

SOME RESULTS OF f -BIHARMONIC MAPS INTO A RIEMANNIAN MANIFOLD OF NON-POSITIVE SECTIONAL CURVATURE

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ABSTRACT. The authors investigate f -biharmonic maps $u : (M, g) \rightarrow (N, h)$ from a Riemannian manifold into a Riemannian manifold with non-positive sectional curvature, and derive that if $\int_M f^p |\tau(u)|^p dv_g < \infty$, $\int_M |\tau(u)|^2 dv_g < \infty$ and $\int_M |du|^2 dv_g < \infty$, then u is harmonic. When u is an isometric immersion, the authors also get that if u satisfies some integral conditions, then it is minimal. These results give an affirmative partial answer to conjecture 4 (generalized Chen's conjecture for f -biharmonic submanifolds).

1. Introduction

In the past several decades harmonic maps have played a central role in geometry and analysis. Let (M^m, g) and (N^n, h) be Riemannian manifolds of dimensions m, n and $u : (M^m, g) \rightarrow (N^n, h)$ be a smooth map. The energy of u is defined by $E(u) = \int_M \frac{|du|^2}{2} dv_g$, where dv_g is the volume element on (M^m, g) . Harmonic maps are the critical maps of $E(\cdot)$. The Euler-Lagrange equation of harmonic maps is $\tau(u) = 0$, where $\tau(u)$ is called the tension field of u . p -harmonic maps [19], exponentially harmonic maps [16], F -harmonic maps and f -harmonic maps are extensions to harmonic maps and many results have been carried out (for instance, see [1–3, 10, 24, 33]).

In 1983, J. Eells and L. Lemaire [13] proposed the problem to consider biharmonic maps which are critical points of the bi-energy functional $E_2(u) = \int_M \frac{|\tau(u)|^2}{2} dv_g$. We see that biharmonic maps are a generalization of harmonic maps. In 1986, G. Y. Jiang [21] studied the first and the second variational formulas of the bi-energy. There have been many studies on biharmonic maps (for instance, see [4–6, 11, 20, 25, 26, 32]). To further generalize the notion of

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harmonic maps, Y. B. Han and S. X. Feng [17] introduced the F -bienergy functional $E_{F,2}(u) = \int_M F\left(\frac{|\tau(u)|^2}{2}\right)dv_g$. The critical points of F -bienergy $E_{F,2}(u)$ are called F -biharmonic maps. If $F(u) = (2u)^{\frac{p}{2}}$, we have p -bienergy functional $E_{p,2}(u) = \int_M |\tau(u)|^p dv_g$. If $F(u) = e^u$, we have exponential bienergy functional $E_{e,2}(u) = \int_M e^{\frac{|\tau(u)|^2}{2}} dv_g$.

A. Lichnerowicz [23] (see also [12]) introduced and studied f -harmonic maps between Riemannian manifolds. The study of f -harmonic maps comes from a physical motivation, since in physics f -harmonic maps can be viewed as stationary solutions to the inhomogeneous Heisenberg spin system (see [22]). W. J. Lu [27] introduced the following functional:

$$E_{2,f}(u) = \int_M f \frac{|\tau(u)|^2}{2} dv_g,$$

where $f : (M, g) \rightarrow (0, +\infty)$ is a smooth function. A map u is called an f -biharmonic map if it is a critical point of the f -bienergy functional.

Recently, N. Nakauchi et al. [31] showed that every biharmonic map of a complete Riemannian manifold into a Riemannian manifold of non-positive curvature whose energy and bi-energy are finite must be harmonic. S. Maeta [29] obtained that biharmonic maps from a complete Riemannian manifold into a non-positive curved manifold with finite $(a+2)$ -bienergy $\int_M |\tau(u)|^{a+2} dv_g < \infty$ ($a \geq 0$) and energy are harmonic. Y. B. Han and W. Zhang [18] obtained that p -biharmonic maps from a complete manifold into a non-positive curved manifold with finite $(a+p)$ -bienergy $\int_M |\tau(u)|^{a+p} dv_g < \infty$ and energy are harmonic. In this paper, we first obtain the following results:

Theorem 1.1 (cf. Theorem 3.1). *Let $u : (M^m, g) \rightarrow (N^n, h)$ be an f -biharmonic map from a compact Riemannian manifold (M^m, g) without boundary into a Riemannian manifold (N^n, h) with non-positive sectional curvature, then u is harmonic.*

Theorem 1.2 (cf. Theorem 3.3). *Let $u : (M^m, g) \rightarrow (N^n, h)$ be an f -biharmonic map from a complete Riemannian manifold (M^m, g) into a Riemannian manifold (N^n, h) with non-positive sectional curvature and let $p \geq 2$ be a non-negative real constant.*

(i) *If*

$$\int_M f^p |\tau(u)|^p dv_g < \infty, \int_M |\tau(u)|^2 dv_g < \infty, \text{ and } \int_M |du|^2 dv_g < \infty,$$

then u is harmonic.

(ii) *If $\text{Vol}(M, g) = \infty$, and $\int_M f^p |\tau(u)|^p dv_g < \infty$, then u is harmonic.*

Chen's conjecture is the most interesting problem in the biharmonic theory. In 1988, Chen [9] raised the following problem:

Conjecture 1. *Any biharmonic submanifold in E^n is minimal.*

There are some affirmative partial answers to Conjecture 1.

Then Chen's conjecture was generalized as follows ([8]): Any biharmonic submanifolds in a Riemannian manifold with non-positive sectional curvature is minimal. There are also some affirmative partial answers to this Conjecture (for instance, see [7, 17, 30, 31]).

Motivated by Chen's conjecture, Y. B. Han [15] proposed the following conjecture:

Conjecture 2. *Any p -biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal.*

Some affirmative partial answers to Conjecture 2 were proved in [15, 18, 28].

