set $R_T(f) = \{x(T; f, u) : u \in L^2(0, T; U)\}$ is a dense subset of H. This kind of equations arise naturally in biology, in physics, control engineering problem, etc.

In the first part of this paper we establish the wellposedness and regularity property for the following equation:

$$
\left\{
\begin{array}{l}
x'(t) + Ax(t) = f(t, x(t)) + k(t), \quad t \in (0, T] \\
x(0) = x_0.
\end{array}
\right.
\tag{1.2}
$$

The existence of solutions for a class of semilinear functional differential equations has been studied by many authors. Recently, Kobayashi et el. [12] introduced the notion of semigroups of locally Lipschitz operators which provide us with mild solutions to the Cauchy problem for semilinear evolution equations. The regularity for the semilinear heat equations has been developed as seen in Section 4.3.1 of Barbu [2] and [11, 16].

In this paper, we propose a different approach of the earlier works (briefly introduced in [10, 18, 22]) about the mild, strong, and classical solutions of Cauchy problems. Our approach is that results of the linear cases of Di Blasio [8] on the L^2-regularity remain valid under the above formulation of the semilinear problem (1.2).

Next, based on the regularity for (1.2), we intend to establish the approximate controllability for (1.1). Approximate controllability for semilinear control systems can be founded in [4, 9–15]. Similar considerations of linear and semilinear systems have been dealt with in many references, linear problems in the book [6] and Nakagiri [15] , the system (1.1) with the uniform bounded nonlinear term in [23], the system (1.1) with the uniform Lipschtz continuous nonlinear term in [11, 14, 17, 24]. However there are few papers treating the systems with local Lipschipz continuity, we can just find a recent article Wang [21]. Among these literatures, in [14, 21], they assumed that the semigroup $S(t)$ generated by A is compact in order to guarantee the compactness of the solution mapping, and the approximate controllability for the equation (1.1) was investigated.

In this paper, in order to show that the main result of [14] is extended to the nonlinear differential equation, we assume that the embedding $D(A) \subset V$ is compact. Then by virtue of the result in Aubin [1], we can take advantage of the fact that the solution mapping $u \in L^2(0, T; U) \mapsto x(T; f, u)$ is compact.

Under natural assumptions such as the local Lipschtiz continuity of nonlinear term, we obtain the approximate controllability for the equation (1.1) when the corresponding linear system is approximately controllable.

The paper is organized as follows. In Section 2, the results of general linear evolution equations besides notations and assumptions are stated. In Section 3, we will obtain that the regularity for parabolic linear equations can also be applicable to (1.2) with nonlinear terms satisfying local Lipschitz continuity. The approach used here is similar to that developed in [11, 18] on the general semilnear evolution equations, which is an important role to extend the theory of practical nonlinear partial differential equations. Thereafter, we investigate the approximate controllability for the problem (1.1) in Section 4. In the proofs of the main theorems, we need some compactness hypothesis. So we make the natural assumption that the embedding $D(A) \subset V$ is compact instead of the compact property of semigroup used in [7, 14]. Finally we give a simple example to which our main result can be applied.

2. Regularity for linear equations

If H is identified with its dual space we may write $V \subset H \subset V^*$ densely and the corresponding injections are continuous. The norm on V, H and V^* will be denoted by $||\cdot||$, $|\cdot|$ and $||\cdot||_*$, respectively. The duality pairing between the element v_1 of V^* and the element v_2 of V is denoted by (v_1, v_2), which is the ordinary inner product in H if $v_1, v_2 \in H$.

For $l \in V^*$ we denote (l, v) by the value $l(v)$ of l at $v \in V$. The norm of l as element of V^* is given by

$$||l||_* = \sup_{v \in V} \frac{|(l, v)|}{||v||}.$$

Therefore, we assume that V has a stronger topology than H and, for brevity, we may regard that

$$||u||_* \le |u| \le ||u||, \quad \forall u \in V. \tag{2.1}$$

Let $a(\cdot, \cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\text{Re } a(u, u) \ge \omega_1 ||u||^2 - \omega_2 |u|^2, \tag{2.2}$$

where $\omega_1 > 0$ and ω_2 is a real number. Let A be the operator associated with this sesquilinear form:

$$(Au, v) = a(u, v), \quad u, \ v \in V.$$

Then $-A$ is a bounded linear operator from V to V^* by the Lax-Milgram Theorem. The realization of A in H which is the restriction of A to

$$D(A) = \{u \in V : Au \in H\}$$

is also denoted by A. From the following inequalities

$$\omega_1 ||u||^2 \le \text{Re}\, a(u, u) + \omega_2 |u|^2 \le C|Au|\,|u| + \omega_2 |u|^2 \le \max\{C, \omega_2\} ||u||_{D(A)} |u|,$$

where

$$||u||_{D(A)} = (|Au|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A)$, it follows that there exists a constant $C_0 > 0$ such that

$$||u|| \le C_0 ||u||_{D(A)}^{1/2} |u|^{1/2}. \tag{2.3}$$

Thus we have the following sequence

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*, \tag{2.4}$$

where each space is dense in the next one which continuous injection.

Lemma 2.1. *With the notations* (2.3), (2.4), *we have*

$$(V, V^*)_{1/2,2} = H,$$
$$(D(A), H)_{1/2,2} = V,$$

where $(V, V^*)_{1/2,2}$ *denotes the real interpolation space between* V *and* V^* (*Section 1.3.3 of* [19]).

It is also well known that A generates an analytic semigroup $S(t)$ in both H and V^*. For the sake of simplicity we assume that $\omega_2 = 0$ and hence the closed half plane $\{\lambda : \operatorname{Re}\lambda \geq 0\}$ is contained in the resolvent set of A.

