Bull. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/BKMS.b160697
pISSN: 1015-8634 / eISSN: 2234-3016

SPECIAL ORTHONORMAL BASIS FOR L^2 FUNCTIONS ON THE UNIT CIRCLE

Young-Bok Chung

Abstract. We compute explicitly the matrices represented by Toeplitz operators on the Hardy space over the unit circle with respect to a special orthonormal basis constructed by author in terms of their symbols. And we also find a necessary condition for the matrix generated by the product of two Toeplitz operators with respect to the basis to be a Toeplitz matrix by a direct calculation and we finally solve commuting problems of two Toeplitz operators in terms of symbols. This is a generalization of the classical results obtained regarding to the orthonormal basis consisting of the monomials.

1. Introduction

It is well known that Toeplitz operators on the Hardy space over the unit disc U are completely classified in terms of Toeplitz matrices when the family $\mathcal{L}_0 = \{\frac{1}{\sqrt{2\pi}} z^p \,|\, p = 0, \pm 1, \pm 2, \ldots\}$ of the monomials is used as the orthonormal basis for $L^2(bU)$ under the Lebesgue measure. Furthermore the matrix representation with respect to the basis $\mathcal{L}_0$ can be recaptured from the symbols of the Toeplitz operators. In fact, if $\varphi \in L^\infty(bU)$ is the symbol of the Toeplitz operator T, then the i-th and j-th entry of the matrix of T with respect to $\mathcal{L}_0$ is equal to the $(i-j)$-th Fourier coefficient $\langle \varphi, \frac{1}{\sqrt{2\pi}} z^{i-j} \rangle$ of the φ (see [3] for details).

What are then the counterparts of the constant $\frac{1}{\sqrt{2\pi}}$ and the monomial functions z^p for the case of general bounded domains? The author recently in [4] constructed a corresponding orthonormal basis $\mathcal{L}$ for $L^2(b\Omega)$ consisting of the Szegő kernel, the Garabedian kernel, and the Ahlfors map over general C^∞ smooth bounded domain Ω and classified Toeplitz operators in terms of Toeplitz matrices with respect to a subset of $\mathcal{L}$. In particular he computed the matrix of T completely in terms of the Fourier coefficients of the symbol and found a necessary condition for the product of two Toeplitz operators to become a Toeplitz operator (see [5] for the reference). On the other hand, when

Received August 24, 2016; Accepted December 26, 2016.
2010 *Mathematics Subject Classification.* Primary 47B35, 30C40.
Key words and phrases. Toeplitz operator, Toeplitz matrices, Hardy space, Szego kernel.

the base domain is the unit disc U, the functions used in the construction of the orthonormal basis $\mathcal{L}$ are easily computable and so we can expect to have more compact result.

In this paper, we work on only the unit disc and prove all the results without using notations for kernel functions. So we would like to write this article as elementarily as possible. In the end computation of the matrices of the Toeplitz operators with respect to $\mathcal{L}$ will be done in terms of the associated symbols in a concrete form without having series expansion. We also obtain a necessary condition for the matrix representation of the product of two Toeplitz operators with resect to $\mathcal{L}$ to be a Toeplitz matrix which is written in terms of symbols in explicit forms. We finally solve commuting problems of two Teoplitz operators about symbols. All the results obtained in the paper are generalizations for the properties held regarding to the classical orthonormal basis $\mathcal{L}_0$.

2. Main results

Let U be the unit disc in the complex plane and let $L^2(bU)$ be the space of square integrable functions on bU with the inner product defined by

$$\langle u, v\rangle = \int_{b\Omega} u\,\overline{v}\; ds$$

where ds is the differential element of arc length on the boundary bU. Let $H^2(bU)$ denote the classical Hardy space of holomorphic functions on U with L^2-boundary values in bU. Let P be the Szegő projection that is the orthogonal projection of $L^2(bU)$ onto $H^2(bU)$ defined by

$$(Pu)(a) = \frac{1}{2\pi}\int_{bU} \frac{zu(z)}{z-a} ds \;\text{ for } u \in L^2(bU),\; a \in U. \tag{2.1}$$

It is well known (see [1], [2]) that the space $L^2(bU)$ has an orthogonal decomposition as a direct sum of the Hardy space $H^2(bU)$ and its orthogonal complement $H^2(bU)^{\perp}$ via the identity

$$u(z) = Pu(z) + \frac{1}{z}\,\overline{P\left(\overline{u(\zeta)}/\zeta\right)(z)} \;\text{ for } u \in L^2(bU) \tag{2.2}$$

which plays a key role in this paper.

