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A CHARACTERIZATION OF THE HYPERBOLIC DISC AMONG CONSTANT WIDTH BODIES

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Abstract. In this paper we prove a condition under which a hyperbolic starshaped set has a center of hyperbolic symmetry. We also give the definition of isometric diameters for a hyperbolic convex set, which behave similar to affine diameters for Euclidean convex sets. Using this concept, we give a definition of constant hyperbolic width and we prove that the only hyperbolic sets with constant hyperbolic width and with a hyperbolic center of symmetry are hyperbolic discs.

1. Introduction

Let Γ denote a Euclidean disc in the plane and let ℓ be a line disjoint from Γ. For every point $X \in \ell$ we consider the two points B_X and C_X in $\partial\Gamma$ such that the rays $\overrightarrow{XB_X}$ and $\overrightarrow{XC_X}$ are tangent to Γ. A well known property of the Euclidean disc is that $XB_X = XC_X$ for every $X \in \ell$, however, it is less known that there is a point P in the interior of Γ such that $XP = XB_X = XC_X$ for every $X \in \ell$ (see Figure 1). This last property of the Euclidean disc can be interpreted as follows: all the circles with center in ℓ and orthogonal to $\partial\Gamma$ are concurrent at the point P.

Consider now the following two facts.

(1) If ℓ is tangent to a convex and closed curve γ and we only know that the two tangent segments to γ from every point in ℓ have equal length, then γ is indeed a Euclidean circle (see for instance [7] for a proof of this fact).

(2) If ℓ is not tangent to γ, then there are convex curves, besides the Euclidean circle, such that the two tangent segments to γ from every point in $\ell \setminus \mathrm{conv}\gamma$ have equal length, where $\mathrm{conv}\gamma$ denotes the convex hull of γ. For instance (taken from [7]), let $\triangle ABC$ be a hyperbolic equilateral triangle (in the Poincaré upper half-plane model $\mathbb{H}^2$ of the hyperbolic plane) and consider the three arcs of hyperbolic circles centered at each one of the vertices and passing through the other two

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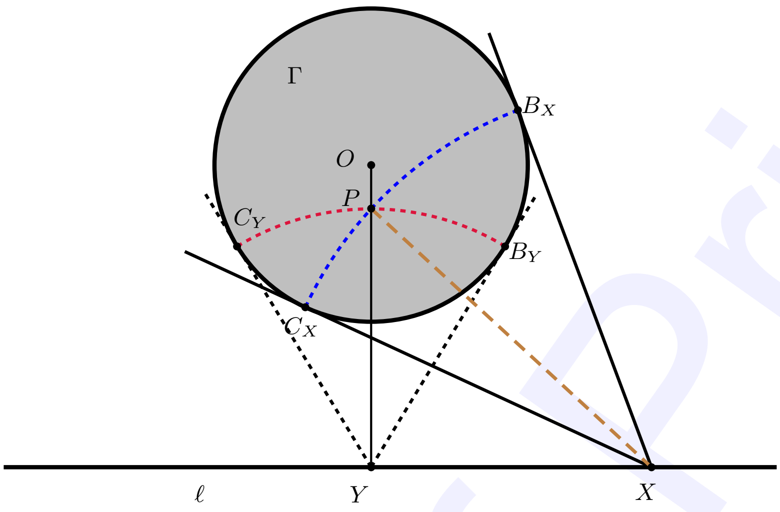

FIGURE 1. The segments XB_X, XC_X, and XP have equal length.

vertices, as shown in Figure 2. In this way we obtain a figure Γ, known as Reuleaux's triangle, with the following property: from every point $X \in \ell$, where ℓ is the *circle of points at infinity* or *the ideal line*, the two segments tangent to Γ from X have the same Euclidean length. Indeed, it was proved in [7] that this property characterizes the bodies of constant hyperbolic width in $\mathbb{H}^2$. The interested reader may consult [6] for more examples of such curves.

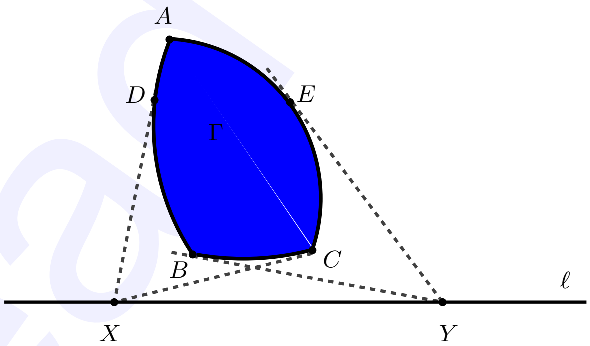

FIGURE 2. The Reuleaux's triangle $\triangle ABC$ has the property of equal tangents.