If X is a Banach space, $L^2(0,T;X)$ is the collection of all strongly measurable square integrable functions from $(0,T)$ into X and $W^{1,2}(0,T;X)$ is the set of all absolutely continuous functions on $[0,T]$ such that their derivative belongs to $L^2(0,T;X)$. $C([0,T];X)$ will denote the set of all continuously functions from $[0,T]$ into X with the supremum norm. If X and Y are two Banach space, $\mathcal{L}(X,Y)$ is the collection of all bounded linear operators from X into Y, and $\mathcal{L}(X,X)$ is simply written as $\mathcal{L}(X)$. Let the solution spaces $\mathcal{W}(T)$ and $\mathcal{W}_1(T)$ of strong solutions be defined by

$$\mathcal{W}(T) = L^2(0,T;D(A)) \cap W^{1,2}(0,T;H),$$
$$\mathcal{W}_1(T) = L^2(0,T;V) \cap W^{1,2}(0,T;V^*).$$

Here, we note that by using interpolation theory, we have

$$\mathcal{W}(T) \subset C([0,T];V), \quad \mathcal{W}_1(T) \subset C([0,T];H).$$

Thus, there exists a constant $M_0 > 0$ such that

$$||x||_{C([0,T];V)} \leq M_0 ||x||_{\mathcal{W}(T)}, \quad ||x||_{C([0,T];H)} \leq M_0 ||x||_{\mathcal{W}_1(T)}. \tag{2.5}$$

The semigroup generated by $-A$ is denoted by $S(t)$ and there exists a constant M such that

$$|S(t)| \leq M, \quad ||s(t)||_* \leq M.$$

The following Lemma is from Lemma 3.6.2 of [18].

Lemma 2.2. *There exists a constant* $M > 0$ *such that the following inequalities hold for all* $t > 0$ *and every* $x \in H$ *or* V^* :

$$|S(t)x| \leq M t^{-1/2} ||x||_*, \quad ||S(t)x|| \leq M t^{-1/2} |x|.$$

First of all, consider the following linear system

$$\begin{cases} x'(t) + Ax(t) = k(t), \\ x(0) = x_0. \end{cases} \tag{2.6}$$

By virtue of Theorem 3.3 of [8] (or Theorem 3.1 of [11], [18]), we have the following result on the corresponding linear equation of (2.6).

Lemma 2.3. *Suppose that the assumptions for the principal operator A stated above are satisfied. Then the following properties hold*:

1) *For $x_0 \in V = (D(A), H)_{1/2,2}$ (see Lemma 2.1) and $k \in L^2(0, T; H)$, $T > 0$, there exists a unique solution x of (2.6) belonging to $\mathcal{W}(T) \subset C([0, T]; V)$ and satisfying*

$$||x||_{\mathcal{W}(T)} \le C_1(||x_0|| + ||k||_{L^2(0,T;H)}), \tag{2.7}$$

where C_1 is a constant depending on T.

2) *Let $x_0 \in H$ and $k \in L^2(0, T; V^*)$, $T > 0$. Then there exists a unique solution x of (2.6) belonging to $\mathcal{W}_1(T) \subset C([0, T]; H)$ and satisfying*

$$||x||_{\mathcal{W}_1(T)} \le C_1(|x_0| + ||k||_{L^2(0,T;V^*)}), \tag{2.8}$$

where C_1 is a constant depending on T.

Lemma 2.4. *Suppose that $k \in L^2(0, T; H)$ and $x(t) = \int_0^t S(t-s)k(s)ds$ for $0 \le t \le T$. Then there exists a constant C_2 such that*

$$||x||_{L^2(0,T;D(A))} \le C_1||k||_{L^2(0,T;H)}, \tag{2.9}$$

$$||x||_{L^2(0,T;H)} \le C_2 T||k||_{L^2(0,T;H)}, \tag{2.10}$$

and

$$||x||_{L^2(0,T;V)} \le C_2\sqrt{T}||k||_{L^2(0,T;H)}. \tag{2.11}$$

Proof. The assertion (2.9) is immediately obtained by (2.7). Since

$$||x||^2_{L^2(0,T;H)} = \int_0^T \left|\int_0^t S(t-s)k(s)ds\right|^2 dt \le M \int_0^T \left(\int_0^t |k(s)|ds\right)^2 dt$$
$$\le M \int_0^T t \int_0^t |k(s)|^2 ds dt \le M\frac{T^2}{2} \int_0^T |k(s)|^2 ds$$

it follows that

$$||x||_{L^2(0,T;H)} \le T\sqrt{M/2}||k||_{L^2(0,T;H)}.$$

From (2.3), (2.9), and (2.10) it holds that

$$||x||_{L^2(0,T;V)} \le C_0\sqrt{C_1 T}(M/2)^{1/4}||k||_{L^2(0,T;H)}.$$

So, if we take a constant $C_2 > 0$ such that

$$C_2 = \max\{\sqrt{M/2}, C_0\sqrt{C_1}(M/2)^{1/4}\},$$

the proof is complete. ∎

3. Semilinear differential equations

Let f be a nonlinear mapping from V into H.

Assumption (F). There exists a function $L : \mathbb{R}_+ \to \mathbb{R}$ such that $L(r_1) \leq L(r_2)$ for $r_1 \leq r_2$ and

$$|f(t,x)| \leq L(r), \quad |f(t,x) - f(t,y)| \leq L(r)||x-y||$$

hold for any $t \in [0,T]$, $||x|| \leq r$ and $||y|| \leq r$.

From now on, we establish the following results on the local solvability of the following equation;

$$\begin{cases} x^{'}(t) + Ax(t) = f(t,x(t)) + k(t), \quad t \in (0,T] \\ x(0) = x_0. \end{cases} \tag{3.1}$$

Let us rewrite $(Fx)(t) = f(t,x(t))$ for each $x \in L^2(0,T;V)$. Then there is a constant, denoted again by $L(r)$, such that

$$||Fx||_{L^2(0,T;H)} \leq L(r)\sqrt{T}, \quad ||Fx_1 - Fx_2||_{L^2(0,T;H)} \leq L(r)||x_1 - x_2||_{L^2(0,T;V)}$$

hold for $x_1,\ x_2 \in B_r(T) = \{x \in L^2(0,T;V) : ||x||_{L^2(0,T;V)} \leq r\}$.