Y. B. Han [16] also proposed the following conjecture:

Conjecture 3. *Any exponentially biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal.*

Some affirmative partial answers to Conjecture 3 were proved in [16].

For f -biharmonic submanifolds, it is natural to consider the following conjecture.

Conjecture 4. *Any f -biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal.*

For f -biharmonic submanifolds, we obtain some results:

Theorem 1.3 (cf. Theorem 4.1). *Let $u : (M, g) \rightarrow (N, h)$ be an f -biharmonic isometric immersion from a complete Riemannian manifold into a Riemannian manifold (N, h) with non-positive sectional curvature and let p, q be two real constants satisfying $2 \leq p < \infty$ and $0 < q \leq p < \infty$. If*

$$\int_M f^p |\vec{H}|^q dv_g < \infty,$$

then u is minimal.

Theorem 1.4 (cf. Theorem 4.2). *Let $u : (M, g) \rightarrow (N, h)$ be an f -biharmonic isometric immersion from a complete Riemannian manifold into a Riemannian manifold (N, h) with non-positive sectional curvature. If*

$$\int_{B_r(x_0)} f^p dv_g \leq C_0(1+r)^s,$$

for some positive integer s , C_0 independent of r and $p \geq 2$, then u is minimal.

Theorem 1.5 (cf. Theorem 4.3). *Let $u : (M, g) \rightarrow (N, h)$ be an f -biharmonic isometric immersion from a complete Riemannian manifold into a Riemannian manifold (N, h) whose sectional curvature is smaller than $-\varepsilon$ for some constant $\varepsilon > 0$ and $\int_{B_r(x_0)} |f\vec{H}|^p dv_g$ ($p \geq 2$) is of at most polynomial growth of r , Then u is minimal.*

Theorem 1.6 (cf. Theorem 4.4). *Let $u : (M, g) \rightarrow (N, h)$ be a complete ε -supper f -biharmonic submanifold in (N, h) for $\varepsilon > 0$. If*

$$\int_M |f\vec{H}|^p dv_g < \infty,$$

where $p \geq 2$, then u is minimal.

2. Preliminaries

In this section we give some necessary notations and terminologies about harmonic maps, biharmonic maps, f -biharmonic maps and f -biharmonic submanifolds.

Let $u : (M^m, g) \rightarrow (N^n, h)$ be a smooth map from an m -dimensional Riemannian manifold (M^m, g) to an n -dimensional Riemannian manifold (N^n, h) . The energy of u is defined by

$$E(u) = \int_M \frac{|du|^2}{2} dv_g,$$

where dv_g is the volume element on (M^m, g) .

The Euler-Lagrange equation of harmonic maps is $\tau(u) = \sum_{i=1}^m \{\tilde{\nabla}_{e_i} du(e_i) - du(\nabla_{e_i} e_i)\} = 0$ where ∇ is the Levi-Civita connection on (M^m, g) and $\tilde{\nabla}$ is the induced Levi-Civita connection of the pullback bundle $u^{-1}TN$. $\{e_i\}_{i=1}^m$ is an orthonormal frame field on (M^m, g) . If $\tau(u) = 0$, then u is called a harmonic map.

In 1983, J. Eells and L. Lemaire [13] proposed the problem to consider the bi-energy functional:

$$E_2(u) = \int_M \frac{|\tau(u)|^2}{2} dv_g.$$

Then, in 1986, G. Y. Jiang [21] obtained the first and the second variational formulas of the bi-energy functional. The Euler-Lagrange equation of the bi-energy functional is

$$\tau_2(u) = -\tilde{\Delta}(\tau(u)) - \sum_i R^N(\tau(u), du(e_i))du(e_i) = 0,$$

where $R^N(X, Y) = [{}^N\nabla_X, {}^N\nabla_Y] - {}^N\nabla_{[X, Y]}$ is the curvature operator on (N, h) . If $\tau_2(u) = 0$, then u is called a biharmonic map.

To generalize the notation of biharmonic maps, W. J. Lu [27] studied the f -bienergy functional

$$E_{2,f}(u) = \int_M f(x) \frac{|\tau(u)|^2}{2} dv_g,$$

where $f : (M, g) \rightarrow (0, +\infty)$ is a smooth function. The Euler-Lagrange equation of $E_{2,f}$ is

$$\tau_{2,f}(u) = -\tilde{\Delta}(f\tau(u)) - \sum_i R^N(f\tau(u), du(e_i))du(e_i) = 0.$$

If $\tau_{2,f}(u) = 0$, then u is called an f -biharmonic map.

Now we briefly recall the submanifold theory. Let $u : (M^m, g) \rightarrow (N^{m+t}, h)$ be an isometric immersion from an m -dimensional Riemannian manifold (M^m, g) into an $(m+t)$ -dimensional Riemannian manifold (N^{m+t}, h) . The second fundamental form $B : TM \otimes TM \rightarrow NM$ is defined by

$$B(X, Y) = {}^N \nabla_X Y - \nabla_X Y, \quad X, Y \in \Gamma(TM).$$

The shape operator $A_\xi : TM \rightarrow TM$ for a unit normal vector field ξ on M is defined by

$${}^N \nabla_X \xi = -A_\xi X + \nabla_X^\perp \xi, \quad X \in \Gamma(TM), \xi \in \Gamma(T^\perp M),$$

where ∇^\perp denotes the normal connection on the normal bundle of M in N . It's well known that B and A_ξ are related by

$$\langle B(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

For any $x \in M$, the mean curvature vector field \vec{H} of M at x is given by

$$\vec{H} = \frac{1}{m} \sum_i B(e_i, e_i).$$

If an isometric immersion $u : (M, g) \rightarrow (N, h)$ is f -biharmonic, then M is called an f -biharmonic submanifold in N . In this case, $\tau(u) = m\vec{H}$. We know that M is an f -biharmonic submanifold in N if and only if

