

## OPERATOR FRACTIONAL BROWNIAN SHEET AND MARTINGALE DIFFERENCES

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**ABSTRACT.** In this paper, inspired by the fractional Brownian sheet of Riemann-Liouville type, we introduce the operator fractional Brownian sheet of Riemann-Liouville type, and study some properties of it. We also present an approximation in law to it based on the martingale differences.

### 1. Introduction

Self-similar processes, first studied rigorously by Lamperti [18] under the name “semi-stable”, are stochastic processes that are invariant in distribution under suitable scaling of time and space. There has been an extensive literature on self-similar processes. We refer to Vervaat [25] for general properties, to Samorodnitsky and Taquq [24, Chaps. 7 and 8] for studies on Gaussian and stable self-similar processes and random fields.

The fractional Brownian motion (fBm) as a well-known self-similar process has been studied extensively. Many results about weak approximation to fBms have been established recently. See [11, 19] and the references therein. We point out that the fBm does not represent a casual time-invariant system as there is no well-defined impulse response function. Hence, based on the Riemann-Liouville fractional integral, Barnes and Allan [4] introduced the fractional Riemann-Liouville (RL) Brownian motion (RL-fBm). RL-fBms share with fBms many properties which include self-similarity, regularity of sample paths, etc. with one notable exception that its increment process is nonstationary. For more information on RL-fBms, refer to Lim [20] and the references therein. On the other hand, there are two typical multiparameter extensions of fBms, one of which is the fractional Brownian sheet introduced by Kamont [16]. Fractional Brownian sheets have been studied extensively as a representative of anisotropic Gaussian random fields. For more information, refer to [2, 3] and [26, 27]. Inspired by the study of RL-fBms and fractional Brownian sheets,

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Received August 25, 2016; Revised July 12, 2017; Accepted October 17, 2017.

2010 *Mathematics Subject Classification.* 60B10, 60G15.

*Key words and phrases.* fractional Brownian sheet, operator fractional Brownian sheet of Riemann-Liouville type, martingale differences, weak convergence.

Dai [9] introduced the multifractional Riemann-Liouville Brownian sheet and studied the weak limit theorem for it.

The definition of self-similarity has been extended to allow scaling by linear operators on multidimensional space  $\mathbb{R}^d$ , and the corresponding processes are called operator self-similar processes. We refer to [17], [18], [22] and the references therein. We note that Didier and Pipiras [13, 14] introduced the operator fractional Brownian motions (ofBm in short) as an extension of fBms and studied their properties. Similar to fBms, weak limit theorems for ofBms have also attracted a lot of interest. Recently, Dai and his coauthors [7, 8, 10] presented some weak limit theorems for some kinds of ofBms.

In contrast to the extensive study on the multiparameter extension of fBms, there is little work studying the multiparameter extension of ofBms. Inspired by the study of the fractional Brownian sheet and the operator fractional Brownian motion of Riemann-Liouville type introduced by Dai [7], we will introduce a new random field, which we call the operator fractional Brownian sheet of Riemann-Liouville type, and present an approximation to it.

Most of the estimates of this paper contain unspecified constants. An unspecified positive and finite constant will be denoted by  $C$ , which may not be the same in each occurrence. Sometimes we shall emphasize the dependence of these constants upon parameters.

At the end of this section, we point out that all processes considered here are assumed to be proper. We say that a process  $\{X(t); t \in \mathbb{R}_+^d\}$  is proper if for each  $t \in \mathbb{R}_+^d$  the distribution of  $X(t)$  is full; that is, the distribution is not contained in a proper hyperplane.

The rest of this paper is organized as follows. In Section 2, we introduce the operator fractional Brownian sheet of Riemann-Liouville type and state some properties. We present an approximation in law to it in Section 3. A final note is presented at the end of this paper.

## 2. Operator fractional Brownian sheet

In this section, we first introduce the operator fractional Brownian sheet of Riemann-Liouville type and then study some properties of it. For any  $x \in \mathbb{R}^d$ ,  $x^T$  denotes the transpose of  $x$ . Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and  $\{\mathcal{F}_{t,s}; (t,s)^T \in \mathbb{R}_+^2\}$  be a family of sub- $\sigma$ -fields of  $\mathcal{F}$  such that  $\mathcal{F}_{t,s} \subseteq \mathcal{F}_{t',s'}$  for any  $(t,s)^T < (t',s')^T$  with the usual partial order. Moreover, for any stochastic process  $Y = \{Y(t,s); (t,s)^T \in \mathbb{R}_+^2\}$ , we denote by  $\Delta_{(t,s)}Y(t',s')$  the increment of  $Y$  over the rectangle  $(t,t'] \times (s,s']$ , that is,

$$\Delta_{(t,s)}Y(t',s') = Y(t',s') - Y(t,s') - Y(t',s) + Y(t,s).$$

Let  $\sigma(A)$  be the collection of all eigenvalues of a linear operator  $A$  on  $\mathbb{R}^d$ . Let

$$\lambda_A = \min\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\} \text{ and } \Lambda_A = \max\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}.$$

Moreover, given any linear operator  $A$  on  $\mathbb{R}^d$  and  $t > 0$ , we define the power operator

$$t^A = \sum_{k=0}^{\infty} (\log t)^k \frac{A^k}{k!}.$$

Next, we recall the operator fractional Brownian motion of Riemann-Liouville type introduced by Dai [7]. Let  $D$  be a linear operator on  $\mathbb{R}^d$  with  $0 < \lambda_D, \Lambda_D < 1$ . We define the operator fractional Brownian motion of Riemann-Liouville type  $\tilde{X} = \{\tilde{X}(t); t \in \mathbb{R}_+\}$  with exponent  $D$  by

$$(2.1) \quad \tilde{X}(t) = \int_0^t (t-u)^{D-I/2} dW(u),$$

where  $W(u) = \{W^1(u), \dots, W^d(u)\}^T$  is a standard  $d$ -dimensional Brownian motion and  $I$  is the  $d \times d$  identity matrix.

Based on (2.1), we can define the operator fractional Brownian sheet of Riemann-Liouville type  $X = \{X(t, s); (t, s)^T \in \mathbb{R}_+^2\}$  as follows.

**Definition 2.1.** Let  $\tilde{B}$  be the standard Brownian sheet. The operator fractional Brownian sheet of Riemann-Liouville type  $X = \{X(t, s); (t, s)^T \in \mathbb{R}_+^2\}$  is defined by

$$(2.2) \quad X(t, s) = \int_0^t \int_0^s (t-u)^{\frac{D}{2}-\frac{1}{2}} (s-v)^{\frac{D}{2}-\frac{1}{2}} B(du, dv),$$

where  $B(du, dv) = (B^1(du, dv), \dots, B^d(du, dv))^T$  with  $B^i$  being independent copies of  $\tilde{B}$ , and  $D$  is a linear operator on  $\mathbb{R}^d$  with  $0 < \lambda_D, \Lambda_D < 1$ .