However, we are interested in the converse question: suppose we have a simple, closed, and differentiable curve γ, in the Euclidean plane, such that there exists a point P in the region enclosed by γ and a line ℓ with the property that for every point $X \in \ell$ the two tangent segments XB_X and XC_X have the same length as XP. Is γ a Euclidean circle? In other words, does this property characterize the Euclidean circle?

By the mentioned facts we can see now that the hypothesis over the existence of the point P is not superfluous. Indeed, we shall prove in the next section that P is a center of hyperbolic symmetry for γ and so, we relate this question to the following well known theorem in Euclidean Convex Geometry (see for instance [13]): *the only centrally symmetric convex body of constant width is the Euclidean disc.*

We first prove the following: Let γ be a simple, differentiable and closed curve in the hyperbolic plane, which is starshaped with respect to a point P. Then γ has center of hyperbolic symmetry at P if the geodesics tangent to γ at the endpoints of every chord through P form equal alternate angles with the chord. We also prove that if the angles are equal to $\pi/2$, then γ is a hyperbolic circle, thus we give answer to the aforesaid question and prove the following.

Theorem 1. *Let K be a convex body in the Euclidean plane with differentiable boundary $\gamma = \partial K$. Let ℓ be a line disjoint from γ and let P be a point in its interior. Suppose that the two tangent segments to γ, XB_X and XC_X, from every point $X \in \ell$ have length equal to XP and the arc of the circle through the points B_X, P, and C_X is contained in K. Then K is a Euclidean disc.*

On the other hand, in this paper we also give the definition of isometric diameters for hyperbolic convex sets, which are the counterparts of affine diameters for Euclidean convex sets. We then use them to define hyperbolic convex sets of constant width and we prove that the hyperbolic convex sets of constant width with hyperbolic center of symmetry are necessarily hyperbolic discs.

2. Hyperbolic characterization of circles

Consider the Poincaré disk model for the two-dimensional hyperbolic space $(\mathbb{D}, d_h)$ where

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

and d_h denotes the hyperbolic distance function. The straight lines or geodesics on this space are given either by diameters of the disk $\mathbb{D}$ or the segments contained in $\mathbb{D}$ of circles orthogonal to the boundary of the disk. For every pair of points $z, w \in \mathbb{D}$, there exists a unique geodesic $L_{z,w}$ passing through them and we denote the geodesic segment joining z and w by $[z, w]$. A subset of $\mathbb{D}$ is called convex if the geodesic segment that joins any two of its points is contained in the set. A convex set is called strictly convex, if its boundary does not contain geodesic segments. If necessary, we will also use the upper-half plane model for the hyperbolic plane $(\mathbb{H}, d_h)$, where

$$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}.$$

For more information about the hyperbolic plane and its properties, the interested reader may consult [1]. In this paper a convex body K is a compact and convex set with non-empty interior. In such a case, its boundary is a simple closed curve γ for which we fix an orientation. For each point $z \in \gamma$ we denote by L_z a supporting line of K at the point z, that is, L_z intersects γ at z and K is contained in one of the closed half-planes determined by L_z.

In planar Euclidean geometry, an affine diameter of a convex body K is a chord $AB \subset K$ such that there are parallel supporting lines of K through the points A and B. We would like to find chords for a given hyperbolic convex body which behave as Euclidean affine diameters. In order to find such chords we proceed as follows: let z be any point in γ and L_z a supporting line; by applying an appropriate Möbius transformation, we may assume that L_z corresponds to the diameter of $\mathbb{D}$ formed by the real interval $(-1, 1)$ and that $z = 0$. Let $w \in \gamma$ be a point such that the tangent vector of a supporting line L_w at the point w is parallel to the interval $(-1, 1)$; in such a case we say that z and w are *opposite points with respect to* L_z. Moreover, since geodesics through the center of $\mathbb{D}$ are Euclidean segments, it follows that $L_{z,w}$ intersects L_w and L_z with the same alternate angles. In this paper we say that the geodesics L_z and L_w are *parallel* (see Figure 3).