Let $L^\infty(bU)$ be the space of essentially bounded measurable functions on the unit circle bU and let φ be in $L^\infty(bU)$. The operator T_φ defined on $H^2(bU)$ by

$$T_\varphi(h) = P(\varphi\, h), \quad h \in H^2(bU).$$

is called the Toeplitz operator with symbol φ and so the operator T is the compression of a multiplication operator on the circle to the Hardy space $H^2(bU)$. Note that the Toeplitz operators are linear in the symbol, that is, $T_{\alpha\varphi+\beta\psi} = \alpha T_\varphi + \beta T_\psi$ for $\alpha, \beta \in \mathbb{C}$ and for $\varphi, \psi \in L^\infty(bU)$ and they also have many algebraic properties as multiplication operators do (see [3] for more details).

The following theorem proved by the author [4] is a reformulated result in the case of the unit disc which will be often referred in the remaining part of the paper.

Theorem 2.1. *Let a be in the unit disc U. Define for $k \in \mathbb{Z}$ the complex rational function E_k by*

$$E_k = \sqrt{\frac{1-|a|^2}{2\pi}}\,\frac{(z-a)^k}{(1-\overline{a}z)^{k+1}}.$$

Then

1. $\mathcal{L} = \{E_k \,|\, k \in \mathbb{Z}\}$ *is an orthonormal basis for* $L^2(bU)$,
2. $\mathcal{L}^+ = \{E_k \,|\, k \in \mathbb{Z}, k \geq 0\}$ *is an orthonormal basis for* $H^2(bU)$, *and*
3. $\mathcal{L}^- = \{E_k \,|\, k \in \mathbb{Z}, k \leq -1\}$ *is an orthonormal basis for* $H^2(bU)^\perp$.

Observe that given $a \in U$, the function E_k in $\mathcal{L}^+$ is a meromorphic rational function with a single pole at $1/\overline{a}$ on the whole plane and hence it is a holomorphic function in a neighborhood of the closed unit disc. Notice also that when $a = 0$, the family $\mathcal{L}$ in the above theorem is just the classical orthonormal basis of the monomials $\sqrt{1/2\pi}\, z^k$.

In order to compute the matrix represented by the Toeplitz operator T_φ with respect to $\mathcal{L}^+$, we need several lemmas about the inner products $< E_p E_l, E_m >$ for integers p, l, m. The first lemma says that the inner products have an additive property for those subscript integers.

Lemma 2.2. *For $p, l, m \in \mathbb{Z}$,*

$$\langle E_p E_l, E_m\rangle = \langle E_{p+l-m} E_0, E_0\rangle.$$

Proof. The proof is easily done by using the fact that

$$\overline{\left(\frac{z-a}{1-\overline{a}z}\right)} = \frac{1-\overline{a}z}{z-a} \text{ for } z \in bU.$$

In fact,

$$\begin{aligned}
&\langle E_p E_l, E_m\rangle \\
&= \sqrt{\frac{1-|a|^2}{2\pi}}^3 \int_{bU} \frac{(z-a)^p}{(1-\overline{a}z)^{p+1}}\,\frac{(z-a)^l}{(1-\overline{a}z)^{l+1}}\,\frac{(1-\overline{a}z)^m}{(z-a)^m}\overline{\left(\frac{1}{1-\overline{a}z}\right)}ds \\
&= \sqrt{\frac{1-|a|^2}{2\pi}}^3 \int_{bU} \frac{(z-a)^{p+l-m}}{(1-\overline{a}z)^{p+l-m+1}}\,\frac{1}{(1-\overline{a}z)}\overline{\left(\frac{1}{1-\overline{a}z}\right)}ds \\
&= \langle E_{p+l-m} E_0, E_0\rangle.
\end{aligned}$$

□

A one-way infinite matrix $M = [m_{ij}], i, j = 0, 1, 2, \ldots$ is called a Toeplitz matrix of order $k \in \mathbb{N}$ if

$$m_{i+k,j+k} = m_{i,j},\ i, j = 0, 1, 2, \ldots$$

and so these matrices are k-tuple diagonal-constant matrices. It is easy to see that the matrix $[T_\varphi]$ of the Toeplitz operator T_φ with respect to the orthonormal basis $\mathcal{L}^+$ is a Toeplitz matrix of order 1 (see [4] for general bounded domains). In fact, it follows from Lemma 2.2 that $\langle E_p E_{l+1}, E_{m+1}\rangle = \langle E_{p+(l+1)-(m+1)}E_0, E_0\rangle = \langle E_{p+l-m}E_0, E_0\rangle$. Hence we have proved the following corollary.