Theorem 3.1. *Let Assumption* (F) *be satisfied. Assume that* $x_0 \in H$, $k \in L^2(0,T;V^*)$. *Then, there exists a time* $T_0 \in (0,T)$ *such that the equation* (3.1) *admits a solution*

$$x \in L^2(0,T_0;V) \cap W^{1,2}(0,T_0;V^*) \subset C([0,T_0];H). \tag{3.2}$$

Proof. For a solution of (3.1) in the wider sense, we are going to find a solution of the following integral equation

$$x(t) = S(t)x_0 + \int_0^t S(t-s)\{(Fx)(s) + k(s)\}ds. \tag{3.3}$$

To prove a local solution, we will use the successive iteration method. First, put

$$x_0(t) = S(t)x_0 + \int_0^t S(t-s)k(s)ds$$

and define $x_{j+1}(t)$ as

$$x_{j+1}(t) = x_0(t) + \int_0^t S(t-s)(Fx_j)(s)ds. \tag{3.4}$$

By virtue of Lemma 2.3, we have $x_0(\cdot) \in \mathcal{W}_1(t)$, so that

$$||x_0||_{\mathcal{W}_1(t)} \leq C_1(|x_0| + ||k||_{L^2(0,t;V^*)}), \tag{3.5}$$

where C_1 is a constant in Lemma 2.3. Choose $r > C_1(|x_0| + ||k||_{L^2(0,t;V^*)})$. Putting $p(t) = \int_0^t S(t-s)(Fx_0)(s)ds$, by (2.11) of Lemma 2.4, we have

$$||p||_{L^2(0,t;V)} \leq C_2\sqrt{t}||Fx_0||_{L^2(0,t;H)} \leq C_2 L(r)t, \tag{3.6}$$

so that, from(3.5) and (3.6),

$$||x_1||_{L^2(0,t;V)} \le r + C_2 L(r)t \le 2r$$

for any $0 \le t \le r(C_2L(r))^{-1}$. By induction, it can be shown that for all $j = 1, 2, \dots$

$$||x_j||_{L^2(0,t;V)} \le 2r, \quad 0 \le t \le r(C_2L(r))^{-1}. \tag{3.7}$$

Hence, from the equation

$$x_{j+1}(t) - x_j(t) = \int_0^t S(t-s)\{f(t, x_j(s)) - f(t, x_{j-1}(s))\}ds$$

From (2.11), (3.7) and Assumption (F), we can observe that the inequality

$$\begin{aligned} ||x_{j+1} - x_j||_{L^2(0,t;V)} &\le C_2\sqrt{t}||Fx_j - Fx_{j-1}||_{L^2(0,t;H)} \\ &\le (C_2L(2r)\sqrt{t})^j ||x_1 - x_0||_{L^2(0,t;V)} \end{aligned}$$

holds for any $0 \le t \le r(C_2L(2r))^{-1}$. Choose $T_0 > 0$ satisfying

$$T_0 < \min\{r(C_2L(r))^{-1}, \, r(C_2L(2r))^{-1}, \, (C_2L(2r))^{-2}\}.$$

Then $\{x_j\}$ is strongly convergent to a function x in $L^2(0, T_0; V)$ uniformly on $0 \le t \le T_0$. By letting $j \to \infty$ in (3.4), we obtain (3.3) and thereby have proved (3.2). Next we prove the uniqueness of the solution. Let $\epsilon > 0$ be given. For $\epsilon \le t \le T_0$, set

$$x^\epsilon(t) = S(t)x_0 + \int_0^{t-\epsilon} S(t-s)\{f(s, x^\epsilon(s)) + k(s)\}ds. \tag{3.8}$$

Then we have $x^\epsilon \in \mathcal{W}_1(T_0)$ and for $x^\epsilon, \ y^\epsilon \in B_r(T_0)$ which is a ball with radius r in $L^2(0, T_0; V)$, since

$$\begin{aligned} x(t) - x^\epsilon(t) = &\int_0^t S(t-s)\{f(s, x(s)) - f(s, x^\epsilon(s))\}ds \\ &+ \int_{t-\epsilon}^t S(t-s)\{f(s, x^\epsilon(s)) + k(s)\}ds, \end{aligned}$$

with aid of Lemma 2.4, we have

$$\begin{aligned} &||x_{j+1} - x^\epsilon_{j+1}||_{L^2(0,T_0;V)} \\ &\le (C_2L(r)\sqrt{T_0})^j ||x_0 - x_0^\epsilon||_{L^2(0,T_0;V)} + C_2\sqrt{\epsilon}(L(r) + ||k||_{L^2(0,T_0;H)}). \end{aligned}$$

Hence, it holds that $x^\epsilon \to x$ as $\epsilon \to 0$ in $L^2(0, T_0; V)$. Suppose y is another solution of (3.1) and y_ϵ is defined as (3.8). Let $x^\epsilon, \ y^\epsilon \in B_r$. Then from Lemma 2.2, it follows that

$$\begin{aligned} &||x^\epsilon - y^\epsilon||_{L^2(0,T_0;V)} \\ &\le \left[\int_0^{T_0} \Big|\Big|\int_0^{s-\epsilon} S(s-\tau)\{(Fx^\epsilon)(\tau) - (Fy^\epsilon)(\tau)\}d\tau\Big|\Big|^2 ds\right]^{1/2} \end{aligned}$$

$$\le M\left[\int_0^{T_0}\left(\int_0^{s-\epsilon}(s-\tau)^{-1/2}|Fx^\epsilon(\tau)-Fy^\epsilon(\tau)|d\tau\right)^2 ds\right]^{1/2}$$
$$\le ML(r)\left[\int_0^{T_0}\int_0^{s-\epsilon}(s-\tau)^{-1}d\tau\int_0^{s-\epsilon}||x^\epsilon(\tau)-y^\epsilon(\tau)||^2 d\tau ds\right]^{1/2}$$
$$\le ML(r)\log\frac{T_0}{\epsilon}\int_0^{T_0}||x^\epsilon-y^\epsilon||_{L^2(0,s;V)}ds,$$

so that by using Gronwall's inequality, independently of ϵ, we get $x^\epsilon = y^\epsilon$ in $L^2(0,T_0;V)$, which proves the uniqueness of solution of (3.1) in $\mathcal{W}_1(T_0)$. □

From now on, we give a norm estimation of the solution of (3.1) and establish the global existence of solutions with the aid of norm estimations.