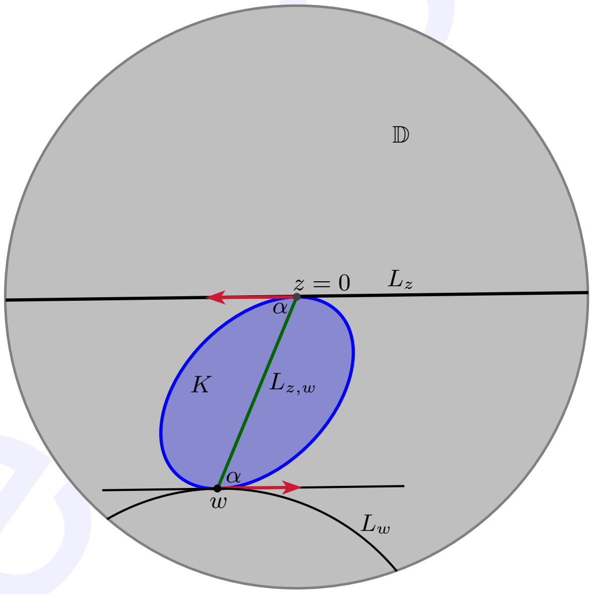

FIGURE 3. The segment $[z, w]$ is an isometric diameter.

Definition 1. An *isometric diameter* of K, is a chord $[z,w]$ joining two opposite points z and w.

For opposite points z, w, the angle of intersection between $L_{z,w}$ and each one of the parallel geodesics L_z and L_w, is of course preserved in magnitude by isometries under the hyperbolic plane. Hence the name isometric diameter. In general, for each z in the boundary of a convex body and a supporting line L_z there may exist several opposite points with respect to L_z. Note that in the case of Euclidean geometry, if a convex body K is strictly convex and has differentiable boundary γ, then to every point of γ there corresponds a unique opposite point. This is not true in the hyperbolic plane, as we can see in the following example.

Example 1. Let Γ denote an arc of a circle intersecting the ideal boundary of $\mathbb{D}$ with angles different to $\pi/2$. Then $\mathbb{D}\setminus\Gamma$ has exactly two connected components. One of them is convex, namely, the connected component whose boundary has interior angles greater than $\pi/2$ at the points in $\Gamma \cap \partial\mathbb{D}$. Let Γ_1 and Γ_2 be two arcs of circles as illustrated in Figure 4, with 0 in Γ_1, $\alpha < \pi/2$ and $\beta > \pi/2$.

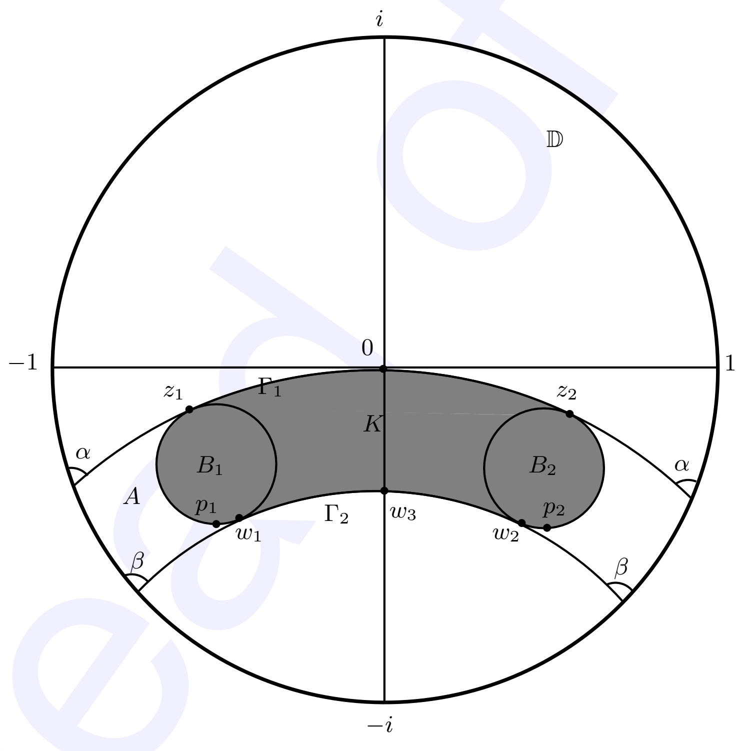

FIGURE 4. A convex region with more than one isometric diameter through a point.