Corollary 2.3. *Let a be fixed in the unit disc U. For $\varphi \in L^\infty(bU)$, the matrix $[T_\varphi]$ of the Toeplitz operator T_φ on the Hardy space $H^2(bU)$ with respect to the orthonormal basis $\mathcal{L}^+$ is a Toeplitz matrix of order 1.*

In the next two lemmas we compute the inner products $\langle E_p E_l, E_m\rangle$ which do not vanish.

Lemma 2.4.

$$\langle E_0 E_0, E_0\rangle = \frac{1}{\sqrt{2\pi(1-|a|^2)}}.$$

Proof. It follows from the identities $1-\overline{a}z = \overline{z}^{-1}(\overline{z}-\overline{a})$ and $dz = izds$ on bU that

$$\begin{aligned}
\langle E_0 E_0, E_0\rangle &= \sqrt{\frac{1-|a|^2}{2\pi}}^{\,3} \int_{bU} \frac{\overline{z}^2}{\overline{(z-a)^2}} \frac{1}{\overline{1-\overline{a}z}}\, ds \\
&= \sqrt{\frac{1-|a|^2}{2\pi}}^{\,3}\, \overline{\int_{bU} \frac{z^2}{(z-a)^2} \frac{1}{1-\overline{a}z}\, ds} \\
&= \sqrt{\frac{1-|a|^2}{2\pi}}^{\,3}\, \overline{\frac{1}{i}\int_{bU} \frac{1}{(z-a)^2} \frac{z}{1-\overline{a}z}\, dz}.
\end{aligned}$$

Thus by the Residue theorem, the last identity is equal to

$$\sqrt{\frac{1-|a|^2}{2\pi}}^{\,3}\, \overline{2\pi \operatorname{Res}\left(\frac{1}{(z-a)^2}\frac{z}{1-\overline{a}z}; a\right)} = \sqrt{\frac{1-|a|^2}{2\pi}}^{\,3} \frac{2\pi}{(1-|a|^2)^2}$$

which proves the lemma. □

Lemma 2.5.

$$\langle E_{-1} E_0, E_0\rangle = \frac{\overline{a}}{\sqrt{2\pi(1-|a|^2)}}.$$

Proof. Similarly as in the proof of Lemma 2.4, the identity above can be verified as follows.

$$\begin{aligned}
\langle E_{-1} E_0, E_0\rangle &= \sqrt{\frac{1-|a|^2}{2\pi}}^{\,3} \int_{bU} \frac{1}{z-a}\frac{1}{1-\overline{a}z}\frac{1}{\overline{1-\overline{a}z}}\, ds \\
&= \sqrt{\frac{1-|a|^2}{2\pi}}^{\,3} \int_{bU} \frac{1}{z-a}\frac{1}{1-\overline{a}z}\frac{z}{z-a}\, ds
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\frac{1-|a|^2}{2\pi}}^3 \frac{1}{i} \int_{bU} \frac{1}{(z-a)^2} \frac{1}{1-\overline{a}z} dz \\
&= \sqrt{\frac{1-|a|^2}{2\pi}}^3 \; 2\pi \operatorname{Res}\left(\frac{1}{(z-a)^2} \frac{1}{1-\overline{a}z}; a \right) \\
&= \sqrt{\frac{1-|a|^2}{2\pi}}^3 \; \frac{2\pi\overline{a}}{(1-|a|^2)^2}
\end{aligned}
$$

which proves the lemma. □

Now in the next two lemmas we show that the inner product $\langle E_m E_0, E_0 \rangle = 0$ for all $m \neq 0, -1$.