*Remark 2.1.* Let  $x_+ = \max\{x, 0\}$ . From (2.2) and Mason and Xiao [21], we get that  $X$  is an  $\mathbb{R}^d$ -valued Gaussian random field with mean zero vector and for any  $(t_1, t_2)^T, (s_1, s_2)^T \in \mathbb{R}_+^2$

$$(2.3) \quad \begin{aligned} \mathbb{E}[X(t_1, t_2)X^T(s_1, s_2)] &= \int_0^\infty \int_0^\infty [(t_1-u)_+(t_2-v)_+]^{\frac{D}{2}-\frac{1}{2}} \\ &\quad \cdot [(s_2-v)_+(s_1-u)_+]^{\frac{D^*}{2}-\frac{1}{2}} dudv, \end{aligned}$$

where  $D^*$  is the adjoint operator of  $D$ .

It is obvious that the equation (2.2) is well defined. Next, we study some properties of the random field  $X$ . We first introduce the following notation. Let  $\|x\|_2$  denote the usual Euclidean norm of  $x \in \mathbb{R}^d$ . Similar to Dai, Shen and Kong [12],  $\text{End}(\mathbb{R}^d)$  denotes the set of linear operators on  $\mathbb{R}^d$  (endomorphisms). Furthermore, we will not distinguish an operator  $D \in \text{End}(\mathbb{R}^d)$  from its associated matrix relative to the standard basis of  $\mathbb{R}^d$ . For any  $A \in \text{End}(\mathbb{R}^d)$ , let  $\|A\| = \max_{\|x\|_2=1} \|Ax\|_2$  be the operator norm of  $A$ . Next, we recall the definition of operator self-similar processes. Recall that an  $\mathbb{R}^d$ -valued stochastic

process  $\tilde{Y} = \{\tilde{Y}(t); t \in \mathbb{R}_+^2\}$  is said to be operator self-similar (o.s.s.) if it is continuous in law at each  $t \in \mathbb{R}_+^2$ , and there exists  $D \in \text{End}(\mathbb{R}^d)$  such that

$$\{\tilde{Y}(ct)\} \stackrel{\mathcal{D}}{=} \{c^D \tilde{Y}(t)\} \text{ for all } c > 0,$$

where  $\stackrel{\mathcal{D}}{=}$  denotes the equality of all finite-dimensional distributions.

**Theorem 2.1.** *The random field  $X = \{X(t, s); (t, s)^T \in \mathbb{R}_+^2\}$  is an operator self-similar Gaussian random field with exponent  $D$ . Moreover,  $X$  has a version with continuous sample paths a.s..*

*Proof.* We first check the operator self-similarity. For every  $c > 0$ , we have

$$\begin{aligned} X(ct, cs) &= \int_0^\infty \int_0^\infty (ct - u)_+^{\frac{D}{2} - \frac{1}{2}} (cs - v)_+^{\frac{D}{2} - \frac{1}{2}} dB(u, v) \\ &\stackrel{\mathcal{D}}{=} c^{D-I} \int_0^{ct} \int_0^{cs} \left(t - \frac{u}{c}\right)^{\frac{D}{2} - \frac{1}{2}} \left(s - \frac{v}{c}\right)^{\frac{D}{2} - \frac{1}{2}} dB(u, v) \\ &\stackrel{\mathcal{D}}{=} c^D X(t, s), \end{aligned}$$

since

$$(2.4) \quad B(cu, cv) \stackrel{\mathcal{D}}{=} c^I B(u, v) \text{ and } z^D y^D = (zy)^D \text{ for any } z > 0, y > 0.$$

Next, we check the sample continuity. Choose any  $t = (t_1, t_2)^T, s = (s_1, s_2)^T \in \mathbb{R}_+^2$ . Without loss of generality, we assume that  $s < t$  with the usual partial order, and  $\|t - s\|_2 \leq 1$ . By some calculations, we have

$$(2.5) \quad \begin{aligned} \Delta_s X(t) &= \int_0^\infty \int_0^\infty \left( (t_1 - u)_+^{\frac{D}{2} - \frac{1}{2}} - (s_1 - u)_+^{\frac{D}{2} - \frac{1}{2}} \right) \\ &\quad \cdot \left( (t_2 - v)_+^{\frac{D}{2} - \frac{1}{2}} - (s_2 - v)_+^{\frac{D}{2} - \frac{1}{2}} \right) B(du, dv). \end{aligned}$$

Hence,

$$(2.6) \quad \|\Delta_s X(t)\|_2^2 = \sum_{i=1}^d \left( \int_0^\infty \int_0^\infty \sum_{j=1}^d F_{i,j}(t, s, u, v) B^j(du, dv) \right)^2,$$

where

$$\begin{aligned} F(t, s, u, v) &= \left( (t_1 - u)_+^{\frac{D}{2} - \frac{1}{2}} - (s_1 - u)_+^{\frac{D}{2} - \frac{1}{2}} \right) \left( (t_2 - v)_+^{\frac{D}{2} - \frac{1}{2}} - (s_2 - v)_+^{\frac{D}{2} - \frac{1}{2}} \right) \\ &= (F_{i,j}(t, s, u, v))_{d \times d}. \end{aligned}$$

Noting that  $\int_0^\infty \int_0^\infty \sum_{j=1}^d F_{i,j}(t, s, u, v) B^j(du, dv)$  is a Gaussian random variable, we get from (2.6) that for any even  $k \in \mathbb{N}$

$$(2.7) \quad \mathbb{E} \left[ \|\Delta_s X(t)\|_2^k \right] \leq C \left[ \int_0^\infty \int_0^\infty \|F(t, s, u, v)\|^2 dudv \right]^{\frac{k}{2}}.$$

On the other hand, we have

$$(2.8) \quad \|F(t, s, u, v)\|^2 \leq C \|F_1(t, s, u, v)\|^2 \times \|F_2(t, s, u, v)\|^2,$$

where

$$F_1(t, s, u, v) = (t_1 - u)_+^{\frac{D}{2} - \frac{I}{2}} - (s_1 - u)_+^{\frac{D}{2} - \frac{I}{2}},$$

and

$$F_2(t, s, u, v) = (t_2 - v)_+^{\frac{D}{2} - \frac{I}{2}} - (s_2 - v)_+^{\frac{D}{2} - \frac{I}{2}}.$$