Theorem 3.2. *Under the assumption* (F) *for the nonlinear mapping f, there exists a unique solution x of* (3.1) *such that*

$$x\in\mathcal{W}_1(T)\equiv L^2(0,T;V)\cap W^{1,2}(0,T;V^*)\subset C([0,T];H),\quad T>0$$

for any $x_0\in H$, $k\in L^2(0,T;V^*)$. Moreover, there exists a constant C_3 such that

$$||x||_{\mathcal{W}_1}\le C_3(1+|x_0|+||k||_{L^2(0,T;V^*)}), \tag{3.9}$$

where C_3 is a constant depending on T.

Proof. Let x be a solution of (3.1) on $[0,T_0]$ and let y be the solution of the following linear functional differential equation parabolic type;

$$\begin{cases} y'(t)+Ay(t)=k(t), & t\in(0,T_0],\\ y(0)=x_0.\end{cases}$$

Then we have

$$\begin{cases} d(x-y)(t)/dt+A(x-y)(t)=(Fx)(t), & t\in(0,T_0],\\ (x-y)(0)=0.\end{cases}$$

From Theorem 3.1, it follows that $x,\ y$ belong to $L^2(0,T_0;V)$, and so that we assume $x,\ y\in B_r(T_0)=\{x\in L^2(0,T_0;V): ||x||_{L^2(0,T_0;V)}\le r\}$. Let $T_1\le T_0$ be such that

$$C_0C_1L(r)(T_1/\sqrt{2})^{1/2}<1. \tag{3.10}$$

Then, noting that $||Fx||_{L^2(0,T_1;H)}\le L(r)\sqrt{T_1}$ and 1) of Lemma 2.3, it holds

$$\begin{aligned}||x-y||_{\mathcal{W}(T_1)} &\le C_1||Fx||_{L^2(0,T_1;H)}\\ &\le C_1(L(r)||x||_{L^2(0,T_1;V)}+||f(\cdot,0)||_{L^2(0,T_1;H)})\\ &\le C_1L(r)\big(||x-y||_{L^2(0,T_1:V)}+||y||_{L^2(0,T_1;V)}+\sqrt{T_1}\big),\end{aligned}$$

and so we obtain

$$||x-y||_{L^2(0,T_1;V)}\le C_0||x-y||^{\frac12}_{L^2(0,T_1;D(A))}||x-y||^{\frac12}_{L^2(0,T_1;H)}$$

$$\le C_0\|x-y\|^{\frac{1}{2}}_{L^2(0,T_1;D(A))}\{\frac{T_1}{\sqrt{2}}\|x-y\|_{W^{1,2}(0,T_1;H)}\}^{\frac{1}{2}}$$
$$\le C_0(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\|x-y\|_{L^2(0,T_1;D(A))\cap W^{1,2}(0,T_1;H)}$$
$$\le C_0C_1L(r)(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}(\|x-y\|_{L^2(0,T_1;V)}+\|y\|_{L^2(0,T_1;V)}+\sqrt{T_1}).$$

Therefore we have

$$\|x-y\|_{L^2(0,T_1;V)}\le\frac{C_0C_1L(r)(T_1/\sqrt{2})^{1/2}}{1-C_0C_1L(r)(T_1/\sqrt{2})^{1/2}}(\|y\|_{L^2(0,T_1;V)}+\sqrt{T_1}),$$

$$\|x\|_{L^2(0,T_1;V)}\le\frac{\|y\|_{L^2(0,T_1;V)}+C_0C_1L(r)T_1 2^{1/4}}{1-C_0C_1L(r)T_1/2^{1/2}}. \tag{3.11}$$

and hence with the aid of Lemma 2.3 and (3.11) we obtain

$$\begin{aligned}\|x\|_{\mathcal{W}_1(T_1)} &\le C_1(|x_0|+\|Fx\|_{L^2(0,T_1;V^*)}+\|k\|_{L^2(0,T_1;V^*)})\\ &\le C_1(|x_0|+L(r)\|x\|_{L^2(0,T_1;V)}+L(r)\sqrt{T_1}+\|k\|_{L^2(0,T_1;V^*)})\\ &\le C_1\{L(r)\sqrt{T_1}+|x_0|+\|k\|_{L^2(0,T_1;V^*)}\\ &\qquad+\frac{L(r)(\|y\|_{L^2(0,T_1;V)}+C_0C_1L(r)T_1 2^{1/4})}{1-C_0C_1L(r)T_1/2^{1/2}}\} \qquad (3.12)\\ &\le C_1\big[L(r)\sqrt{T_1}+|x_0|+\|k\|_{L^2(0,T_1;V^*)}\\ &\qquad+\frac{L(r)\{C_1(|x_0|+\|k\|_{L^2(0,T_1;V^*)})+C_0C_1L(r)T_1 2^{1/4}\}}{1-C_0C_1L(r)T_1/2^{1/2}}\big]\\ &\le C_3(1+|x_0|+\|k\|_{L^2(0,T_1;V^*)})\end{aligned}$$

for some constant C_3. Now from (2.5) and (3.12), it follows that

$$|x(T_1)|\le\|x\|_{C([0,T_1];H)}\le M_0\|x\|_{\mathcal{W}_1(T_1)}\le M_0C_3(1+|x_0|+\|k\|_{L^2(0,T_1;V^*)}).$$

So, we can solve the equation in $[T_1,2T_1]$ and obtain an analogous estimate to (3.12). Since the condition (3.10) is independent of initial values, the solution of (3.1) can be extended the internal $[0,nT_1]$ for a natural number n, i.e., for the initial $u(nT_1)$ in the interval $[nT_1,(n+1)T_1]$, as analogous estimate (3.12) holds for the solution in $[0,(n+1)T_1]$. □

Remark 3.3. Let Assumption (F) be satisfied and $(x_0,k)\in D(A)\times L^2(0,T;H)$. Then by the argument of the proof of Theorem 3.2 term by term, we also obtain that there exists a unique solution x of (3.1) such that

$$x\in L^2(0,T;D(A))\cap W^{1,2}(0,T;H)\subset C([0,T];V).$$

Moreover, there exists a constant C_3 such that

$$\|x\|_{\mathcal{W}(T)}\le C_3(1+\|x_0\|+\|k\|_{L^2(0,T;H)}),$$

where C_3 is a constant depending on T.