Lemma 2.6. *For $k \geq 1$,*

$$\langle E_k E_0, E_0 \rangle = 0.$$

Proof.

$$
\begin{aligned}
\langle E_k E_0, E_0 \rangle &= \sqrt{\frac{1-|a|^2}{2\pi}}^3 \int_{bU} \frac{(z-a)^k}{(1-\overline{a}z)^{k+1}} \frac{1}{1-\overline{a}z} \frac{1}{\overline{1-\overline{a}z}} ds \\
&= \sqrt{\frac{1-|a|^2}{2\pi}}^3 \frac{1}{i} \int_{bU} \frac{(z-a)^k}{(1-\overline{a}z)^{k+2}} \frac{1}{\overline{z}(z-a)} \frac{1}{z} dz \\
&= \sqrt{\frac{1-|a|^2}{2\pi}}^3 \frac{1}{i} \int_{bU} \frac{(z-a)^{k-1}}{(1-\overline{a}z)^{k+2}} dz.
\end{aligned}
$$

Since the integrand $\frac{(z-a)^{k-1}}{(1-\overline{a}z)^{k+2}}$ in the integral is holomorphic in a neighborhood of the closed unit disc, the last identity vanishes by the Cauchy theorem. □

Lemma 2.7. *For $k \geq 2$,*

$$\langle E_{-k} E_0, E_0 \rangle = 0.$$

Proof.

$$
\begin{aligned}
\langle E_{-k} E_0, E_0 \rangle &= \sqrt{\frac{1-|a|^2}{2\pi}}^3 \int_{bU} \frac{(z-a)^{-k}}{(1-\overline{a}z)^{-k+1}} \frac{1}{1-\overline{a}z} \frac{1}{\overline{1-\overline{a}z}} ds \\
&= \sqrt{\frac{1-|a|^2}{2\pi}}^3 \frac{1}{i} \int_{bU} \frac{(1-\overline{a}z)^{k-1}}{(z-a)^k} \frac{1}{1-\overline{a}z} \frac{1}{\overline{z}(z-a)} \frac{1}{z} dz \\
&= \sqrt{\frac{1-|a|^2}{2\pi}}^3 \frac{1}{i} \int_{bU} \frac{(1-\overline{a}z)^{k-2}}{(z-a)^{k+1}} dz.
\end{aligned}
$$

Here the residue of the integrand at $z = a$ is equal to $\frac{1}{k!} \frac{d^k}{dz^k} (1-\overline{a}z)^{k-2}|_{z=a}$ which must be zero because the analytic polynomial $(1-\overline{a}z)^{k-2}$ has degree not more than $k-2$ and so the proof is done. □

Now we are ready to prove the following theorem which represents the matrix of the Toeplitz operator T_φ completely in terms of the Fourier coefficients of the symbol φ.

Theorem 2.8. *Let a be fixed in the unit disc U. Let $\varphi \in L^\infty(bU)$ have the Fourier series representation $\varphi = \sum_{p=-\infty}^{\infty} \alpha_p E_p$ with respect to the orthonormal basis $\mathcal{L}$. Then for nonnegative integers m and l, the entry of m-th row and l-th column of the matrix $[T_\varphi]$ of the Toeplitz operator T_φ on the Hardy space $H^2(bU)$ with respect to the orthonormal basis $\mathcal{L}^+$ is equal to*

$$[T_\varphi]_{ml} = \frac{1}{\sqrt{2\pi(1-|a|^2)}}\alpha_{m-l} + \frac{\overline{a}}{\sqrt{2\pi(1-|a|^2)}}\alpha_{m-l-1}. \tag{2.3}$$

Proof. Since the matrix $[T_\varphi]$ is a Toeplitz matrix of order 1, for $m \geq l$, it follows that

$$[T_\varphi]_{ml} = [T_\varphi]_{m-1,l-1} = [T_\varphi]_{m-2,l-2} = \cdots = [T_\varphi]_{m-l,0}.$$