Now, we look at

$$\int_0^\infty \|F_1(t, s, u, v)\|^2 dudv.$$

By using the same method as in Dai, Hu and Lee [10], we have

$$(2.9) \quad \int_0^\infty \|(t_1 - u)_+^{\frac{D}{2} - \frac{I}{2}} - (s_1 - u)_+^{\frac{D}{2} - \frac{I}{2}}\|^2 du \leq C(t_1 - s_1)^{\lambda_D - \delta}.$$

Similarly,

$$(2.10) \quad \int_0^\infty \|F_2(t, s, u, v)\|^2 dv \leq C(t_2 - s_2)^{\lambda_D - \delta}.$$

From Maejima and Mason [21], and (2.7)-(2.10), we can get that for any  $\delta > 0$  with  $\lambda_D - \delta > 0$ ,

$$(2.11) \quad \begin{aligned} \mathbb{E} \left[ \|\Delta_s X(t)\|_2^k \right] &\leq C \left[ (t_1 - s_1)^{\lambda_D - \delta} \times (t_2 - s_2)^{\lambda_D - \delta} \right]^{\frac{k}{2}} \\ &\leq C \|t - s\|_2^{(\lambda_D - \delta)k}. \end{aligned}$$

The sample continuity follows from Garsia [15] and (2.11).  $\square$

### 3. Limit theorem

One aim of this paper is to present an approximation in law to the operator fractional Brownian sheet of Riemann-Liouville type  $X$  via the martingale differences. In order to reach it, we first recall some facts about the martingale differences. Similar to Wang, Yan and Yu [27], we use the definitions and notations introduced in the basic work of Cairoli and Walsh [6] on stochastic calculus in the plane. For any  $\mathbf{n} = (n_1, n_2)^T \in N_0 \times M_0$  with  $N_0 = \{1, \dots, n_0\}$  and  $M_0 = \{1, \dots, m_0\}$ , let  $\tilde{\mathcal{F}}_{\mathbf{n}} := \mathcal{F}_{n_1, m_0} \vee \mathcal{F}_{n_0, n_2}$ , the  $\sigma$ -fields generated by  $\mathcal{F}_{n_1, m_0}$  and  $\mathcal{F}_{n_0, n_2}$ . Now, we recall the definition of the strong martingale.

**Definition 3.1.** An integrable process  $Y = \{Y(\mathbf{n}), \mathbf{n} \in N_0 \times M_0\}$  is called a strong martingale if:

- (i)  $Y$  is adapted;
- (ii)  $Y$  vanishes on the axes;
- (iii)  $\mathbb{E}[\Delta_{\mathbf{n}} Y(\mathbf{m}) | \tilde{\mathcal{F}}_{\mathbf{n}}] = 0$  for any  $\mathbf{n} \leq \mathbf{m} \in N_0 \times M_0$  with the usual partial order.

Let  $\{\xi^{(n)} = (\xi_{i,j}^{(n)}, \mathcal{F}_{i,j}^{(n)})\}_{n \in \mathbb{N}}$  be a sequence such that for all

$$\mathbb{E}[\xi_{i+1,j+1}^{(n)} | \mathcal{F}_{i,j}^{(n)}] = 0,$$

where  $\mathcal{F}_{i,j}^{(n)} = \mathcal{F}_{i,n}^{(n)} \vee \mathcal{F}_{n,j}^{(n)}$  with  $\mathcal{F}_{k,l}^{(n)}$  being the  $\sigma$ -fields generated by all  $\xi_{r,s}^{(n)}$ ,  $r \leq k$ ,  $s \leq l$ . Then we call  $\{\xi^{(n)} = (\xi_{i,j}^{(n)}, \mathcal{F}_{i,j}^{(n)})\}_{n \in \mathbb{N}}$  a martingale differences sequence.

It is well known that if the martingale differences sequence  $\{\xi^{(n)}\}$  satisfies the following condition

$$\sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=1}^{\lfloor ns \rfloor} (\xi_{i,j}^{(n)})^2 \rightarrow t \cdot s$$

in the sense of  $\mathcal{L}^1$ , then the sequence

$$\sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=1}^{\lfloor ns \rfloor} \xi_{i,j}^{(n)}$$

converges weakly to the Brownian sheet, as  $n$  goes to infinity (see for example, Morkvenas [23].) Recently, Wang, Yan and Yu [27] extended this work to the fractional Brownian sheet. If  $\{\xi^{(n)}\}$  is a square integrable martingale differences sequence satisfying the following two conditions:

$$(3.1) \quad \lim_{n \rightarrow \infty} n(\xi_{i,j}^{(n)}) = 1 \text{ a.s.}$$

for any  $1 \leq i, j \leq n$ , and

$$(3.2) \quad \max_{1 \leq i, j \leq n} |\xi_{i,j}^{(n)}| \leq \frac{C}{n} \text{ a.s.}$$

for some  $C \geq 1$ , then, based on  $\{\xi^{(n)}\}$ , the authors of [27] constructed a sequence to converge weakly to the fractional Brownian sheet. Inspired by these results, we want to study the weak limit theorem for the operator fractional Brownian sheet of Riemann-Liouville type  $X$  introduced in Definition 2.1. Similar to Wang, Yan and Yu [27], we assume that  $\frac{1}{2} < \lambda_D, \Lambda_D < 1$  in the rest of this paper.

Define

$$(3.3) \quad \eta_{i,j}^{(n)} = (\xi_{i,j,1}^{(n)}, \dots, \xi_{i,j,d}^{(n)})^T,$$

and

$$(3.4) \quad B_n(t, s) = \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=1}^{\lfloor ns \rfloor} \eta_{i,j}^{(n)},$$

where  $\xi_{i,j,k}^{(n)}$ ,  $k = 1, \dots, d$ , are independent copies of  $\xi_{i,j}^{(n)}$ .

From the above arguments, we obtain that  $\{(\eta_{i,j}^{(n)}, \mathcal{F}_{i,j}^{(n)})\}_{n \in \mathbb{N}}$  is still a sequence of square integrable martingale differences on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .

For any  $n \geq 1$  and  $(t, s)^T \in [0, 1]^2$ , define

$$\begin{aligned} X_n(t, s) &= \int_0^t \int_0^s (t-u)_+^{\frac{D}{2}-\frac{1}{2}} (s-v)_+^{\frac{D}{2}-\frac{1}{2}} B_n(du, dv) \\ (3.5) \quad &= n^2 \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=1}^{\lfloor ns \rfloor} \eta_{i,j}^{(n)} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} (t-u)_+^{\frac{D}{2}-\frac{1}{2}} (s-v)_+^{\frac{D}{2}-\frac{1}{2}} dudv. \end{aligned}$$

Then, we have the following approximation. As a prelude to giving the result, let

$$\mathcal{D}([0, 1]^2) = \mathcal{D}([0, 1]^2, \mathbb{R}^d).$$

**Theorem 3.1.** *Let  $\frac{1}{2} < \lambda_D, \Lambda_D < 1$ . The sequence of processes  $\{X_n(t, s); (t, s)^T \in [0, 1]^2\}$  given by (3.5) converges weakly, as  $n \rightarrow \infty$  in  $\mathcal{D}([0, 1]^2)$ , to the operator fractional Brownian sheet of Riemann-Liouville type  $\{X(t, s); (t, s)^T \in [0, 1]^2\}$  given by (2.2).*

The proof of Theorem 3.1 is based on a series of technical results.

**Lemma 3.1.** *Let  $\{X_n(t, s)\}$  be the family of processes defined by (3.5). Then for any  $s = (s_1, s_2)^T < t = (t_1, t_2)^T < u = (u_1, u_2)^T \in [0, 1]^2$ ,*

$$(3.6) \quad \mathbb{E} \left[ \left\| \Delta_s X_n(t) \right\|_2 \left\| \Delta_t X_n(u) \right\|_2 \right]^2 \leq C(u_2 - s_2)^{2H} (u_1 - s_1)^{2H},$$

where  $H = \lambda_D - \delta$  with  $0 < \delta < \lambda_D - \frac{1}{2}$ .

*Proof:* From (3.5), we have

$$\begin{aligned} &\Delta_s X_n(t) \\ &= \int_{s_1}^{t_1} \int_{s_2}^{t_2} \left( (t_1 - u)_+^{\frac{D}{2}-\frac{1}{2}} - (s_1 - u)_+^{\frac{D}{2}-\frac{1}{2}} \right) \\ &\quad \cdot \left( (t_2 - v)_+^{\frac{D}{2}-\frac{1}{2}} - (s_2 - v)_+^{\frac{D}{2}-\frac{1}{2}} \right) B_n(du, dv) \\ &= \sum_{i=1}^{\lfloor nt_1 \rfloor} \sum_{j=1}^{\lfloor nt_2 \rfloor} n^2 \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} \left( \left( \frac{\lfloor nt_1 \rfloor}{n} - u \right)_+^{\frac{D}{2}-\frac{1}{2}} - \left( \frac{\lfloor ns_1 \rfloor}{n} - u \right)_+^{\frac{D}{2}-\frac{1}{2}} \right) \\ &\quad \times \left( \left( \frac{\lfloor nt_2 \rfloor}{n} - v \right)_+^{\frac{D}{2}-\frac{1}{2}} - \left( \frac{\lfloor ns_2 \rfloor}{n} - v \right)_+^{\frac{D}{2}-\frac{1}{2}} \right) dudv \eta_{i,j}^{(n)}. \end{aligned}$$

It follows from (3.2) and the Cauchy-Schwarz inequality that

$$\begin{aligned} (3.7) \quad &\mathbb{E} \left[ \left\| \Delta_s X_n(t) \right\|_2 \right]^4 \\ &= \mathbb{E} \left[ \left\| \sum_{i=1}^{\lfloor nt_1 \rfloor} \sum_{j=1}^{\lfloor nt_2 \rfloor} n^2 \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} \left( \left( \frac{\lfloor nt_1 \rfloor}{n} - u \right)_+^{\frac{D}{2}-\frac{1}{2}} - \left( \frac{\lfloor ns_1 \rfloor}{n} - u \right)_+^{\frac{D}{2}-\frac{1}{2}} \right) \right. \right. \end{aligned}$$

$$\begin{aligned}
& \times \left( \left( \frac{\lfloor nt_2 \rfloor}{n} - v \right)_+^{\frac{D}{2} - \frac{I}{2}} - \left( \frac{\lfloor ns_2 \rfloor}{n} - v \right)_+^{\frac{D}{2} - \frac{I}{2}} \right) dudv \eta_{i,j}^{(n)} \Big\|_2 \Big]^4 \\
& \leq C \sum_{i=1}^{\lfloor nt_1 \rfloor} \left( \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left\| \left( \frac{\lfloor nt_1 \rfloor}{n} - u \right)_+^{\frac{D}{2} - \frac{I}{2}} - \left( \frac{\lfloor ns_1 \rfloor}{n} - u \right)_+^{\frac{D}{2} - \frac{I}{2}} \right\|^2 du \right)^2 \\
& \quad \times \sum_{j=1}^{\lfloor nt_2 \rfloor} \left( \int_{\frac{j-1}{n}}^{\frac{j}{n}} \left\| \left( \frac{\lfloor nt_2 \rfloor}{n} - v \right)_+^{\frac{D}{2} - \frac{I}{2}} - \left( \frac{\lfloor ns_2 \rfloor}{n} - v \right)_+^{\frac{D}{2} - \frac{I}{2}} \right\|^2 dv \right)^2 \\
& \leq C \left( \int_0^{t_1} \left\| \left( \frac{\lfloor nt_1 \rfloor}{n} - u \right)_+^{\frac{D}{2} - \frac{I}{2}} - \left( \frac{\lfloor ns_1 \rfloor}{n} - u \right)_+^{\frac{D}{2} - \frac{I}{2}} \right\|^2 du \right)^2 \\
& \quad \times \left( \int_0^{t_2} \left\| \left( \frac{\lfloor nt_2 \rfloor}{n} - v \right)_+^{\frac{D}{2} - \frac{I}{2}} - \left( \frac{\lfloor ns_2 \rfloor}{n} - v \right)_+^{\frac{D}{2} - \frac{I}{2}} \right\|^2 dv \right)^2 \\
& \leq C \left( \int_0^1 \left\| \left( \frac{\lfloor nt_1 \rfloor}{n} - u \right)_+^{\frac{D}{2} - \frac{I}{2}} - \left( \frac{\lfloor ns_1 \rfloor}{n} - u \right)_+^{\frac{D}{2} - \frac{I}{2}} \right\|^2 du \right)^2 \\
& \quad \times \left( \int_0^1 \left\| \left( \frac{\lfloor nt_2 \rfloor}{n} - v \right)_+^{\frac{D}{2} - \frac{I}{2}} - \left( \frac{\lfloor ns_2 \rfloor}{n} - v \right)_+^{\frac{D}{2} - \frac{I}{2}} \right\|^2 dv \right)^2.
\end{aligned}$$