The region A, delimited by the two arcs Γ_1 and Γ_2, is convex since it is the intersection of two convex sets in $\mathbb{D}$. Let z_1 and z_2 be two points in Γ_1 that are symmetric with respect to the imaginary axis. Let B_1 be the disc that is tangent to Γ_1 and Γ_2 at the points z_1 and w_1, respectively, and similarly we define the disc B_2 and the point w_2. Let K be the shaded region (as shown in Figure 4) bounded by the arcs $\widehat{z_1 z_2}$ of Γ_1, $\widehat{z_2 w_2}$ of ∂B_2, $\widehat{w_2 w_1}$ of Γ_2, and $\widehat{w_1 z_1}$ of ∂B_1. It is readily seen that K is a bounded strictly convex set with differentiable boundary. Let w_3 be the intersection of Γ_2 with the imaginary axis and let p_j, for $j = 1, 2$, be the points in ∂B_1 and ∂B_2, respectively, with the same real part as the centers of B_1 and B_2, and with the smallest imaginary part. We have that p_1, p_2, and w_3 are points opposite to 0 for the convex body K.

There are also convex bodies in $\mathbb{D}$ with the property that for every point in its boundary there correspond a unique opposite point. In order to show this we make use of horocycles. A horocycle in $\mathbb{D}$ is a circle that is tangent to the boundary of $\mathbb{D}$. Given two points $z, w \in \mathbb{D}$, there exist two horocycles passing through them. Two such horocycles are symmetric with respect to the geodesic $L_{z,w}$, in the sense that one is the image of the other when reflecting along $L_{z,w}$. The domain of intersection of the two discs bounded by the horocycles is called the horocyclic lentil. There is another notion of convexity given by L. A. Santaló in [12]: a subset $K \subseteq \mathbb{D}$ is called *h-convex* if for any two points $z, w \in K$, the two segments of finite length of the two horocycles determined by these points are contained in the set. We have that the horocyclic lentil determined by two points in an h-convex set K is also contained in K. Since the geodesic segment joining z and w is contained in the interior of the horocyclic lentil, we conclude that any h-convex set is convex in $\mathbb{D}$. The horocyclic lentil is convex in the Euclidean sense because it is the intersection of two discs. We conclude that an h-convex set is also convex in the Euclidean sense. Moreover, the boundary of an h-convex set does not contain geodesic nor Euclidean segments. Note that since all isometries of the hyperbolic plane map horocycles to horocycles, the property of being h-convex is preserved by such maps. In particular, if K is a compact h-convex set, then it is an Euclidean convex set and the construction of opposite points given above guarantees that for every boundary point z of K, there exists a unique opposite point w.

Let z and w be two opposite points of K, then there are two parallel supporting lines L_z and L_w forming alternate angles with $L_{z,w}$. The geodesics L_z and L_w must be disjoint. Otherwise, L_z, L_w and $L_{z,w}$ would bound a hyperbolic triangle whose angles add up at least π leading to a contradiction. Since L_z and L_w are disjoint, we know that there is a unique geodesic $L_{z,w}^{\perp}$ which is orthogonal to both, L_z and L_w.

Definition 2. Let K be a convex body and z, w two opposite points in γ. The width of K with respect to each pair of parallel lines L_z and L_w is the distance between L_z and L_w, i.e., the length of the segment of $L_{z,w}^{\perp}$ joining L_z and L_w.

We say that K is of constant width λ if for every pair of parallel supporting lines L_z and L_w the distance between L_z and L_w is λ.

Curves of constant width were studied earlier by some authors. In particular, Fillmore introduced in [5] a notion of constant width for h-convex sets as follows: Let K be an h-convex set with smooth boundary and fix a point O in $\mathbb{D}$. Consider the geodesic $\ell(\theta)$ through O and $e^{i\theta}$ and let $a(\theta), b(\theta) \in \mathbb{D}$ be the points where $\ell(\theta)$ intersects the horocycles through $e^{i\theta}$ which are tangent to K. The width of K in the direction θ is the length of the segment $[a(\theta), b(\theta)]$. Under this approach, a set K is said to have constant width λ, if the width in the direction θ is λ for all θ.