Thus by Lemmas 2.2, 2.6, and 2.7, the last identity yields

$$\begin{aligned}[T_\varphi]_{m-l,0} &= \sum_{p=-\infty}^{\infty} \alpha_p \langle E_p E_0, E_{m-l}\rangle = \sum_{p=-\infty}^{\infty} \alpha_p \langle E_{p-m+l} E_0, E_0\rangle \\ &= \alpha_{m-l}\langle E_0 E_0, E_0\rangle + \alpha_{m-l-1}\langle E_{-1} E_0, E_0\rangle\end{aligned}$$

which is exactly equal to the formula of Theorem 2.8. For $m < l$, similarly as in the case of $m \geq l$, we obtain

$$\begin{aligned}[T_\varphi]_{ml} = [T_\varphi]_{0,l-m} &= \sum_{p=-\infty}^{\infty} \alpha_p \langle E_p E_{l-m}, E_0\rangle \\ &= \sum_{p=-\infty}^{\infty} \alpha_p \langle E_{p+l-m} E_0, E_0\rangle \\ &= \alpha_{m-l}\langle E_0 E_0, E_0\rangle + \alpha_{m-l-1}\langle E_{-1} E_0, E_0\rangle\end{aligned}$$

which proves the identity (2.3) of the theorem. □

The matrix $[T_\varphi]$ can be written in the more compact form using a lower shift matrix as follows.

Corollary 2.9. *Let a be fixed in the unit disc U. Let $\varphi \in L^\infty(bU)$ have the Fourier series representation $\varphi = \sum_{p=-\infty}^{\infty} \alpha_p E_p$ with respect to the orthonormal basis $\mathcal{L}$. Then the matrix $[T_\varphi]$ of the Toeplitz operator T_φ on the Hardy space $H^2(bU)$ with respect to the orthonormal basis $\mathcal{L}^+$ is equal to*

$$[T_\varphi] = \frac{1}{\sqrt{2\pi(1-|a|^2)}} A(I + \overline{a}L)$$

where A is the matrix composing of the Fourier coefficients of the symbol φ given by

$$A = \begin{bmatrix} \alpha_0 & \alpha_{-1} & \alpha_{-2} & \cdots \\ \alpha_1 & \alpha_0 & \alpha_{-1} & \cdots \\ \alpha_2 & \alpha_1 & \alpha_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

I is the identity matrix, and L is the one-way infinite lower shift matrix given by

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Proof. The proof is easily done by Theorem 2.8. In fact, it follows that

$$\begin{aligned} &[T_\varphi] \\ &= \begin{bmatrix} \frac{1}{\sqrt{2\pi(1-|a|^2)}}(\alpha_0 + \overline{a}\alpha_{-1}) & \frac{1}{\sqrt{2\pi(1-|a|^2)}}(\alpha_{-1} + \overline{a}\alpha_{-2}) & \cdots \\ \frac{1}{\sqrt{2\pi(1-|a|^2)}}(\alpha_1 + \overline{a}\alpha_0) & \frac{1}{\sqrt{2\pi(1-|a|^2)}}(\alpha_0 + \overline{a}\alpha_{-1}) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \\ &= \frac{1}{\sqrt{2\pi(1-|a|^2)}} \begin{bmatrix} \alpha_0 & \alpha_{-1} & \alpha_{-2} & \cdots \\ \alpha_1 & \alpha_0 & \alpha_{-1} & \cdots \\ \alpha_2 & \alpha_1 & \alpha_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\ &\qquad + \frac{\overline{a}}{\sqrt{2\pi(1-|a|^2)}} \begin{bmatrix} \alpha_{-1} & \alpha_{-2} & \alpha_{-3} & \cdots \\ \alpha_0 & \alpha_{-1} & \alpha_{-2} & \cdots \\ \alpha_1 & \alpha_0 & \alpha_{-1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\ &= \frac{1}{\sqrt{2\pi(1-|a|^2)}} A \\ &\quad + \frac{\overline{a}}{\sqrt{2\pi(1-|a|^2)}} \begin{bmatrix} \alpha_0 & \alpha_{-1} & \alpha_{-2} & \cdots \\ \alpha_1 & \alpha_0 & \alpha_{-1} & \cdots \\ \alpha_2 & \alpha_1 & \alpha_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \end{aligned}$$