$$(1) \quad -\tilde{\Delta}(f\vec{H}) - \sum_i R^N(f\vec{H}, e_i)e_i = 0.$$

From (2), we obtain the sufficient and necessary condition for M to be an f -biharmonic submanifold in N as follows:

$$(2) \quad \Delta^\perp(f\vec{H}) - \sum_i B(e_i, A_{f\vec{H}}e_i) + [\sum_i R^N(f\vec{H}, e_i)e_i]^\perp = 0,$$

$$(3) \quad Tr_g(\nabla_{(\cdot)} A_{f\vec{H}}(\cdot)) + Tr_g[A_{\nabla^\perp(f\vec{H})}(\cdot)] - [\sum_i R^N(f\vec{H}, e_i)e_i]^\top = 0.$$

We also need the following lemma.

Lemma 2.1 (Gaffney [14]). *Let (M, g) be a complete Riemannian manifold. If a C^1 1-form α satisfies that $\int_M |\alpha| dv_g < \infty$ and $\int_M (\delta\alpha) dv_g < \infty$, or equivalently, a C^1 vector X defined by $\alpha(Y) = \langle X, Y \rangle$ satisfies that $\int_M |X| dv_g < \infty$ and $\int_M \operatorname{div}(X) dv_g < \infty$, then $\int_M (\delta\alpha) dv_g = \int_M \operatorname{div}(X) dv_g = 0$.*

3. f -biharmonic maps in a Riemannian manifold of non-positive sectional curvature

In this section, we obtain some results as follows:

Theorem 3.1. *Let $u : (M^m, g) \rightarrow (N^n, h)$ be an f -biharmonic map from a compact Riemannian manifold (M^m, g) without boundary into a Riemannian manifold (N^n, h) with non-positive sectional curvature, then u is harmonic.*

Proof. From (1), we have

$$\begin{aligned} \frac{1}{2}\Delta|f\tau(u)|^2 &= |\tilde{\nabla}(f\tau(u))|^2 + \langle \tilde{\Delta}[f\tau(u)], f\tau(u) \rangle \\ &= |\tilde{\nabla}(f\tau(u))|^2 - \sum_i \langle R^N(f\tau(u), du(e_i))du(e_i), f\tau(u) \rangle \\ &\geq |\tilde{\nabla}(f\tau(u))|^2. \end{aligned}$$

From Green theorem and the compactness of (M, g) , we have

$$(4) \quad 0 = \int_M \frac{1}{2}\Delta|f\tau(u)|^2 dv_g = \int_M |\tilde{\nabla}(f\tau(u))|^2 dv_g.$$

Then, for every $X \in \Gamma(TM)$, we have

$$\tilde{\nabla}_X|f\tau(u)| = 0.$$

Let $Y = \sum_i h(du(e_i), f\tau(u))e_i$, we have

$$\begin{aligned} (5) \quad \operatorname{div}(Y) &= \sum_k g(\nabla_{e_k} Y, e_k) \\ &= \sum_k [h(\tilde{\nabla}_{e_k} du(e_k), f\tau(u)) - h(du(\nabla_{e_k} e_k), f\tau(u))] \\ &= h(\tau(u), f\tau(u)) = f|\tau(u)|^2. \end{aligned}$$

From (6), we have

$$0 = \int_M \operatorname{div}(Y) dv_g = \int_M f|\tau(u)|^2 dv_g.$$

Since $f > 0$ in M , so we have $\tau(u) = 0$. \square

Corollary 3.2. *Any f -biharmonic function in a compact manifold (M, g) without boundary is constant.*

Proof. From Theorem 3.1, u is an f -biharmonic function if and only if u is a harmonic function. On the other hand, any harmonic function in a compact manifold (M, g) is constant, so we have $u = C$. \square

Theorem 3.3. *Let $u : (M^m, g) \rightarrow (N^n, h)$ be an f -biharmonic map from a complete Riemannian manifold (M^m, g) into a Riemannian manifold (N^n, h) with non-positive sectional curvature and let $p \geq 2$ be a non-negative real constant.*

(i) *If*

$$\int_M f^p |\tau(u)|^p dv_g < \infty, \int_M |\tau(u)|^2 dv_g < \infty \quad \text{and} \quad \int_M |du|^2 dv_g < \infty,$$

then u is harmonic.

(ii) *If $\operatorname{Vol}(M, g) = \infty$ and $\int_M f^p |\tau(u)|^p dv_g < \infty$, then u is harmonic.*

Proof. Take a fixed point $x_0 \in M$ and for every $r > 0$, let us consider the following cut off function $\lambda(x)$ on M :

$$(6) \quad \begin{cases} 0 \leq \lambda(x) \leq 1, & x \in M, \\ \lambda(x) = 1, & x \in B_r(x_0), \\ \lambda(x) = 0, & x \in M - B_{2r}(x_0), \\ |\nabla \lambda| \leq \frac{C}{r}, & x \in M, \end{cases}$$

where $B_r(x_0) = \{x \in M : d(x, x_0) < r\}$, C is a positive constant and d is the distance of M . From (1), we have

$$(7) \quad \begin{aligned} & \int_M \langle -\tilde{\Delta}(f\tau(u)), \lambda^2 |f\tau(u)|^{p-2} f\tau(u) \rangle dv_g \\ &= \int_M \lambda^2 f^p |\tau(u)|^{p-2} \sum_i \langle R^N(\tau(u), du(e_i)) du(e_i), \tau(u) \rangle dv_g \leq 0, \end{aligned}$$

where the inequality follows from the sectional curvature of (N, h) is non-positive. From (8), we have