From Dai, Hu and Lee [10], we obtain that

$$\int_0^1 \left\| \left( \frac{\lfloor nt_1 \rfloor}{n} - u \right)_+^{\frac{D}{2} - \frac{I}{2}} - \left( \frac{\lfloor ns_1 \rfloor}{n} - u \right)_+^{\frac{D}{2} - \frac{I}{2}} \right\|^2 du \leq C \left( \frac{\lfloor nt_1 \rfloor}{n} - \frac{\lfloor ns_1 \rfloor}{n} \right)^H,$$

where  $H = \lambda_D - \delta$ . Then (3.7) can be bounded by

$$(3.8) \quad C \left( \frac{\lfloor nt_1 \rfloor - \lfloor ns_1 \rfloor}{n} \right)^{2H} \left( \frac{\lfloor nt_2 \rfloor - \lfloor ns_2 \rfloor}{n} \right)^{2H}.$$

Hence, for any  $s < t < u \in [0, 1]^2$ , we have

$$\begin{aligned}
(3.9) \quad \mathbb{E} \left[ \left\| \Delta_s X(t) \right\|_2 \left\| \Delta_t X(u) \right\|_2 \right]^2 & \leq C \left[ \mathbb{E} \left[ \left\| \Delta_t X(u) \right\|_2^4 \right] \right]^{\frac{1}{2}} \left[ \mathbb{E} \left[ \left\| \Delta_s X(t) \right\|_2^4 \right] \right]^{\frac{1}{2}} \\
& \leq C \left( \frac{\lfloor nt_1 \rfloor - \lfloor ns_1 \rfloor}{n} \right)^H \left( \frac{\lfloor nt_2 \rfloor - \lfloor ns_2 \rfloor}{n} \right)^H \\
& \quad \times \left( \frac{\lfloor nu_1 \rfloor - \lfloor nt_1 \rfloor}{n} \right)^H \left( \frac{\lfloor nu_2 \rfloor - \lfloor nt_2 \rfloor}{n} \right)^H.
\end{aligned}$$

Hence, if  $u_2 - s_2 \geq \frac{1}{n}$ , then

$$(3.10) \quad \left| \frac{\lfloor nu_2 \rfloor - \lfloor ns_2 \rfloor}{n} \right|^{2H} \leq C |u_2 - s_2|^{2H}.$$

Conversely, if  $u_2 - s_2 < \frac{1}{n}$ , then either  $u_2$  and  $t_2$  or  $t_2$  and  $s_2$  belong to a same subinterval  $[\frac{m}{n}, \frac{m+1}{n})$  for some integer  $m$ . Hence (3.10) still holds. The other term follows a similar discussion. The proof is now completed.  $\square$

Since  $X_n(t, s)$ ,  $n \in \mathbb{N}$ , are null on the axes, by using the criterion given by Bickel and Wichura [5], and Lemma 3.1, we can get the following lemma.

**Lemma 3.2.** *The sequence  $\{X_n(t, s); (t, s)^T \in [0, 1]^2\}$  is tight in  $\mathcal{D}([0, 1]^2)$ .*



Now, in order to prove Theorem 3.1, it suffices to show the following lemma which states that the law of all possible weak limits is the law of the operator fractional Brownian sheet of Riemann-Liouville type  $X$ .

**Lemma 3.3.** *The family of random fields  $X_n(t, s)$  defined by (3.5) converges, as  $n$  tends to infinity, to the operator fractional Brownian sheet of Riemann-Liouville type  $X$  in the sense of finite-dimensional distributions.*

In order to prove Lemma 3.3, we need a technical result. Before we present this result, we first introduce the following notation.

$$(t - u)_+^{\frac{D}{2} - \frac{1}{2}} = (\tilde{K}_{i,j}(t, u))_{d \times d}$$

and

$$\left(\frac{\lfloor nt \rfloor}{n} - u\right)_+^{\frac{D}{2} - \frac{1}{2}} = (\tilde{K}_{i,j}^n(t, u))_{d \times d}.$$

**Lemma 3.4.** *For any  $(t_k, s_k)^T, (t_l, s_l)^T \in [0, 1]^2$  and  $q, m \in \{1, \dots, d\}$ , we have that*

$$(3.11) \quad n^4 \sum_{i=1}^n \sum_{j=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} \tilde{K}_{q,m}^n(t_k, u) \tilde{K}_{m,q}^n(s_k, v) dudv \cdot \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} \tilde{K}_{q,m}^n(t_l, u) \tilde{K}_{m,q}^n(s_l, v) dudv (\xi_{i,j,q}^{(n)})^2$$

converges to

$$(3.12) \quad \int_0^1 \int_0^1 \tilde{K}_{q,m}(t_k, u) \tilde{K}_{m,q}(s_k, v) \tilde{K}_{q,m}(t_l, u) \tilde{K}_{m,q}(s_l, v) dudv, \quad a.s.$$

as  $n$  tends to infinity.

*Proof.* It is obvious that (3.11) is equivalent to

$$(3.13) \quad n^2 \sum_{i=1}^n n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \tilde{K}_{q,m}^n(t_k, u) du \int_{\frac{i-1}{n}}^{\frac{i}{n}} \tilde{K}_{q,m}^n(t_l, u) du \cdot \sum_{j=1}^n n \int_{\frac{j-1}{n}}^{\frac{j}{n}} \tilde{K}_{m,q}^n(s_k, v) dv \int_{\frac{j-1}{n}}^{\frac{j}{n}} \tilde{K}_{m,q}^n(s_l, v) dv (\xi_{i,j,q}^{(n)})^2.$$

By using the same method as the proof of Lemma 8 in Dai, Hu and Lee [10], we can prove the lemma.  $\square$

Next, we prove Lemma 3.3.

*Proof of Lemma 3.3.* Let  $a_1, \dots, a_Q \in \mathbb{R}$  and  $(t_1, s_1)^T, \dots, (t_Q, s_Q)^T \in [0, 1]^2$ . Next, we prove that the random vector

$$Y_n = \sum_{k=1}^Q a_k X_n(t_k, s_k)$$

converges in distribution, as  $n$  tends to infinity, to the Gaussian random vector

$$\tilde{X} = \sum_{k=1}^Q a_k X(t_k, s_k).$$

By the well-known Cramér-Wold device, see Whitt [28] for example, in order to prove the above statement, we only need to show that as  $n \rightarrow \infty$

$$(3.14) \quad bY_n \xrightarrow{\mathcal{D}} b\tilde{X},$$

where  $b = (b_1, b_2, \dots, b_d)$  and  $\xrightarrow{\mathcal{D}}$  denotes convergence in distribution.