□

The product of two Toeplitz operators is not a Toeplitz operator in general and so we find a necessary condition for the product to be a Toeplitz operator in terms of Toeplitz matrices. Using the identity (2.3), we obtain

$$
\begin{aligned}
&[T_\varphi T_\psi]_{m+1,l+1} = \sum_{p=0}^{\infty} [T_\varphi]_{m+1,p}[T_\psi]_{p,l+1} \\
&= \sum_{p=0}^{\infty} \left(\frac{1}{\sqrt{2\pi(1-|a|^2)}} \alpha_{m+1-p} + \frac{\overline{a}}{\sqrt{2\pi(1-|a|^2)}} \alpha_{m-p} \right) \cdot \\
&\qquad\qquad \left(\frac{1}{\sqrt{2\pi(1-|a|^2)}} \beta_{p-l-1} + \frac{\overline{a}}{\sqrt{2\pi(1-|a|^2)}} \beta_{p-l-2} \right) \\
&= \sum_{p=0}^{\infty} \left(\frac{1}{\sqrt{2\pi(1-|a|^2)}} \alpha_{m-p} + \frac{\overline{a}}{\sqrt{2\pi(1-|a|^2)}} \alpha_{m-p-1} \right) \cdot \\
&\qquad\qquad \left(\frac{1}{\sqrt{2\pi(1-|a|^2)}} \beta_{p-l} + \frac{\overline{a}}{\sqrt{2\pi(1-|a|^2)}} \beta_{p-l-1} \right) \\
&\quad + \left(\frac{1}{\sqrt{2\pi(1-|a|^2)}} \alpha_{m+1} + \frac{\overline{a}}{\sqrt{2\pi(1-|a|^2)}} \alpha_{m} \right) \cdot \\
&\qquad\qquad \left(\frac{1}{\sqrt{2\pi(1-|a|^2)}} \beta_{-l-l} + \frac{\overline{a}}{\sqrt{2\pi(1-|a|^2)}} \beta_{-l-2} \right) \\
&= [T_\varphi T_\psi]_{m,l} + \frac{1}{2\pi(1-|a|^2)} (\alpha_{m+1} + \overline{a}\alpha_m)(\beta_{-l-1} + \overline{a}\beta_{-l-2}).
\end{aligned}
$$

Hence we have proved the following.

Corollary 2.10. *Let a be fixed in the unit disc U. Let $\varphi, \psi \in L^\infty(bU)$ have the Fourier series representations $\varphi = \sum_{p=-\infty}^{\infty} \alpha_p E_p$ and $\psi = \sum_{q=-\infty}^{\infty} \beta_p E_p$ with respect to the orthonormal basis $\mathcal{L}$. Let T_φ and T_ψ be the Toeplitz operators with symbols φ and ψ on the Hardy space $H^2(bU)$. Then for nonnegative integers m and l, the entries of the matrix of the product $T_\varphi T_\psi$ with respect to the orthonormal basis $\mathcal{L}^+$ satisfy*

$$
\begin{aligned}
[T_\varphi T_\psi]_{m+1,l+1} &= [T_\varphi T_\psi]_{m,l} \\
&\quad + \frac{1}{2\pi(1-|a|^2)} (\alpha_{m+1} + \overline{a}\alpha_m)(\beta_{-l-1} + \overline{a}\beta_{-l-2}).
\end{aligned}
\tag{2.4}
$$

Definition. We say that for a C^∞ smooth bounded domain Ω, a function φ in $L^\infty(b\Omega)$ is analytic with respect to an orthonormal basis $\mathcal{B} = \{u_m \,|\, m \in \mathbb{Z}\}$ for $L^2(b\Omega)$ if for all negative integers m, the Fourier coefficient $\langle \varphi, u_m \rangle = 0$ and that φ is co-analytic with respect to $\mathcal{B}$ if the function $\varphi - \sum_{p=-\infty}^{-1} \langle \varphi, u_p \rangle u_p$ is a constant.

Observe that for the case of the unit disc $\Omega = U$, the orthonormal basis $\mathcal{L} = \{E_m \,|\, m \in \mathbb{Z}\}$ becomes the family $\mathcal{L}_0$ consisting of monomials $\frac{1}{\sqrt{2\pi}} z^k$ when $a = 0$. Hence in this case, since $\overline{z^m} = z^{-m}$ and the orthogonal projection $P^\perp(\varphi)$

is equal to $\overline{zP\left(\overline{\varphi(\zeta)}\big/\zeta\right)(z)}$ by (2.2), it is easy to see that co-analyticity of φ with respect to $\mathcal{L}_0$ is equivalent to saying that the conjugate of φ is holomorphic in $H^2(bU)$ and so the definition of analyticity(co-analyticity) with respect to $\mathcal{L}_0$ is consistent with the classical one.