$$(8) \quad \begin{aligned} 0 &\geq \int_M \langle -\tilde{\Delta}(f\tau(u)), \lambda^2 |f\tau(u)|^{p-2} f\tau(u) \rangle dv_g \\ &= \int_M \langle \tilde{\nabla}(f\tau(u)), \tilde{\nabla}(\lambda^2 |f\tau(u)|^{p-2} f\tau(u)) \rangle dv_g \\ &= \int_M \sum_{i=1}^m [\langle \tilde{\nabla}_{e_i}(f\tau(u)), \tilde{\nabla}_{e_i}(\lambda^2 |f\tau(u)|^{p-2} f\tau(u)) \rangle] dv_g \\ &= \int_M \sum_{i=1}^m \langle \tilde{\nabla}_{e_i}(f\tau(u)), 2\lambda e_i(\lambda) |f\tau(u)|^{p-2} f\tau(u) \\ &\quad + \lambda^2 e_i[|f\tau(u)|^{p-2}] f\tau(u) + \lambda^2 |f\tau(u)|^{p-2} \tilde{\nabla}_{e_i}[f\tau(u)] \rangle dv_g \\ &= \int_M \sum_{i=1}^m 2\lambda e_i(\lambda) |f\tau(u)|^{p-2} \langle \tilde{\nabla}_{e_i}[f\tau(u)], f\tau(u) \rangle dv_g \\ &\quad + \int_M \sum_{i=1}^m (p-2) \lambda^2 |f\tau(u)|^{p-4} \langle \tilde{\nabla}_{e_i}[f\tau(u)], f\tau(u) \rangle^2 dv_g \\ &\quad + \int_M \sum_{i=1}^m \lambda^2 |f\tau(u)|^{p-2} \langle \tilde{\nabla}_{e_i}[f\tau(u)], \tilde{\nabla}_{e_i}[f\tau(u)] \rangle dv_g \\ &\geq \int_M \sum_{i=1}^m 2\lambda e_i(\lambda) |f\tau(u)|^{p-2} \langle \tilde{\nabla}_{e_i}[f\tau(u)], f\tau(u) \rangle dv_g \\ &\quad + \int_M \sum_{i=1}^m \lambda^2 |f\tau(u)|^{p-2} \langle \tilde{\nabla}_{e_i}[f\tau(u)], \tilde{\nabla}_{e_i}[f\tau(u)] \rangle dv_g, \end{aligned}$$

where the inequality follows from

$$\int_M \sum_{i=1}^m \lambda^2 |f\tau(u)|^{p-4} \langle \tilde{\nabla}_{e_i}[f\tau(u)], f\tau(u) \rangle^2 dv_g \geq 0.$$

From (9), we have

$$(9) \quad \begin{aligned} & \int_M \sum_{i=1}^m \lambda^2 |f\tau(u)|^{p-2} \langle \tilde{\nabla}_{e_i}[f\tau(u)], \tilde{\nabla}_{e_i}[f\tau(u)] \rangle dv_g \\ & \leq - \int_M \sum_{i=1}^m 2\lambda e_i(\lambda) |f\tau(u)|^{p-2} \langle \tilde{\nabla}_{e_i}[f\tau(u)], f\tau(u) \rangle dv_g. \end{aligned}$$

By using Young's inequality, we have

$$(10) \quad \begin{aligned} & - \int_M \sum_{i=1}^m 2\lambda e_i(\lambda) |f\tau(u)|^{p-2} \langle \tilde{\nabla}_{e_i}[f\tau(u)], f\tau(u) \rangle dv_g \\ & \leq \frac{1}{2} \int_M \sum_{i=1}^m \lambda^2 |f\tau(u)|^{p-2} |\tilde{\nabla}_{e_i}[f\tau(u)]|^2 dv_g + 2 \int_M |\nabla\lambda|^2 f^p |\tau(u)|^p dv_g. \end{aligned}$$

From (10) and (11), we have

$$(11) \quad \begin{aligned} & \int_M \sum_{i=1}^m \lambda^2 |f\tau(u)|^{p-2} \langle \tilde{\nabla}_{e_i}[f\tau(u)], \tilde{\nabla}_{e_i}[f\tau(u)] \rangle dv_g \\ & \leq 4 \int_M |\nabla\lambda|^2 f^p |\tau(u)|^p dv_g \leq \frac{4C^2}{r^2} \int_M f^p |\tau(u)|^p dv_g. \end{aligned}$$

By assumption $\int_M f^p |\tau(u)|^p dv_g < \infty$, letting $r \rightarrow \infty$ in (12), we have

$$\int_M \sum_{i=1}^m f^{p-2} |\tau(u)|^{p-2} \langle \tilde{\nabla}_{e_i}[f\tau(u)], \tilde{\nabla}_{e_i}[f\tau(u)] \rangle dv_g = 0.$$

So we obtain that $f|\tau(u)|$ is constant. If $|\tau(u)| \neq 0$, we get

$$\int_M f^p |\tau(u)|^p = |f\tau(u)|^p \text{Vol}(M) = \infty,$$

which yields a contradiction. So we have $|\tau(u)| = 0$, i.e., u is harmonic. We derive that (ii) is tenable.