For conciseness of the paper, let

$$(t-u)_+^{\frac{D}{2}-\frac{1}{2}}(s-v)_+^{\frac{D}{2}-\frac{1}{2}} = K(t, s, u, v) = (K_1(t, s, u, v), \dots, K_d(t, s, u, v))^T,$$

where

$$K_j(t, s, u, v) = (K_{j,1}(t, s, u, v), \dots, K_{j,d}(t, s, u, v)).$$

Then, we have

$$\begin{aligned} bY_n &= \sum_{q=1}^d \sum_{k=1}^Q \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=1}^{\lfloor ns \rfloor} n^2 \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} a_k b_q K_q \left( \frac{\lfloor nt_k \rfloor}{n}, \frac{\lfloor ns_k \rfloor}{n}, u, v \right) \eta_{i,j}^{(n)} dudv \\ &= \sum_{m=1}^d \sum_{q=1}^d \sum_{k=1}^Q \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=1}^{\lfloor ns \rfloor} n^2 \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} a_k b_q K_{q,m} \left( \frac{\lfloor nt_k \rfloor}{n}, \frac{\lfloor ns_k \rfloor}{n}, u, v \right) \xi_{i,j,m}^{(n)} dudv, \end{aligned}$$

and

$$b\tilde{X} = \sum_{m=1}^d \sum_{q=1}^d \sum_{k=1}^Q \int_0^1 \int_0^1 a_k b_q K_{q,m}(t_k, s_k, u, v) B^m(du, dv).$$

Since  $\xi_{i,j,m}^{(n)}$ ,  $m = 1, \dots, d$ , are independent, in order to prove (3.14), we only need to show

$$\begin{aligned} &\sum_{q=1}^d \sum_{k=1}^Q \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=1}^{\lfloor ns \rfloor} n^2 \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} a_k b_q K_{q,m} \left( \frac{\lfloor nt_k \rfloor}{n}, \frac{\lfloor ns_k \rfloor}{n}, u, v \right) \xi_{i,j,m}^{(n)} dudv \xrightarrow{\mathcal{D}} \\ (3.15) \quad &\sum_{q=1}^d \sum_{k=1}^Q \int_0^1 \int_0^1 a_k b_q K_{q,m}(t_k, s_k, u, v) B^m(du, dv). \end{aligned}$$

For convenience, we introduce the following notation.

$$Y_{i,j}^{(n)} = \sum_{q=1}^d \sum_{k=1}^Q n^2 \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} a_k b_q K_{q,m} \left( \frac{\lfloor nt_k \rfloor}{n}, \frac{\lfloor ns_k \rfloor}{n}, u, v \right) \xi_{i,j,m}^{(n)} dudv.$$

Then, (3.15) can be rewritten as

$$(3.16) \quad \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=1}^{\lfloor ns \rfloor} Y_{i,j}^{(n)} \xrightarrow{\mathcal{D}} \sum_{q=1}^d \sum_{k=1}^Q \int_0^1 \int_0^1 a_k b_q K_{q,m}(t_k, s_k, u, v) B^m(du, dv).$$

Inspired by Wang, Yan and Yu [27], in order to prove (3.16), we first prove the following Lindeberg condition

$$(3.17) \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^n \mathbb{E} \left[ (Y_{i,j}^{(n)})^2 1_{\{|Y_{i,j}^{(n)}| > \varepsilon\}} \middle| \mathcal{F}_{i-1,j-1}^{(n)} \right] = 0$$

for all  $\varepsilon > 0$ .

In fact, we have that

$$(3.18) \quad \begin{aligned} & \left( n^2 \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} K_{q,m} \left( \frac{\lfloor nt_k \rfloor}{n}, \frac{\lfloor st_k \rfloor}{n}, u, v \right) \xi_{i,j,m}^{(n)} dudv \right)^2 \\ & \leq n^4 (\xi_{i,j,m}^{(n)})^2 \left( \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} |K_{q,m} \left( \frac{\lfloor nt_k \rfloor}{n}, \frac{\lfloor st_k \rfloor}{n}, u, v \right)| dudv \right)^2 \\ & \leq C n^2 (\xi_{i,j,m}^{(n)})^2 \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} |K_{q,m} \left( \frac{\lfloor nt_k \rfloor}{n}, \frac{\lfloor st_k \rfloor}{n}, u, v \right)|^2 dudv. \end{aligned}$$

It is easy to verify that there exists some  $\delta > 0$  with  $\lambda_D - \delta > 0$  such that

$$(3.19) \quad \int_{\frac{i-1}{n}}^{\frac{i}{n}} \|(t-u)_+^{\frac{D}{2}-\frac{1}{2}}\|^2 du \leq C \int_{\frac{n-1}{n}}^1 (1-u)_+^{\lambda_D-1-\delta} du,$$

since  $0 < \lambda_D - \delta < 1$  and  $t \in [0, 1]$ .

Noting the form of  $K$ , we get from (3.18) and (3.19) that

$$(3.20) \quad \left( n^2 \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} K_{q,m} \left( \frac{\lfloor nt_k \rfloor}{n}, \frac{\lfloor st_k \rfloor}{n}, u, v \right) \xi_{i,j,m}^{(n)} dudv \right)^2 \leq C n^2 (\xi_{i,j,m}^{(n)})^2 \delta_n^2,$$

where

$$\delta_n = \int_{\frac{n-1}{n}}^1 (1-u)_+^{\lambda_D-1-\delta} du.$$

It follows from (3.18) and (3.20) that

$$(3.21) \quad (Y_{i,j}^{(n)})^2 \leq C \sum_{q=1}^d \sum_{k=1}^Q n^2 a_k^2 b_q^2 (\xi_{i,j,m}^{(n)})^2 \delta_n^2.$$

On the other hand,

$$(3.22) \quad \{|Y_{i,j}^{(n)}| > \varepsilon\} = \{|Y_{i,j}^{(n)}|^2 > \varepsilon^2\}.$$

Hence, from (3.21) and (3.22),

$$(3.23) \quad \{|Y_{i,j}^{(n)}| > \varepsilon\} \subseteq \{C n^2 (\xi_{i,j,m}^{(n)})^2 \delta_n^2 > \varepsilon^2\}.$$