Theorem 2.11. *Let a be fixed in the unit disc U. Let $\varphi, \psi \in L^\infty(bU)$ have the Fourier series representations $\varphi = \sum_{p=-\infty}^{\infty} \alpha_p E_p$ and $\psi = \sum_{q=-\infty}^{\infty} \beta_p E_p$ with respect to the orthonormal basis $\mathcal{L}$. Let T_φ and T_ψ be the Toeplitz operators with symbols φ and ψ on the Hardy space $H^2(bU)$. If the matrix $[T_\varphi T_\psi]$ of the product of T_φ and T_ψ with respect to $\mathcal{L}^+$ is a Toeplitz matrix of order 1, then either φ is co-analytic with respect to $\mathcal{L}$ or ψ is analytic with respect to $\mathcal{L}$. In fact, in the case, either*

$$\varphi = \frac{\alpha_0}{\sqrt{2\pi(1-|a|^2)}} + \sum_{p=-\infty}^{-1} \alpha_p E_p \tag{2.5}$$

or

$$\psi = \frac{\overline{a}\,\beta_{-1}}{\sqrt{2\pi(1-|a|^2)}} + \sum_{q=0}^{\infty} \beta_q E_q \tag{2.6}$$

Proof. Suppose that the matrix $[T_\varphi T_\psi]$ of the product of T_φ and T_ψ with respect to $\mathcal{L}^+$ is a Toeplitz matrix of order 1. Then it follows from the identity (2.4) that for all $m, l \geq 0$,

$$(\alpha_{m+1} + \overline{a}\alpha_m)(\beta_{-l-1} + \overline{a}\beta_{-l-2}) = 0$$

which yields either for all $m \geq 0$, $\alpha_{m+1} + \overline{a}\alpha_m = 0$ or for all $l \geq 0$, $\beta_{-l-1} + \overline{a}\beta_{-l-2} = 0$. On the other hand, the Fourier coefficients of φ and ψ can be written as follows. For $m \geq 0$,

$$\begin{aligned}
\alpha_m = \langle P\varphi, E_m\rangle &= \sqrt{\frac{1-|a|^2}{2\pi}} \int_{bU} (P\varphi) \frac{\overline{z-a}^{m}}{\overline{1-\overline{a}z}^{m+1}} ds \\
&= \sqrt{\frac{1-|a|^2}{2\pi}} \frac{1}{i} \int_{bU} (P\varphi) \frac{\overline{z}^m (1-\overline{a}z)^m}{\overline{z}^{m+1}(z-a)^{m+1}} \frac{1}{z} dz \\
&= \sqrt{\frac{1-|a|^2}{2\pi}} \frac{1}{i} \int_{bU} \frac{1}{(z-a)^{m+1}} \left[(P\varphi)(1-\overline{a}z)^m\right] dz \\
&= \frac{\sqrt{2\pi(1-|a|^2)}}{m!} \left[(P\varphi)(1-\overline{a}z)^m\right]^{(m)}(a).
\end{aligned}$$

Here we used the identity $\overline{z} = 1/z$ for $|z| = 1$ and the Residue theorem. Thus

$$\begin{aligned}
\alpha_{m+1} &= \frac{\sqrt{2\pi(1-|a|^2)}}{(m+1)!} \left[(P\varphi)(1-\overline{a}z)^{m+1}\right]^{(m+1)}(a) \\
&= \frac{\sqrt{2\pi(1-|a|^2)}}{(m+1)!} \left[(P\varphi)'(1-\overline{a}z)^{m+1} - (m+1)\overline{a}(P\varphi)(1-\overline{a}z)^m\right]^{(m)}(a)
\end{aligned}$$

$$= \frac{\sqrt{2\pi(1-|a|^2)}}{(m+1)!}\sum_{j=0}^{m}(P\varphi)^{(j+1)}(a)[(1-\overline{a}z)^{m+1}]^{(m-j)}(a) - \overline{a}\alpha_m$$

which yields

$$\alpha_{m+1} + \overline{a}\alpha_m = \frac{\sqrt{2\pi(1-|a|^2)}}{(m+1)!}\sum_{j=0}^{m}(P\varphi)^{(j+1)}(a)[(1-\overline{a}z)^{m+1}]^{(m-j)}(a). \tag{2.7}$$