An alternate approach to define curves of constant width in $\mathbb{D}$ was already developed by Santaló in [11]: For a convex set K, let z be a point in ∂K. Let L_z be the supporting geodesic of K passing through z and let $L_z^{\perp}$ be the orthogonal geodesic to L_z at the point z. Choose another supporting geodesic L_w (passing through a point w in ∂K) that is orthogonal to $L_z^{\perp}$, then the breadth of K corresponding to z is the hyperbolic distance between L_w and L_z, i.e., the length of the segment of $L_z^{\perp}$ joining L_w and L_z. Under this approach, the set K is said to have constant width λ if the breadth of K with respect to every point in ∂K is λ.

Both approaches define the width as the distance between supporting lines (horocycles or geodesics) that can be regarded as parallel is some sense (e.g. geodesic parallels). In this paper we follow a similar path but we consider supporting lines that are parallel in a different sense.

It should be mention that a different notion of curve of constant width was given by Leichtweiss in [8]. In that paper, he defines a strip as a region which is invariant against a one-parameter subgroup of motions of the hyperbolic plane. He then defines the width of a curve in a direction θ as the width of the supporting strip in this direction.

Another necessary definition is the following.

Definition 3. We say that p is a center of hyperbolic symmetry for a set K, if for every $z \in K$ the point $\tilde{z} \in L_{pz}$ such that $d_h(\tilde{z}, p) = d_h(p, z)$, is also in K. In this case, we say that K is symmetric with respect to p.

A set K in $\mathbb{D}$ is called starshaped with respect to $p \in K$, if for every point $z \in K$ the segment $[p, z]$ is contained in K. Our first result here is the following property characterizing centrally symmetric starshaped sets.

Theorem 2. *Let K be a compact starshaped set with respect to a point p with boundary of class C^1. Suppose that every chord of K through p has the property that the geodesic tangent lines of K at the endpoints of the chord, make equal alternate angles with the chord. Then p is a center of hyperbolic symmetry for K.*

Proof. By applying a conformal isometry of $\mathbb{D}$, to say σ, we may assume that $p = 0$. It follows that all the chords of $\sigma(K)$ through 0 are Euclidean segments and hence $\sigma(K)$ is a Euclidean starshaped set with respect to 0. We now use a standard argument. Parameterize the boundary of $\sigma(K)$, denoted again by γ, in polar coordinates by $(\theta, r(\theta))$, where $\theta \in [0, 2\pi]$. We know that the angle $\phi(\theta)$ between the tangent vector at $(\theta, r(\theta))$ and the radial vector $\overrightarrow{r(\theta)}$ satisfies (see for instance [10])

$$\cot \phi(\theta) = \frac{r'(\theta)}{r(\theta)}. \tag{1}$$

Since for any two points z, w, which are endpoints of a chord of $\sigma(K)$ through 0, the geodesic $L_{z,w}$ is a Euclidean line that intersects the tangent vectors at z and w with the same angle, we must have $\phi(\theta) = \phi(\theta + \pi)$ for every θ. Thus

$$\frac{r'(\theta)}{r(\theta)} = \frac{r'(\theta+\pi)}{r(\theta+\pi)}.$$

Equivalently

$$\frac{d}{d\theta}\left[\frac{r(\theta)}{r(\theta+\pi)}\right] = \frac{r'(\theta)r(\theta+\pi) - r(\theta)r'(\theta+\pi)}{r^2(\theta+\pi)} = 0.$$

From this equality we can see that there is a nonnegative constant λ such that $r(\theta) = \lambda \cdot r(\theta + \pi)$ for every $\theta \in [0, 2\pi]$. In particular $r(0) = \lambda \cdot r(\pi)$ and $r(\pi) = \lambda \cdot r(2\pi) = \lambda \cdot r(0)$. It follows that $\lambda = 1$ and then $r(\theta) = r(\theta + \pi)$ for every $\theta \in [0, 2\pi]$. Therefore, for every $z = r(\theta)e^{i\theta}$ in γ we have that for $\tilde{z} = r(\theta + \pi)e^{i(\theta+\pi)}$,

$$d_h(0, \tilde{z}) = \log \frac{1 + |r(\theta+\pi)|}{1 - |r(\theta+\pi)|} = d_h(0, z).$$

We conclude that $\sigma(K)$ has center of symmetry at 0. Since σ^{-1} is also an isometry, we conclude that K has center of symmetry at $p = \sigma^{-1}(0)$. □

The following corollary is the analogue of a theorem by P. C. Hammer [4] for the Euclidean plane.