Consequently,

$$\mathbb{E} \left[ (Y_{i,j}^{(n)})^2 1_{\{|Y_{i,j}^{(n)}| > \varepsilon\}} \middle| \mathcal{F}_{i-1,j-1}^{(n)} \right]$$

$$\begin{aligned}
&\leq C\mathbb{E}\left[n^2(\xi_{i,j,m}^{(n)})^2\delta_n^2\mathbf{1}_{\{Cn^2(\xi_{i,j,m}^{(n)})^2\delta_n^2>\varepsilon^2\}}\middle|\mathcal{F}_{i-1,j-1}^{(n)}\right] \\
(3.24) \quad &\leq C\delta_n^2\mathbb{E}\left[\mathbf{1}_{\{Cn^2(\xi_{i,j,m}^{(n)})^2\delta_n^2>\varepsilon^2\}}\middle|\mathcal{F}_{i-1,j-1}^{(n)}\right]
\end{aligned}$$

for all  $i, j = 1, 2, \dots, n$ . Hence, from (3.1) and (3.24),

$$\begin{aligned}
&\sum_{i=1}^n\sum_{j=1}^n\mathbb{E}\left[(Y_{i,j}^{(n)})^2\mathbf{1}_{\{|Y_{i,j}^{(n)}|>\varepsilon\}}\middle|\mathcal{F}_{i-1,j-1}^{(n)}\right] \\
&\leq\sum_{i=1}^n\sum_{j=1}^nC\delta_n^2\mathbb{E}\left[\mathbf{1}_{\{Cn^2(\xi_{i,j,m}^{(n)})^2\delta_n^2>\varepsilon^2\}}\middle|\mathcal{F}_{i-1,j-1}^{(n)}\right] \\
&\leq C\delta_n^2\sum_{i=1}^n\sum_{j=1}^n\mathbb{E}[\mathbf{1}_{\{C\delta_n^2>\varepsilon^2\}}]\rightarrow 0 \quad (n\rightarrow\infty),
\end{aligned}$$

because  $\delta_n \rightarrow 0$  implies that  $\mathbf{1}_{\{C\delta_n^2>\varepsilon^2\}} = 0$  for large enough  $n$ .

In order to prove (3.14), we also need to show that

$$(3.25) \quad \sum_{i=1}^n\sum_{j=1}^n\left[Y_{i,j}^{(n)}\right]^2\stackrel{\mathbb{P}}{\rightarrow}\mathbb{E}\left[\sum_{q=1}^d\sum_{k=1}^Q\int_0^1\int_0^1a_kb_qK_{q,m}(t_k,s_k,u,v)B^m(du,dv)\right],$$

where  $\stackrel{\mathbb{P}}{\rightarrow}$  denotes convergence in probability. For convenience, we define

$$\tilde{B}^m(t,s,u,v)=\sum_{q=1}^db_qK_{q,m}(t,s,u,v).$$

Note that the right-hand side of (3.25) is equivalent to

$$(3.26) \quad \sum_{i,j=1}^Qa_ia_j\int_0^1\int_0^1\tilde{B}^m(t_i,s_i,u,v)\tilde{B}^m(t_j,t_j,u,v)dudv.$$

Next, we look at the left-hand side of (3.25). In fact, we have

$$(3.27) \quad Y_{i,j}^{(n)}=\sum_{k=1}^Qa_k\int_{\frac{i-1}{n}}^{\frac{i}{n}}\int_{\frac{j-1}{n}}^{\frac{j}{n}}\tilde{B}^m\left(\frac{\lfloor nt_k\rfloor}{n},\frac{\lfloor ns_k\rfloor}{n},u,v\right)\xi_{i,j,m}^{(n)}dudv.$$

Hence,

$$\begin{aligned}
&\left(Y_{i,j}^{(n)}\right)^2=\left(\xi_{i,j,m}^{(n)}\right)^2\sum_{k,l=1}^Qa_ka_l\int_{\frac{i-1}{n}}^{\frac{i}{n}}\int_{\frac{j-1}{n}}^{\frac{j}{n}}\tilde{B}^m\left(\frac{\lfloor nt_k\rfloor}{n},\frac{\lfloor ns_k\rfloor}{n},u,v\right)dudv \\
(3.28) \quad &\cdot\int_{\frac{i-1}{n}}^{\frac{i}{n}}\int_{\frac{j-1}{n}}^{\frac{j}{n}}\tilde{B}^m\left(\frac{\lfloor nt_l\rfloor}{n},\frac{\lfloor ns_l\rfloor}{n},u,v\right)dudv.
\end{aligned}$$

Here, we point out that the entry  $K_{q,m}(t, s, u, v)$  takes the form of

$$\sum_{i=1}^d \tilde{K}_{q,i}(t, u) \tilde{K}_{q,i}(s, v) \tilde{K}_{i,m}(t, u) \tilde{K}_{i,m}(s, v).$$

Hence, it follows from Lemma 3.4 and (3.26)-(3.28) that (3.25) holds.

From the above arguments, we can easily get that the lemma holds.  $\square$

Now, we prove Theorem 3.1.

*Proof of Theorem 3.1.* Theorem 3.1 is a direct consequence of Lemmas 3.2 and 3.3, because tightness and the convergence of finite dimensional distributions imply weak convergence (see Bickel and Wichura [5]).  $\square$

#### 4. Final note

In this work, based on the fractional Brownian motion of Riemann-Liouville type, we introduce the operator fractional Brownian sheet of Riemann-Liouville type  $X$  and present an approximation to it via martingale differences. In Definition 2.1,  $\lambda_D$  and  $\Lambda_D$  are assumed to be at  $(0, 1)$ . In fact, if we only want to define a random field  $X$ ,  $\lambda_D$  and  $\Lambda_D$  can be at a larger interval than  $(0, 1)$ . However, in this paper, we also need the random field  $X$  to enjoy some nice properties. It follows from Mason and Xiao [22] that for an operator self-similar random field  $\{\hat{X}(t, s)\}$  with exponent  $\hat{D}$ , if  $\lambda_{\hat{D}} > 0$ , then  $\hat{X}(0, 0) = (0, \dots, 0)^T$  a.s. Furthermore, if  $\hat{X}(1, 0)$  is proper and  $\mathbb{E}[\|\hat{X}(1, 0)\|_2] < \infty$ , then  $\Lambda_{\hat{D}} \leq 1$ . In this paper, the operator fractional Brownian sheet of Riemann-Liouville type  $\{X(t, s)\}$  is assumed to be proper for  $(t, s)^T = (1, 0)^T$ , and  $X(0, 0) = (0, \dots, 0)^T$  a.s. Hence, we assume  $0 < \lambda_D, \Lambda_D < 1$  in (2.2).