For (i), we assume that

$$\int_M f^p |\tau(u)|^p dv_g < \infty, \int_M |\tau(u)|^2 dv_g < \infty, \int_M |du|^2 dv_g < \infty.$$

We define a 1-form

$$(12) \quad \alpha(X) = |f\tau(u)|^{\frac{p}{2}-1} \langle du(X), f\tau(u) \rangle,$$

where $X \in \Gamma(TM)$. We note that

$$\begin{aligned}
 (13) \quad \int_M |\alpha| dv_g &= \int_M \left[\sum_{i=1}^m |\alpha(e_i)|^2 \right]^{\frac{1}{2}} dv_g \\
 &= \int_M \left\{ \sum_{i=1}^m [|\mathcal{f}\tau(u)|^{\frac{p}{2}-1} \langle du(e_i), \mathcal{f}\tau(u) \rangle]^2 \right\}^{\frac{1}{2}} dv_g \\
 &\leq \int_M |\mathcal{f}\tau(u)|^{\frac{p}{2}} |du| dv_g \leq \left[\int_M f^p |\tau(u)|^p dv_g \right]^{\frac{1}{2}} \left[\int_M |du|^2 dv_g \right]^{\frac{1}{2}} < \infty.
 \end{aligned}$$

We compute

$$\begin{aligned}
 -\delta\alpha &= \sum_{i=1}^m (\nabla_{e_i} \alpha)(e_i) = \sum_{i=1}^m [\nabla_{e_i} \alpha(e_i) - \alpha(\nabla_{e_i} e_i)] \\
 &= \sum_{i=1}^m \nabla_{e_i} [|\mathcal{f}\tau(u)|^{\frac{p}{2}-1} \langle du(e_i), \mathcal{f}\tau(u) \rangle] - \sum_{i=1}^m |\mathcal{f}\tau(u)|^{\frac{p}{2}-1} \langle du(\nabla_{e_i} e_i), \mathcal{f}\tau(u) \rangle \\
 &= \sum_{i=1}^m |\mathcal{f}\tau(u)|^{\frac{p}{2}-1} \langle \tilde{\nabla}_{e_i} du - du(\nabla_{e_i} e_i), \mathcal{f}\tau(u) \rangle = |\mathcal{f}\tau(u)|^{\frac{p}{2}} |\tau(u)|,
 \end{aligned}$$

where the third equality follows from that $|\mathcal{f}\tau(u)|$ is constant and $\tilde{\nabla}_X[\mathcal{f}\tau(u)] = 0$, for all $X \in \Gamma(TM)$. We have

$$\int_M (-\delta\alpha) dv_g = \int_M |\mathcal{f}\tau(u)|^{\frac{p}{2}} |\tau(u)| dv_g \leq \left[\int_M f^p |\tau(u)|^p dv_g \right]^{\frac{1}{2}} \left[\int_M |\tau(u)|^2 dv_g \right]^{\frac{1}{2}}.$$

From $\int_M f^p |\tau(u)|^p dv_g < \infty$ and $\int_M |\tau(u)|^2 dv_g < \infty$, we know the function $-\delta\alpha$ is also integrable over M .

From this and (14), applying Lemma 2.1 for the 1-form α , we have

$$0 = \int_M (-\delta\alpha) dv_g = \int_M f^{\frac{p}{2}} |\tau(u)|^{\frac{p}{2}+1} dv_g.$$

So we have $\tau(u) = 0$, i.e., u is harmonic. \square

4. f -biharmonic submanifolds in a Riemannian manifold of non-positive sectional curvature

Theorem 4.1. *Let $u : (M, g) \rightarrow (N, h)$ be an f -biharmonic isometric immersion from a complete Riemannian manifold into a Riemannian manifold (N, h) with non-positive sectional curvature and let p, q be two real constants satisfying $2 \leq p < \infty$ and $0 < q \leq p < \infty$. If*

$$\int_M f^p |\vec{H}|^q dv_g < \infty,$$

then u is minimal.

Proof. From (3), we have

$$\begin{aligned}
\Delta|f\vec{H}|^2 &= \Delta\langle f\vec{H}, f\vec{H} \rangle = 2\langle \Delta^\perp(f\vec{H}), f\vec{H} \rangle + 2|\nabla^\perp(f\vec{H})|^2 \\
&= 2|\nabla^\perp(f\vec{H})|^2 + 2\sum_{i=1}^m \langle B(A_{f\vec{H}}e_i, e_i), f\vec{H} \rangle - 2\sum_{i=1}^m \langle R^N(f\vec{H}, e_i)e_i, f\vec{H} \rangle \\
(14) \quad &\geq 2|\nabla^\perp(f\vec{H})|^2 + 2\sum_{i=1}^m \langle B(A_{f\vec{H}}e_i, e_i), f\vec{H} \rangle,
\end{aligned}$$

where the inequality follows from the sectional curvature of N is non-positive. Now we proof the following inequality:

$$(15) \quad \sum_{i=1}^m \langle B(A_{f\vec{H}}e_i, e_i), f\vec{H} \rangle \geq mf^2|\vec{H}|^4.$$

Let $x \in M$, if $\vec{H} = 0$, we are done. If $\vec{H}(x) \neq 0$, we have at x ,

$$\begin{aligned}
\sum_{i=1}^m \langle B(A_{f\vec{H}}e_i, e_i), f\vec{H} \rangle &= \sum_{i=1}^m f^2|\vec{H}|^2 \langle B(A_{\frac{f\vec{H}}{|\vec{H}|}}e_i, e_i), \frac{\vec{H}}{|\vec{H}|} \rangle \\
&= \sum_{i=1}^m f^2|\vec{H}|^2 \langle A_{\frac{f\vec{H}}{|\vec{H}|}}e_i, A_{\frac{f\vec{H}}{|\vec{H}|}}e_i \rangle = \sum_{i,j=1}^m f^2|\vec{H}|^2 |\langle B(e_i, e_j), \frac{\vec{H}}{|\vec{H}|} \rangle|^2 \geq mf^2|\vec{H}|^4.
\end{aligned}$$

From (15) and (16), we have

$$(16) \quad \Delta|f\vec{H}|^2 \geq 2|\nabla^\perp(f\vec{H})|^2 + 2mf^2|\vec{H}|^4.$$