The following inequality is refereed to as the Young inequality.

Lemma 3.4 (Young inequality)**.** *Let $a>0,\ b>0$ and $1/p+1/q=1$ where $1\le p<\infty$ and $1<q<\infty$. Then for every $\lambda>0$ one has*

$$ab \le \frac{\lambda^p a^p}{p} + \frac{b^q}{\lambda^q q}\ .$$

From the following result, we obtain that the solution mapping is continuous, which is useful for physical applications of the given equation.

Theorem 3.5. *Let the assumption* (F) *be satisfied and $(x_0,k)\in V\times L^2(0,T;H)$. Then the solution x of the equation* (3.1) *belongs to $x\in\mathcal{W}\equiv L^2(0,T;D(A))\cap W^{1,2}(0,T;H)$ and the mapping*

$$V\times L^2(0,T;H)\ni(x_0,k)\mapsto x\in\mathcal{W}(T)$$

is Lipschitz continuous.

Proof. From Theorem 3.2, it follows that if $(x_0,k)\in V\times L^2(0,T;H)$ then x belongs to $\mathcal{W}(T)$. Let $(x_{0i},k_i)\in V\times L^2(0,T;H)$ and $x_i\in\mathcal{W}(T)$ be the solution of (3.1) with (x_{0i},k_i) in place of (x_0,k) for $i=1,\ 2$. Hence, we assume that x_i belongs to a ball $B_r(T)=\{y\in\mathcal{W}(T):||y||_{\mathcal{W}(T)}\le r\}$. By virtue of 1) in Lemma 2.3 and Assumption (F) we have

$$\begin{aligned}&||x_1-x_2||_{\mathcal{W}(T)}\\ &\le C_1\{||x_{01}-x_{02}||+||Fx_1-Fx_2||_{L^2(0,T;H)}+||k_1-k_2||_{L^2(0,T;H)}\}\\ &\le C_1\{||x_{01}-x_{02}||+||k_1-k_2||_{L^2(0,T;H)}+L(r)||x_1-x_2||_{L^2(0,T:V)}\}.\end{aligned}\tag{3.13}$$

Since

$$x_1(t)-x_2(t)=x_{01}-x_{02}+\int_0^t(\dot{x}_1(s)-\dot{x}_2(s))ds,$$

we get

$$||x_1-x_2||_{L^2(0,T;H)}\le\sqrt{T}|x_{01}-x_{02}|+\frac{T}{\sqrt{2}}||x_1-x_2||_{W^{1,2}(0,T;H)}.$$

Hence, by (2.3) and Lemma 3.1, we get

(3.14)

$$\begin{aligned}&||x_1-x_2||_{L^2(0,T;V)}\\ &\le C_0||x_1-x_2||^{1/2}_{L^2(0,T;D(A))}||x_1-x_2||^{1/2}_{L^2(0,T;H)}\\ &\le C_0||x_1-x_2||^{1/2}_{L^2(0,T;D(A))}\\ &\quad\times\{T^{1/4}|x_{01}-x_{02}|^{1/2}+(\frac{T}{\sqrt{2}})^{1/2}||x_1-x_2||^{1/2}_{W^{1,2}(0,T;H)}\}\\ &\le C_0T^{1/4}|x_{01}-x_{02}|^{1/2}||x_1-x_2||^{1/2}_{L^2(0,T;D(A))}+C_0(\frac{T}{\sqrt{2}})^{1/2}||x_1-x_2||_{\mathcal{W}(T)}\end{aligned}$$

$$\leq 2^{-7/4}C_0|x_{01}-x_{02}|+2C_0(\frac{T}{\sqrt{2}})^{1/2}||x_1-x_2||_{\mathcal{W}(T)}.$$

Combining (3.13) and (3.14) we obtain

$$(3.15) \qquad \begin{aligned} ||x_1-x_2||_{\mathcal{W}(T)} \leq C_1\{||x_{01}-x_{02}||+||k_1-k_2||_{L^2(0,T;H)}\} \\ +2^{-7/4}C_0C_1L(r)|x_{01}-x_{02}| \\ +2C_0C_1(\frac{T}{\sqrt{2}})^{1/2}L(r)||x_1-x_2||_{\mathcal{W}(T)}. \end{aligned}$$

Suppose that $(x_{0i},k_i)\mapsto(x_0,k)$ in $V\times L^2(0,T;H)$, and let x_n and x be the solutions (3.1) with (x_{0i},k_i) and (x_0,k) respectively. Let $0<T_2\leq T$ be such that

$$2C_0C_1(T_2/\sqrt{2})^{1/2}L(r)<1.$$

Then by virtue of (3.15) with T replaced by T_2 we see that

$$x_n\mapsto x\in\mathcal{W}(T_2)\equiv L^2(0,T_2;D(A))\cap W^{1,2}(0,T_2;H).$$

This implies that $(x_n(T_2),(x_n)_{T_2})\mapsto(x(T_2),x_{T_2})$ in $V\times L^2(0,T;D(A))$. Hence the same argument shows that $x_n\mapsto x$ in

$$L^2(T_2,\min\{2T_2,T\};D(A))\cap W^{1,2}(T_2,\min\{2T_2,T\};H).$$

Repeating this process we conclude that $x_n\mapsto x$ in $\mathcal{W}(T)$. □

Remark 3.6. The result of Theorem 3.3 is important to apply for the control problems and the optimal control theory for technologically given cost functions. In particular, under the assumptions stated in Remark 3.1, from using the similar way to the proof of Theorem 3.3 it follows that if $(x_0,k)\in H\times L^2(0,T;V^*)$, then the solution x of the equation (3.1) belongs to $x\in\mathcal{W}_1(T)\equiv L^2(0,T;V)\cap W^{1,2}(0,T;V^*)\subset C([0,T];H)$ and the mapping

$$H\times L^2(0,T;V^*)\ni(x_0,k)\mapsto x\in\mathcal{W}_1(T)$$

is continuous.