On the other hand, we get from Ayache, Lèger and Pontier [1] that, in the one dimensional case ( $d = 1$ ), a fractional Brownian sheet  $\{W^{\alpha, \beta}(t, s)\}$  with two parameters  $\alpha, \beta \in (0, 1)$  can be defined as

$$(4.1) \quad W^{\alpha, \beta}(t, s) = \int_{\mathbb{R}^2} f_{\alpha}(t, u) f_{\beta}(s, v) \tilde{B}(dv, du),$$

where  $f_H(t, u) = (t - u)_+^{H - \frac{1}{2}} - (-u)_+^{H - \frac{1}{2}}$ . Hence in the one-dimensional case ( $d = 1$ ),  $X$  defined by (2.2) is a special kind of fractional Brownian sheets (with  $\alpha = \beta$ ) of Riemann-Liouville type. In fact, inspired by (4.1), one could like to define the operator fractional Brownian sheet of Riemann-Liouville type  $\hat{X} = \{\hat{X}(t, s); (t, s)^T \in \mathbb{R}_+^2\}$  by

$$\hat{X}(t, s) = \int_0^{\infty} \int_0^{\infty} (t - u)_+^{\frac{\hat{D}}{2} - \frac{1}{2}} (s - v)_+^{\frac{\hat{D}}{2} - \frac{1}{2}} B(du, dv),$$

where  $\hat{D}$  is a linear operator on  $\mathbb{R}^d$  with  $0 < \lambda_{\hat{D}}, \Lambda_{\hat{D}} < 1$ . It is easy to verify that the random field  $\hat{X}$  is well defined. However, in such case, we can not get Theorem 3.1 according to our method.

**Acknowledgments** We thank the referee and the editor for their time and comments. Dai was supported by the National Natural Science Foundation of China (No. 11361007), the Shandong Natural Science Foundation (No. ZR2014 AM021) and the Fostering Project of Dominant Discipline and Talent Team of Shandong Province Higher Education Institutions. Shen was supported by the Distinguished Young Scholars Foundation of Anhui Province (No. 1608085J06), and the Top Talent Project of University Discipline (Speciality) (No. gxb-jZD03).

## References

- [1] A. Ayache, S. Lèger, and M. Pontier, *Drap brownien fractionnaire*, Potential Anal. **17** (2002), no. 1, 31–43.
- [2] X. Bardina and M. Jolis, *Weak approximation of the Brownian sheet from a Poisson process in the plane*, Bernoulli **6** (2000), no. 4, 653–665.
- [3] X. Bardina, M. Jolis, and C. A. Tudor, *Weak convergence to the fractional Brownian sheet and other two-parameter Gaussian processes*, Statist. Probab. Lett. **65** (2003), no. 4, 317–329.
- [4] J. A. Barnes and D. W. Allan, *A statistical model of flicker noise*, Proc. IEEE. **54** (1966), 176–178.
- [5] P. J. Bickel and M. J. Wichura, *Convergence criteria for multiparameter stochastic processes and some applications*, Ann. Math. Statist. **42** (1971), 1656–1670.
- [6] R. Cairoli and J. B. Walsh, *Stochastic integrals in the plane*, Acta Math. **134** (1975), 111–183.
- [7] H. Dai, *Convergence in law to operator fractional Brownian motion of Riemann-Liouville type*, Acta Math. Sin. (Engl. Ser.) **29** (2013), no. 4, 777–788.
- [8] ———, *Convergence in law to operator fractional Brownian motions*, J. Theoret. Probab. **26** (2013), no. 3, 676–696.
- [9] ———, *Approximation to multifractional Riemann-Liouville Brownian sheet*, Comm. Statist. Theory Methods **44** (2015), no. 7, 1399–1410.
- [10] H. Dai, T.-C. Hu, and J.-Y. Lee, *Operator fractional Brownian motion and martingale differences*, Abstr. Appl. Anal. **2014**, Art. ID 791537, 8 pp.
- [11] H. Dai and Y. Li, *A note on approximation to multifractional Brownian motion*, Sci. China Math. **54** (2011), no. 10, 2145–2154.
- [12] H. Dai, G. Shen, and L. Kong, *Limit theorems for functionals of Gaussian vectors*, Front. Math. China **12** (2017), no. 4, 821–842.
- [13] G. Didier and V. Pipiras, *Integral representations and properties of operator fractional Brownian motions*, Bernoulli **17** (2011), no. 1, 1–33.
- [14] ———, *Exponents, symmetry groups and classification of operator fractional Brownian motions*, J. Theoret. Probab. **25** (2012), no. 2, 353–395.
- [15] A. M. Garsia, *Continuity properties of Gaussian processes with multidimensional time parameter*, in Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory, 369–374, Univ. California Press, Berkeley, CA.
- [16] A. Kamont, *On the fractional anisotropic Wiener field*, Probab. Math. Statist. **16** (1996), no. 1, 85–98.
- [17] R. G. Laha and V. K. Rohatgi, *Operator self-similar stochastic processes in  $\mathbb{R}^d$* , Stochastic Process. Appl. **12** (1982), no. 1, 73–84.
- [18] J. Lamperti, *Semi-stable stochastic processes*, Trans. Amer. Math. Soc. **104** (1962), 62–78.

- [19] Y. Li and H. Dai, *Approximations of fractional Brownian motion*, Bernoulli **17** (2011), no. 4, 1195–1216.
- [20] S. C. Lim, *Fractional Brownian motion and multifractional Brownian motion of Riemann-Liouville type*, J. Phys. A **34** (2001), no. 7, 1301–1310.
- [21] M. Maejima and J. D. Mason, *Operator-self-similar stable processes*, Stochastic Process. Appl. **54** (1994), no. 1, 139–163.
- [22] J. D. Mason and Y. Xiao, *Sample path properties of operator-self-similar Gaussian random fields*, Theory Probab. Appl. **46** (2002), no. 1, 58–78.
- [23] R. Morkvenas, *The invariance principle for martingales in the plane*, Litovsk. Mat. Sb. **24** (1984), no. 4, 127–132.
- [24] G. Samorodnitsky and M. S. Taqqu, *Stable non-Gaussian random processes*, Stochastic Modeling, Chapman & Hall, New York, 1994.
- [25] W. Vervaat, *Sample path properties of self-similar processes with stationary increments*, Ann. Probab. **13** (1985), no. 1, 1–27.
- [26] Z. Wang, L. Yan, and X. Yu, *Weak approximation of the fractional Brownian sheet from random walks*, Electron. Commun. Probab. **18** (2013), no. 90, 13 pp.
- [27] ———, *Weak convergence to the fractional Brownian sheet using martingale differences*, Statist. Probab. Lett. **92** (2014), 72–78.
- [28] W. Whitt, *Stochastic-process limits*, Springer Series in Operations Research, Springer-Verlag, New York, 2002.

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