Take a fixed point $x_0 \in M$ and for every $r > 0$, let us consider the following cut off function $\lambda(x)$ on M :

$$(17) \quad \begin{cases} 0 \leq \lambda(x) \leq 1, & x \in M, \\ \lambda(x) = 1, & x \in B_r(x_0), \\ \lambda(x) = 0, & x \in M - B_{2r}(x_0), \\ |\nabla\lambda| \leq \frac{C}{r}, & x \in M, \end{cases}$$

where $B_r(x_0) = \{x \in M : d(x, x_0) < r\}$, C is a positive constant and d is the distance of M . From (17), we have

$$\begin{aligned}
(18) \quad & - \int_M \nabla(\lambda^{a+4}|f\vec{H}|^a) \nabla|f\vec{H}|^2 dv_g = \int_M \lambda^{a+4}|f\vec{H}|^a \Delta|f\vec{H}|^2 dv_g \\
& \geq 2 \int_M \lambda^{a+4}|f\vec{H}|^a |\nabla^\perp(f\vec{H})|^2 dv_g + 2m \int_M \lambda^{a+4}|f\vec{H}|^a f^2|\vec{H}|^4 dv_g,
\end{aligned}$$

where a is a positive constant to be determined later. On the other hand, we have

$$\begin{aligned}
 & - \int_M \nabla(\lambda^{a+4}|f\vec{H}|^a)\nabla|f\vec{H}|^2 dv_g \\
 & = -2(a+4) \int_M \lambda^{a+3}\nabla\lambda|f\vec{H}|^a\langle\nabla^\perp(f\vec{H}), f\vec{H}\rangle dv_g \\
 (19) \quad & - 2a \int_M \lambda^{a+4}|f\vec{H}|^{a-2}\langle\nabla^\perp(f\vec{H}), f\vec{H}\rangle^2 dv_g \\
 & \leq -2(a+4) \int_M \lambda^{a+3}\nabla\lambda|f\vec{H}|^a\langle\nabla^\perp(f\vec{H}), f\vec{H}\rangle dv_g.
 \end{aligned}$$

From (19) and (20), we have

$$\begin{aligned}
 & 2 \int_M \lambda^{a+4}|f\vec{H}|^a|\nabla^\perp(f\vec{H})|^2 dv_g + 2m \int_M \lambda^{a+4}|f\vec{H}|^a f^2 |\vec{H}|^4 dv_g \\
 & \leq -2(a+4) \int_M \lambda^{a+3}\nabla\lambda|f\vec{H}|^a\langle\nabla^\perp(f\vec{H}), f\vec{H}\rangle dv_g \\
 (20) \quad & \leq \int_M \lambda^{a+4}|f\vec{H}|^a|\nabla^\perp(f\vec{H})|^2 dv_g + (a+4)^2 \int_M \lambda^{a+2}|f\vec{H}|^{a+2}|\nabla\lambda|^2 dv_g.
 \end{aligned}$$

So we have

$$\begin{aligned}
 & \int_M \lambda^{a+4}|f\vec{H}|^a|\nabla^\perp(f\vec{H})|^2 dv_g + 2m \int_M \lambda^{a+4}|f\vec{H}|^a f^2 |\vec{H}|^4 dv_g \\
 (21) \quad & \leq (a+4)^2 \int_M \lambda^{a+2} f^{a+2} |\vec{H}|^{a+2} |\nabla\lambda|^2 dv_g.
 \end{aligned}$$

From Young's inequality, we have

$$\begin{aligned}
 & (a+4)^2 \int_M f^{a+2} \lambda^{a+2} |\vec{H}|^{a+2} |\nabla\lambda|^2 dv_g \\
 & = (a+4)^2 \int_M f^{a+2} \lambda^s |\vec{H}|^s \lambda^{a+2-s} |\vec{H}|^{a+2-s} |\nabla\lambda|^2 dv_g \\
 & \leq \int_M \lambda^{a+4} |\vec{H}|^{a+4} f^{a+2} dv_g \\
 (22) \quad & + C(a, s) \int_M f^{a+2} \lambda^{(a+2-s)\frac{a+4}{a+4-s}} |\vec{H}|^{(a+2-s)\frac{a+4}{a+4-s}} |\nabla\lambda|^{2\frac{a+4}{a+4-s}} dv_g,
 \end{aligned}$$

where $s \in (0, a+2)$ and $C(a, s)$ is a constant depending on a, s . From (22) and (23), we have

$$\begin{aligned}
 & \int_M \lambda^{a+4}|f\vec{H}|^a|\nabla^\perp(f\vec{H})|^2 dv_g + (2m-1) \int_M f^{a+2} \lambda^{a+4} |\vec{H}|^{a+4} dv_g \\
 & \leq C(a, s) \int_M f^{a+2} \lambda^{(a+2-s)\frac{a+4}{a+4-s}} |\vec{H}|^{(a+2-s)\frac{a+4}{a+4-s}} |\nabla\lambda|^{2\frac{a+4}{a+4-s}} dv_g \\
 (23) \quad & \leq C(a, s) \left(\frac{C}{r}\right)^{2\frac{a+4}{a+4-s}} \int_M f^{a+2} \lambda^{(a+2-s)\frac{a+4}{a+4-s}} |\vec{H}|^{(a+2-s)\frac{a+4}{a+4-s}} dv_g.
 \end{aligned}$$

We know that when s varies from 0 to $a+2$, then $(a+2-s)\frac{a+4}{a+4-s}$ varies from $a+2$ to 0. Let $q = (a+2-s)\frac{a+4}{a+4-s}$, then $q \in (0, a+2)$. Let $p = a+2$, from $\int_M f^p |\vec{H}|^q dv_g < \infty$, $2 \leq p < \infty$ and $0 < q \leq p < \infty$, set $r \rightarrow \infty$ in (24), we have

$$\int_M |f\vec{H}|^a |\nabla^\perp(f\vec{H})|^2 dv_g + (2m-1) \int_M f^{a+2} |\vec{H}|^{a+4} dv_g = 0.$$

So we have $\vec{H} = 0$. \square

Theorem 4.2. *Let $u : (M, g) \rightarrow (N, h)$ be an f -biharmonic isometric immersion from a complete Riemannian manifold into a Riemannian manifold (N, h) with non-positive sectional curvature. If*

$$(24) \quad \int_{B_r(x_0)} f^p dv_g \leq C_0(1+r)^s$$

for some positive integer s , C_0 independent of r and $p \geq 2$, then u is minimal.