Corollary 1. *Let p be a point in the interior of a convex body K such that every chord of K through p is an isometric diameter. Then p is a center of hyperbolic symmetry for K.*

Given a convex body K, we say that a chord $[z, w] \subset K$ is a *normal* at z if $[z, w]$ is orthogonal to a supporting line L_z. In case that $[z, w]$ is also orthogonal to a supporting line L_w at w, we say that $[z, w]$ is a *double normal*. Every double normal $[z, w]$ is also an isometric diameter. A well known result in Euclidean convex geometry says that a convex body K is of constant width if and only if every normal is a double normal (see for instance [2]). This characterization was extended to Riemannian manifolds by B. Dekster in [3], which of course contains the case of the Hyperbolic plane. According to Dekster [3], a convex set K in the hyperbolic plane is a body of constant width $\lambda > 0$ if for any $p \in \partial K$ and any exterior unit normal η of K at p, there exists the geodesic segment $[p, q]$ of length λ having direction $-\eta$ such that $K \supset [p, q]$ but K contains no longer segment $[p, q'] \supset [p, q]$. Here we prove the following results with our definition of constant width.

Theorem 3. *Let K be a convex body of constant width λ, and let z, w be opposite points with corresponding parallel supporting lines L_z and L_w. Then the geodesic $L_{z,w}$ containing the isometric diameter $[z, w]$ intersects L_z and L_w at right angles. In other words, $[z, w]$ is a double normal and $d_h(z, w) = \lambda$.*

Proof. Let z', w' be two points in γ such that $d_h(z', w') = \max_{z,w\in\gamma}\{d_h(z, w)\}$. Then the geodesic orthogonal to $L_{z',w'}$ passing through z' intersects γ only at z'. Otherwise there would exist a point $\tilde{z}$ in γ such that $d_h(\tilde{z}, w') > d_h(z', w')$. Thus, the orthogonal geodesic to $L_{z',w'}$ passing through z' is a supporting line for K at z' and it may be denoted by $L_{z'}$. Similarly, there exists a supporting line $L_{w'}$ that is orthogonal to $L_{z',w'}$ at w'. It follows that $[z', w']$ is an isometric diameter intersecting L_z and L_w at right angles. Since K is of constant width λ, we must have that the length of $[z', w']$ is λ, i.e., $d_h(z', w') = \lambda$.

Now suppose there exist two opposite points z, w in γ such that the geodesic $L_{z,w}$ intersects the corresponding parallel geodesics L_z and L_w with alternate angles different to $\pi/2$. Then the isometric diameter $[z, w]$ has length greater than λ, i.e., $d_h(z, w) > \lambda = d_h(z', w')$, which is a contradiction. □

The maximum distance between a pair of points in a given convex set K is known as the diameter of K, denoted by $\operatorname{diam} K$. We have the following corollary.

Corollary 2. *Let K be a convex body of constant width λ. Then its diameter is equal to λ and for every point $z \in \gamma$ there is a point $w \in \gamma$ such that $d_h(z, w) = \lambda$.*

Proof. The proof that $\operatorname{diam} K = \lambda$ was given in the proof of Theorem 3. For every point $p \in \gamma$ there is at least one opposite point q, and by Theorem 3 and since K is of constant width λ we have that $d_h(p, q) = \lambda$. □

Theorem 4. *Let K be a convex body. Then K is of constant width λ if and only if every normal is a double normal.*

Proof. Suppose first K is of constant width λ and let z be any point in γ. Let L_z be a supporting line of K through z and let $w \in \gamma$ be the point such that $[z, w]$ is orthogonal to L_z at z. If w is opposite to z with respect to L_z, by Theorem 3 we have that $[z, w]$ is a double normal. Else, there is a point $w' \in \gamma$ opposite to z with respect to L_z. Again, by Theorem 3 we have that $[z, w']$ is a double normal. We have that $[z, w]$ and $[z, w']$ are both orthogonal to L_z at z, which is only possible if $w' = w$. Hence $[z, w]$ is a double normal.