4. Approximate controllability

In this section, we make the natural assumption that the embedding $D(A)\subset V$ is compact in order to show that the main result of K. Naito [14] is extended to the nonlinear differential equation. Let U be a Banach space of control variables. Here B is a linear bounded operator from $L^2(0,T;U)$ to $L^2(0,T;H)$, which is called a controller. Consider the following nonlinear equation.

$$(4.1) \qquad \begin{cases} x'(t)+Ax(t)=f(t,x(t))+(Bu)(t), & t\in(0,T], \\ x(0)=x_0. \end{cases}$$

Let $x(T; f, u)$ be a state value of the system (4.1) at time T corresponding to the nonlinear term f and the control u. Let $S(\cdot)$ be the analytic semigroup generated by $-A$. Then the solution $x(t; f, u)$ can be written as

$$x(t; f, u) = S(t)x_0 + \int_0^t S(t-s)\{f(s, x(s, f, u)) + (Bu)(s)\}ds,$$

and in view of Theorem 3.2

$$||x(\cdot; f, u)||_{\mathcal{W}_1(T)} \le C_3(1 + |x_0| + ||B||||u||_{L^2(0,T;U)}). \tag{4.2}$$

We define the reachable sets for the system (4.1) as follows:

$$R_T(f) = \{x(T; f, u) : u \in L^2(0, T; U)\},$$
$$R_T(0) = \{x(T; 0, u) : u \in L^2(0, T; U)\}.$$

Definition. The system (4.1) is said to be approximately controllable at time T if for every desired final state $x_1 \in H$ and $\epsilon > 0$ there exists a control function $u \in L^2(0, T; U)$ such that the solution $x(T; f, u)$ of (4.1) satisfies $|x(T; f, u) - x_1| < \epsilon$, that is, $\overline{R_T(f)} = H$ where $\overline{R_T(f)}$ is the closure of $R_T(f)$ in H.

For $k \in L^2(0, T; H)$ let y_k be the solution of equation with $B = I$.

$$\begin{cases} x'(t) + Ax(t) = f(t, x(t)) + k(t), \quad t \in (0, T] \\ x(0) = x_0. \end{cases} \tag{4.3}$$

Then, the solution of (4.3) is represented as

$$y_k(t) = S(t)x_0 + \int_0^t S(t-s)\{f(s, y_k(\cdot)) + k(s)\}ds.$$

Theorem 4.1. *Let us assume that the embedding $D(A) \subset V$ is compact. For $k \in L^2(0, T; H)$ let y_k be the solution of equation (4.3). Then the mapping $k \mapsto y_k$ is compact from $L^2(0, T; H)$ to $L^2(0, T; V) \subset L^2(0, T; H)$. Moreover, if we define the operator $\mathcal{F}$ by*

$$\mathcal{F}(k) = f(\cdot, y_k),$$

then $\mathcal{F}$ is also a compact mapping from $L^2(0, T; H)$ to itself.

Proof. If $(x_0, k) \in V \times L^2(0, T; H)$, then in view of Theorem 3.2

$$||y_k||_{\mathcal{W}_1(T)} \le C_3(1 + |x_0| + ||k||_{L^2(0,T;H)}).$$

Since $y_k \in L^2(0, T; V)$, we have $f(\cdot, y_k) \in L^2(0, T; H)$. Consequently, by 1) of Lemma 2.3, we know $y \in \mathcal{W}(T) \subset C([0, T]; V)$. Let $||y_k|| < r$. Then it holds that $||f(\cdot, y_k)||_{L^2(0,T;H)} \le L(r)\sqrt{T}$. With aid of 1) of Lemma 2.3, noting that $||y_k||_{L^2(0,T;V)} \le ||y_k||_{\mathcal{W}_1(T)}$, we have

$$\begin{aligned} ||y_k||_{\mathcal{W}(T)} &\le C_1(||x_0|| + ||f(\cdot, y_k) + k||_{L^2(0,T;H)}) \\ &\le C_1\{||x_0|| + L(r)\sqrt{T} + ||k||_{L^2(0,T;H)}\}. \end{aligned}$$

Hence if k is bounded in $L^2(0,T;H)$, then so is y_k in $\mathcal{W}(T) \equiv L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$. Since $D(A)$ is compactly embedded in V by assumption, the embedding

$$\mathcal{W}(T) \equiv L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset L^2(0,T;V)$$

is compact in view of Theorem 2 of Aubin [1]. Hence $k \mapsto y_k$ is compact from $L^2(0,T;H)$ to $L^2(0,T;V)$. Moreover, it is immediately that $\mathcal{F}$ is a compact mapping from $L^2(0,T;H)$ to itself. □

Let

$$N = \{p \in L^2(0,T;H) : \int_0^T S(T-s)p(s)ds = 0\}$$

and denote by $N^\perp$ be the orthogonal complement of N in $L^2(0,T;H)$. We denote the range of the operator B by H_B. We need the following assumption:

Assumption (A). For each $p \in L^2(0,T;H)$ there exists an element $q \in \overline{H}_B$ such that

$$\int_0^T S(T-s)p(s)ds = \int_0^T S(T-s)q(s)ds,$$

that is, $L^2(0,T;H) = \overline{H}_B + N$, where $\overline{H}_B$ is the closure of H_B in $L^2(0,T;H)$.

Here, we remark that under Assumption (A) it is known that $\overline{R_T(0)} = H$ as in [14].