Proof. From (21), we have

$$(25) \quad \begin{aligned} & 2 \int_M \lambda^{a+4} |f\vec{H}|^a |\nabla^\perp(f\vec{H})|^2 dv_g + 2m \int_M \lambda^{a+4} |f\vec{H}|^a f^2 |\vec{H}|^4 dv_g \\ & \leq -2(a+4) \int_M \lambda^{a+3} \nabla \lambda |f\vec{H}|^a \langle \nabla^\perp(f\vec{H}), F'(\frac{m^2 |\vec{H}|^2}{2}) \vec{H} \rangle dv_g. \end{aligned}$$

From Young's inequality, we have

$$(26) \quad \begin{aligned} & -2(a+4) \int_M \lambda^{a+3} \nabla \lambda |f\vec{H}|^a \langle \nabla^\perp(f\vec{H}), f\vec{H} \rangle dv_g \\ & \leq \int_M \lambda^{a+4} |f\vec{H}|^a |\nabla^\perp(f\vec{H})|^2 dv_g + \int_M \lambda^{a+4} f^{a+2} |\vec{H}|^{a+4} dv_g \\ & + C(a) \int_M f^{a+2} |\nabla \lambda|^{a+4} dv_g, \end{aligned}$$

where $C(a)$ is a constant depending on a . From (26) and (27), we have

$$(27) \quad \begin{aligned} & \int_M \lambda^{a+4} |f\vec{H}|^a |\nabla^\perp(f\vec{H})|^2 dv_g + \int_M (2m-1) \lambda^{a+4} f^{a+2} |\vec{H}|^{a+4} dv_g \\ & \leq C(a) \int_M f^{a+2} |\nabla \lambda|^{a+4} dv_g \leq C(a) \frac{C^{a+4}}{r^{a+4}} \int_{B_{2r}(x_0)} f^{a+2} dv_g \\ & \leq C(a) C^{a+4} C_0 \frac{(1+2r)^s}{r^{a+4}}. \end{aligned}$$

Let a be big enough and $r \rightarrow \infty$, then we finish the proof. \square

Theorem 4.3. *Let $u : (M, g) \rightarrow (N, h)$ be an f -biharmonic isometric immersion from a complete Riemannian manifold into a Riemannian manifold (N, h) whose sectional curvature is smaller than $-\varepsilon$ for some constant $\varepsilon > 0$ and $\int_{B_r(x_0)} |f\vec{H}|^p dv_g (p \geq 2)$ is of at most polynomial growth of r . Then u is minimal.*

Proof. From the equation (3), we have

$$\begin{aligned}
 \Delta|f\vec{H}|^2 &= \Delta\langle f\vec{H}, f\vec{H} \rangle = 2\langle \Delta^\perp(f\vec{H}), f\vec{H} \rangle + 2|\nabla^\perp[f\vec{H}]|^2 \\
 &= 2|\nabla^\perp(f\vec{H})|^2 + 2\sum_{i=1}^m \langle B(A_{f\vec{H}}e_i, e_i), f\vec{H} \rangle - 2\sum_{i=1}^m \langle R^N(f\vec{H}, e_i)e_i, f\vec{H} \rangle \\
 &\geq 2|\nabla^\perp(f\vec{H})|^2 + 2m|\vec{H}|^4 f^2 + 2m\varepsilon|f\vec{H}|^2 \\
 &\geq 2|\nabla^\perp(f\vec{H})|^2 + 2m\varepsilon|f\vec{H}|^2,
 \end{aligned}$$

that is

$$(28) \quad \Delta|f\vec{H}|^2 \geq 2|\nabla^\perp(f\vec{H})|^2 + 2m\varepsilon|f\vec{H}|^2.$$

From (29), we have

$$\begin{aligned}
 & - \int_M \nabla[\lambda^2|f\vec{H}|^a] \nabla|f\vec{H}|^2 dv_g = \int_M [\lambda^2|f\vec{H}|^a] \Delta|f\vec{H}|^2 dv_g \\
 (29) \quad & \geq 2 \int_M \lambda^2|f\vec{H}|^a |\nabla^\perp(f\vec{H})|^2 dv_g + 2m\varepsilon \int_M \lambda^2|f\vec{H}|^{a+2} dv_g,
 \end{aligned}$$

where λ is given by (18) and a is a nonnegative constant. We also have

$$\begin{aligned}
 & - \int_M \nabla[\lambda^2|f\vec{H}|^a] \nabla|f\vec{H}|^2 dv_g \\
 &= -4 \int_M \lambda \nabla \lambda |f\vec{H}|^a \langle \nabla^\perp(f\vec{H}), f\vec{H} \rangle dv_g \\
 & \quad - 2a \int_M \lambda^2 |f\vec{H}|^{a-2} \langle \nabla^\perp(f\vec{H}), f\vec{H} \rangle^2 dv_g \\
 &\leq -4 \int_M \lambda \nabla \lambda |f\vec{H}|^a \langle \nabla^\perp(f\vec{H}), f\vec{H} \rangle dv_g \\
 &\leq 2 \int_M \lambda^2 |f\vec{H}|^a |\nabla^\perp(f\vec{H})|^2 dv_g + 2 \int_M |f\vec{H}|^{a+2} |\nabla \lambda|^2 dv_g \\
 &\leq 2 \int_M \lambda^2 |f\vec{H}|^a |\nabla^\perp(f\vec{H})|^2 dv_g + 2 \frac{C^2}{r^2} \int_{B_{2r}(x_0) - B_r(x_0)} |f\vec{H}|^{a+2} dv_g \\
 (30) \quad &\leq 2 \int_M \lambda^2 |f\vec{H}|^a |\nabla^\perp(f\vec{H})|^2 dv_g + 2 \frac{C^2}{r^2} \int_{B_{2r}(x_0)} |f\vec{H}|^{a+2} dv_g.
 \end{aligned}$$