Now, we assume that every normal of K is a double normal. Then, by a theorem proved by Dekster in [3], we have that K has constant width λ in the sense of Dekster, i.e., for any point $w \in \gamma$ and a support line L_w there is a point

$z \in \gamma$ such that $[w, z]$ is orthogonal to L_w at w and $d_h(w, z) = \lambda$. Since $[w, z]$ is a double normal, there exists a supporting line L_z orthogonal to $[w, z]$ at z. It follows that the width of K (with our definition) with respect to the parallel lines L_w and L_z is $d_h(w, z) = \lambda$. This last holds for every point $w \in \gamma$ and any supporting line of K through w, we then conclude that K has constant width λ. □

Theorem 5. *A convex body K is symmetric with respect to a point p and has constant width λ if and only if it is a hyperbolic disc of radius $\lambda/2$ with center at p.*

Proof. Sufficiency is obvious, so we prove necessity only. Suppose that K is of constant width λ and is symmetric with respect to p. By applying an adequate isometry, we may assume that $p = 0$. Let z be any point in γ. Since K is symmetric with respect to 0, the point $-z$ is also in γ. Let L_z be a line supporting K at z and let L_{-z} be the supporting line symmetric to L_z, with respect to 0. We have that $[z, -z]$ is an isometric diameter, hence by Theorem 3, the geodesic $L_{z,-z}$ is orthogonal to L_z and L_{-z}. The same happens for every supporting line L_z at z and its parallel at $-z$, hence there is a unique supporting line at z, i.e., γ is a differentiable curve. Now, since K is of constant width λ, the length of all the segments $[z, -z]$ is λ and then $d_h(0, z) = \lambda/2$ for all z in γ. Therefore we have that K is a hyperbolic disc and consequently it is also a Euclidean disc. □

Finally, we give the proof of Theorem 1 given in the introduction.

Proof of Theorem 1. We may assume that the line ℓ is the ideal boundary of the hyperbolic plane. The hypotheses of the theorem imply that K is a hyperbolic starshaped set with respect to the point P. By applying an isometry, we may assume that $P = 0$. Parameterizing γ by polar coordinates $(\theta, r(\theta))$ and using Equation (1) we obtain $r'(\theta) = 0$ for all θ, i.e., $r(\theta)$ is constant and therefore γ is a circle. □

References

[1] A. F. Beardon, *The Geometry of Discrete Groups*, Springer-Verlag, Berlin Heidelberg New York, 1995.
[2] G. D. Chakerian and H. Groemer, *Convex bodies of constant width*, Convexity and its Applications, Eds. P. M. Gruber and J. M. Wills, Birkhäuser, Basel, 1983.
[3] B. V. Dekster, *Double normals characterize bodies of constant width in Riemannian manifolds*, Geometric analysis and nonlinear partial differential equations (Denton, TX, 1990), 187–201, Lecture Notes in Pure and Appl. Math., 144, Dekker, New York, 1993.
[4] P. C. Hammer, *Diameters of convex bodies*, Proc. Amer. Math. Soc. **5** (1954), 304–306.
[5] J. Fillmore, *Barbier's theorem in the Lobachevsky plane*, Proc. Amer. Math. Soc. **24** (1970), 705–709.
[6] J. Jerónimo-Castro and E. Roldán-Pensado, *A characteristic property of the Euclidean disc*, Period. Math. Hungar. **59** (2009), no. 2, 213–222.
[7] J. Jerónimo-Castro, G. Ruiz-Hernández, and S. Tabachnikov, *The equal tangents property*, Adv. Geom. **14** (2014), no. 3, 447–453.

[8] K. Leichtweiss, *Curves of constant width in the non-Euclidean geometry*, Abh. Math. Sem. Univ. Hamburg **75** (2005), 257–284.
[9] H. Rademacher and O. Toeplitz, *The enjoyment of mathematics*, Princeton University Press, 1957.
[10] V. A. Toponogov, *Differential Geometry of Curves and Surfaces, a Concise Guide*, Birkhäuser, Boston-Basel-Berlin, 2006.
[11] L. A. Santaló, *Note on convex curves in the hyperbolic plane*, Bull. Amer. Math. Soc. **51** (1945), 405–412.
[12] ———, *Convexity in the hyperbolic plane*, Univ. Nac. Tucumán Rev. Ser. A **19** (1969), 173–183.
[13] I. Yaglom and V. Boltyanski, *Convex Figures*, Holt Rinehart and Winston, New York, 1961.

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