Theorem 4.2. *Under Assumptions* (A) *and* (F), *and assuming in addition*

$$\limsup_{r\to\infty} (r - \sqrt{T}\sup\{L(s) : |s| \le r\}) = \infty, \tag{4.4}$$

we have

$$R_T(0) \subset \overline{R_T(f)}.$$

Therefore, if the linear system (4.3) *with* $f = 0$ *is approximately controllable at time* T, *then so is the nonlinear system* (4.1).

Proof. It will be shown that $R_T(0) \subset \overline{R_T(f)}^V$, where $\overline{R_T(f)}^V$ is the closure of $R_T(f)$ in V. For $u \in N^\perp$, let Pu be the unique minimum norm element of $\{u + N\} \cap \overline{H}_B$. Then the proof of Lemma 1 of Naito [14] can be applied to show that P is a linear and continuous operator from $N^\perp$ to $\overline{H}_B$. Let $\tilde{Y} = L^2(0,T;H)/N$ be the quotient space and the norm of a coset $\tilde{u} = u + N \in \tilde{Y}$ is defined of $||\tilde{u}|| = \inf\{|u+f| : f \in N\}$.

We define by Q the isometric isomorphism from $\tilde{Y}$ onto $N^\perp$, that is, $Q\tilde{u}$ is the minimum norm element in $\tilde{u} = \{u + f : f \in N\}$. Let

$$\tilde{\mathcal{F}}\tilde{u} = \mathcal{F}(PQ\tilde{u}) + N, \quad \forall \tilde{u} \in \tilde{Y}.$$

Then $\tilde{\mathcal{F}}$ is a compact mapping from $\tilde{Y}$ to itself by Theorem 4.1. If $(x_0, k) \in V \times L^2(0,T;H)$, we know $y \in \mathcal{W}(T) \subset C([0,T];V)$ by 1) of Lemma 2.3. Let

$$\eta = \int_0^T S(T-s)(Bv)(s)ds \in R_T(0).$$

We are going to show that for every $\epsilon > 0$ there exists w such that

$$||\eta - x(T; f, w)|| \le \epsilon.$$

Put $z = Bv$ and $r_1 = ||B||||v||_{L^2(0,T;U)}$. Then it follows that

$$\tilde{z} = z + N \in V_{r_1} = \{\tilde{x} \in \tilde{Y} : ||\tilde{x}||_{\tilde{Y}} \le r_1\}.$$

From (3.9), noting that $||y_k||_{L^2(0,T;V)} \le C_3(1+||x_0||+||k||_{L^2(0,T;H)})$, we choose a constant $r > 0$ such that

$$r \ge C_3(1 + ||x_0|| + ||k||_{L^2(0,T;H)}).$$

Then it holds that

$$||\mathcal{F}(k)||_{L^2(0,T;H)} \le L(r)\sqrt{T}, \quad ||\tilde{\mathcal{F}}(\tilde{k})||_{\tilde{Y}} \le L(r)\sqrt{T}. \tag{4.5}$$

Let

$$\mathcal{L}(r) = \sup\{L(s) : |s| \le r\}.$$

Then by the assumption (4.4), there exists $r > 0$ such that

$$\mathcal{L}(r)\sqrt{T} + r_1 < r. \tag{4.6}$$

Define an operator J from $\tilde{Y}$ to itself as

$$J(\tilde{u}) = \tilde{z} - \tilde{\mathcal{F}}\tilde{u}, \quad \tilde{u} \in \tilde{Y}. \tag{4.7}$$

Then since $\tilde{z} \in V_{r_1}$ and from (4.6) it follows that

$$||J\tilde{u}|| \le ||\tilde{z}|| + ||\tilde{\mathcal{F}}\tilde{u}|| \le r_1 + L(r)\sqrt{T} \le r_1 + \mathcal{L}(r)\sqrt{T} < r.$$

Hence, J maps bounded closed set V_r into itself. It follows from the Schauder fixed point theorem that there exists a fixed point $\tilde{u}$ of J in V_r, that is, it holds

$$\tilde{z} = \tilde{\mathcal{F}}\tilde{u} + \tilde{u}.$$

Put $u = Q\tilde{u}$ and $u_B = PQ\tilde{u}$. Then we have that $u_B = Pu$ and $u - u_B = u - Pu \in N$. Hence

$$\tilde{z} = \mathcal{F}(u_B) + u + N = \mathcal{F}(u_B) + u_B + N.$$

Therefore,

$$\begin{aligned}\eta &= \int_0^T S(T-s)(\mathcal{F}(u_B)(s) + u_B(s))ds \\ &= \int_0^T S(T-s)(f(s, y_{u_B}) + u_B(s))ds.\end{aligned}$$

Since $u_B \in \overline{H}_B$, there exists a sequence $\{v_n\} \in L^2(0,T;U)$ such that $Bv_n \mapsto u_B$ in $L^2(0,T;H)$. Then by Theorem 3.3 we have that $x(\cdot; f, v_n) \mapsto y_{u_B}$ in $L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$, and hence $x(T; f, v_n) \mapsto y_{u_B}(T) = \eta$ in V. Thus we conclude $\eta \in \overline{R_T(f)}^V$. □

Corollary 4.3. *Under Assumptions* (A) *and* (F)*, and assuming in addition that $f(\cdot,\cdot)$ is continuous and uniformly bounded, we have*

$$R_T(0) \subset \overline{R_T(f)}.$$

Remark 4.4. Let $f : [0,T] \times V \to H$ be continuous in t on $[0,T]$ and uniformly Lipschitz continuous on V. Then as seen in [11], we can prove the results of Theorem 4.2 by using the homotopic property of the Leray-Schauder degree theory(cf. [12, Theorem 4.4.3]).