From (30) and (31), we have

$$2m\varepsilon \int_{B_r(x_0)} |f\vec{H}|^{a+2} dv_g \leq 2 \frac{C^2}{r^2} \int_{B_{2r}(x_0)} |f\vec{H}|^{a+2} dv_g.$$

Letting $g(r) = \int_{B_r(x_0)} |f\vec{H}|^{a+2} dv_g$, we have $g(r) \leq \frac{C_1}{r^2} g(2r)$ where $C_1 = \frac{C^2}{m\varepsilon}$. Then we know $g(r) \leq \frac{C_2}{r^{2n}} g(2^n r)$, where C_2 is a constant independent of r . From the assumption, we know $g(r) \leq C_2(1 + 2^{ns} r^s)$ for some integer $s > 0$. When r

is big enough, we have $g(r) \leq \frac{C_2^2(1+2^{ns}r^s)}{r^{2n}}$. Set $2n > s$, then $\lim_{r \rightarrow \infty} g(r) = 0$, so $\vec{H} = 0$. \square

Definition. Let M be a submanifold in N with the metric $\langle \cdot, \cdot \rangle$, then we call M a ε -super f -biharmonic submanifold, if

$$(31) \quad \langle \Delta(f\vec{H}), f\vec{H} \rangle \geq (\varepsilon - 1)|\nabla(f\vec{H})|^2,$$

where $\varepsilon \in [0, 1]$ is a constant.

Theorem 4.4. Let $u : (M, g) \rightarrow (N, h)$ be a complete ε -super f -biharmonic submanifold in (N, h) for $\varepsilon > 0$. If

$$(32) \quad \int_M |f\vec{H}|^p dv_g < \infty,$$

then u is minimal, where $p \geq 2$.

Proof. From (32), we have

$$\begin{aligned} & (\varepsilon - 1) \int_M \lambda^2 |f\vec{H}|^a |\nabla(f\vec{H})|^2 dv_g \leq \int_M \lambda^2 |f\vec{H}|^a \langle \Delta(f\vec{H}), f\vec{H} \rangle dv_g \\ & = - \int_M \lambda^2 |f\vec{H}|^a |\nabla(f\vec{H})|^2 dv_g - \int_M 2\lambda \nabla \lambda |f\vec{H}|^a \langle \nabla(f\vec{H}), f\vec{H} \rangle dv_g \\ & \quad - a \int_M \lambda^2 |f\vec{H}|^{a-2} \langle \nabla(f\vec{H}), f\vec{H} \rangle^2 dv_g \\ & \leq - \int_M \lambda^2 |f\vec{H}|^a |\nabla(f\vec{H})|^2 dv_g - \int_M 2\lambda \nabla \lambda |f\vec{H}|^a \langle \nabla(f\vec{H}), f\vec{H} \rangle dv_g, \end{aligned}$$

where λ is defined by (18), $a \geq 0$, we have

$$\varepsilon \int_M \lambda^2 |f\vec{H}|^a |\nabla(f\vec{H})|^2 dv_g \leq - \int_M 2\lambda \nabla \lambda |f\vec{H}|^a \langle \nabla(f\vec{H}), f\vec{H} \rangle dv_g.$$

From Young's inequality, we have

$$\begin{aligned} & \varepsilon \int_M \lambda^2 |f\vec{H}|^a |\nabla(f\vec{H})|^2 dv_g \\ & \leq - \int_M 2\lambda \nabla \lambda |f\vec{H}|^a \langle \nabla(f\vec{H}), f\vec{H} \rangle dv_g \\ & \leq \frac{\varepsilon}{2} \int_M \lambda^2 |f\vec{H}|^a |\nabla(f\vec{H})|^2 dv_g + \frac{2}{\varepsilon} \int_M |f\vec{H}|^{a+2} |\nabla \lambda|^2 dv_g, \end{aligned}$$

so

$$(33) \quad \int_M \lambda^2 |f\vec{H}|^a |\nabla(f\vec{H})|^2 dv_g \leq \frac{4}{\varepsilon^2} \frac{C^2}{r^2} \int_M |f\vec{H}|^{a+2} dv_g.$$

Since $\int_M |f\vec{H}|^{a+2} dv_g$ is finite, setting $r \rightarrow \infty$ in (34), we have

$$(34) \quad \int_M |f\vec{H}|^a |\nabla(f\vec{H})|^2 dv_g \leq 0,$$

and then $\vec{H} = 0$ or $\nabla(f\vec{H}) = 0$.

We will prove that $\nabla(f\vec{H}) = 0$ implies $\vec{H} = 0$.

Set $x \in M$ such that $\nabla(f\vec{H}) = 0$. We take an orthonormal basis $\{e_i\}_{i=1}^m$ of $T_x M$, an orthonormal basis $\{v_\alpha\}_{\alpha=1}^t$ of $(T_x M)^\perp$, then we have

$$(35) \quad 0 = \langle \nabla_{e_i}(f\vec{H}), e_j \rangle = -\langle f\vec{H}, B(e_i, e_j) \rangle.$$

From (36), we have

$$0 = \sum_{i=1}^m \langle f\vec{H}, B(e_i, e_i) \rangle = m|\vec{H}|^2 f,$$

so we obtain $\vec{H} = 0$. □

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