Example. We consider the semilinear heat equation dealt with by Naito [14] and Zhou [24]. Let

$$H = L^2(0,\pi),\ V = H_0^1(0,\pi),\ V^* = H^{-1}(0,\pi),$$
$$a(u,v) = \int_0^\pi \frac{du(x)}{dx}\overline{\frac{dv(x)}{dx}}dx$$

and

$$A = -d^2/dx^2 \quad \text{with} \quad D(A) = \{y \in H^2(0,\pi) : y(0) = y(\pi) = 0\}.$$

We consider the following retarded functional differential equation

$$\text{(4.8)} \qquad \begin{cases} \frac{\partial}{\partial t}x(t,y) + Ax(t,y) = f(t,x(t,y)) + Bu(t), \\ x(t,0) = x(t,\pi) = 0, \quad t > 0 \\ x(0,y) = x_0(y). \end{cases}$$

The eigenvalue and the eigenfunction of A are $\lambda_n = -n^2$ and $\phi_n(x) = \sin nx$, respectively. Let

$$U = \{\sum_{n=2}^{\infty} u_n\phi_n : \sum_{n=2}^{\infty} u_n^2 < \infty\},$$
$$Bu = 2u_2\phi_1 + \sum_{n=2}^{\infty} u_n\phi_n, \quad \text{for} \quad u = \sum_{n=2}^{\infty} u_n \in U,$$
$$T > 0.$$

In [14] Naito showed that the operator B is one to one, $R(B)$ is closed and $L^2(0,T) = R(B) + N$, where $R(B)$ is the range of the operator B. It follows that the operator B satisfies Assumption (A).

We assume that the nonlinear operator $f : [0,T] \times V \to H$ is continuous and there is a constant $0 < \gamma < 1$ and a function $k \in L^2[0,T]$ such that

$$|f(t,x)| \le k(s)||x||^\gamma, \quad \forall (t,x) \in [0,T] \times V.$$

Hence Assumption (F) and (4.4) are satisfied. Therefore, all the conditions stated Theorem 4.2 are satisfied. So the semilinear system (4.8) is approximately controllable at time T.

References

[1] J. P. Aubin, *Un thèoréme de compacité*, C. R. Acad. Sci. **256** (1963), 5042–5044.
[2] V. Barbu, *Analysis and Control of Nonlinear Infinite Dimensional Systems*, Academic Press Limited, 1993.
[3] A. E. Bashirov and N. I. Mahmudov, *On concepts of controllability for deterministic and stochastic systems*, SIAM J. Control Optim. **37** (1999), no. 6, 1808–1821.
[4] M. Benchohra, L. Górniewicz, S. K. Ntouyas, and A. Ouahab, *Controllability results for implusive functional differential inclusions*, Rep. Math. Phys. **54** (2004), no. 2, 211–228.
[5] M. Benchohra and A. Ouahab, *Controllability results for functional semilinear differential inclusion in Frechet spaces*, Nonlinear Anal. **61** (2005), no. 3, 405–423.
[6] R. F. Curtain and H. Zwart, *An Introduction to Infinite Dimensional Linear Systems Theory*, Springer-Velag, New-York, 1995.
[7] J. P. Dauer and N. I. Mahmudov, *Exact null controllability of semilinear integrodifferential systems in Hilbert spaces*, J. Math. Anal. Appl. **299** (2004), no. 2, 322–333.
[8] G. Di Blasio, K. Kunisch, and E. Sinestrari, *L^2-regularity for parabolic partial integrodifferential equations with delay in the highest-order derivatives*, J. Math. Anal. Appl. **102** (1984), no. 1, 38–57.
[9] L. Górniewicz, S. K. Ntouyas, and D. O'Reran, *Controllability of semilinear differential equations and inclusions via semigroup theory in Banach spaces*, Rep. Math. Phys. **56** (2005), no. 3, 437–470.
[10] M. L. Heard, *An abstract parabolic Volterra integro-differential equation*, J. Appl. Math. **17** (1981), 175–202.
[11] J. M. Jeong, Y. C. Kwun, and J. Y. Park, *Approximate controllability for semilinear retarded functional differential equations*, J. Dynam. Control Systems **5** (1999), no. 3, 329–346.
[12] Y. Kobayashi, T. Matsumoto, and N. Tanaka, *Semigroups of locally Lipschitz operators associated with semilinear evolution equations*, J. Math. Anal. Appl. **330** (2007), no. 2, 1042–1067.
[13] N. I. Mahmudov, *Approximate controllability of semilinear deterministic and stochastic evolution equations in abstract spaces*, SIAM J. Control Optim. **42** (2006), 175–181.
[14] K. Naito, *Controllability of semilinear control systems dominated by the linear part*, SIAM J. Control Optim. **25** (1987), no. 3, 715–722.
[15] S. Nakagiri, *Controllability and identifiability for linear time-delay systems in Hilbert space*, Control theory of distributed parameter systems and applications, Lecture Notes in Control and Inform. Sci., 159, Springer, Berlin, 1991.
[16] A. Pazy, *Semigroups of Linear Operators and Applications to partial Differential Equations*, Springer-Verlag, New-York, Inc. 1983.
[17] N. Sukavanam and N. K. Tomar, *Approximate controllability of semilinear delay control system*, Nonlinear Funct. Anal. Appl. **12** (2007), no. 1, 53–59.
[18] H. Tanabe, *Equations of Evolution, Pitman-London*, 1979.
[19] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, 1978.
[20] L. Wang, *Approximate controllability and approximate null controllability of semilinear systems*, Commun. Pure Appl. Anal. **5** (2006), no. 4, 953–962.
[21] ______, *Approximate controllability for integrodifferential equations and multiple delays*, J. Optim. Theory Appl. **143** (2009), no. 1, 185–206.
[22] G. Webb, *Continuous nonlinear perturbations of linear accretive operator in Banach spaces*, J. Funct. Anal. **10** (1972), 191–203.
[23] M. Yamamoto and J. Y. Park, *Controllability for parabolic equations with uniformly bounded nonlinear terms*, J. Optim. Theory Appl. **66** (1990), no. 3, 515–32.
[24] H. X. Zhou, *Approximate controllability for a class of semilinear abstract equations*, SIAM J. Control Optim. **21** (1983), no. 4, 551–565.

Jin-Mun Jeong
Department of Applied Mathematics
Pukyong National University
Busan 608-737, Korea
E-mail address: jmjeong@pknu.ac.kr

Yong Han Kang
Institute of Liberal Education
Catholic University of Daegu
Daegu 712-702, Korea
E-mail address: yonghann@cu.ac.kr