Now as the first case, suppose that $\alpha_{m+1} + \overline{a}\alpha_m = 0$ for $m \geq 0$. If $a = 0$, then $\alpha_m = 0$ for all $m \geq 1$ and so $\varphi - \sum_{p=-\infty}^{-1} \alpha_p E_p = \alpha_0 E_0 = \alpha_0/\sqrt{2\pi}$ which is a constant and hence φ is co-analytic. For $a \neq 0$, it follows from the identity (2.7) that

$$\begin{aligned} &\sum_{j=0}^{m}(P\varphi)^{(j+1)}(a)[(1-\overline{a}z)^{m+1}]^{(m-j)}(a) \\ &= \sum_{j=0}^{m}(P\varphi)^{(j+1)}(a)(m+1)m(m-1)\cdots(j+2)(1-|a|^2)^{j+1}(-\overline{a})^{m-j} = 0, \end{aligned} \tag{2.8}$$

from which it is easy to see by induction on m that for all $k \geq 1$, $(P\varphi)^{(k)}(a) = 0$. Then the holomorphic function $P\varphi$ must be the constant $(P\varphi)(a)$ and so since the function $\varphi - (P\varphi)(a)$ belongs to the orthogonal complement $H^2(bU)^\perp$ of the Hardy space, $\varphi = (P\varphi)(a) + \sum_{-\infty}^{-1} \alpha_p E_p$. Furthermore, from the identity (2.1) the constant $(P\varphi)(a)$ is equal to

$$\langle \varphi, \frac{1}{2\pi(1-\overline{a}z)}\rangle = \frac{1}{\sqrt{2\pi(1-|a|^2)}}\langle \varphi, E_0\rangle = \frac{\alpha_0}{\sqrt{2\pi(1-|a|^2)}}$$

which proves the statement of Theorem 2.11 with the former assumption.

Next suppose that the latter condition

$$\beta_{-l-1} + \overline{a}\beta_{-l-2} = 0 \text{ for } l \geq 0 \tag{2.9}$$

holds. If $a = 0$, then $\beta_{-l-1} = 0$ for all $l \geq 0$. Then the function $\psi = \sum_{q=0}^{\infty} \beta_q E_q$ is an analytic function with respect to $\mathcal{L}$. Next suppose that $a \neq 0$. The Fourier coefficients β_{-1} and β_{-2} can be computed as follows. Let $\Lambda = P\left(\overline{\psi(\zeta)}\big/\zeta\right)$ denote the projection of the function $\overline{\psi(\zeta)}\big/\zeta$ appearing in the orthogonal decomposition (2.2) of ψ for simplicity. It follows from (2.2) that

$$\begin{aligned} \beta_{-1} &= \langle \psi, E_{-1}\rangle = \langle \frac{\overline{\Lambda(z)}}{z}, E_{-1}\rangle \\ &= \sqrt{\frac{1-|a|^2}{2\pi}}\,\overline{\int_{bU}\frac{z\Lambda(z)}{z-a}ds} = \sqrt{\frac{1-|a|^2}{2\pi}}\,\overline{\frac{1}{i}\int_{bU}\frac{\Lambda(z)}{z-a}dz} \\ &= \sqrt{2\pi(1-|a|^2)}\,\overline{\Lambda(a)} \end{aligned} \tag{2.10}$$

and

$$\beta_{-2} = \langle \psi, E_{-2}\rangle = \langle \frac{\overline{\Lambda(z)}}{z}, E_{-2}\rangle$$

$$\begin{aligned}
&= \sqrt{\frac{1-|a|^2}{2\pi}}\,\overline{\int_{bU} \frac{z\Lambda(z)(1-\overline{a}z)}{(z-a)^2} ds} \\
&= \sqrt{\frac{1-|a|^2}{2\pi}}\,\overline{\frac{1}{i}\int_{bU} \frac{\Lambda(z)(1-\overline{a}z)}{(z-a)^2} dz} \\
&= \sqrt{2\pi(1-|a|^2)}\,\overline{[(1-\overline{a}z)\Lambda(z)]'(a)} \\
&= \sqrt{2\pi(1-|a|^2)}(1-|a|^2)\overline{\Lambda'(a)} + \sqrt{2\pi(1-|a|^2)}\,\overline{(-\overline{a})\Lambda(a)} \\
&= \sqrt{2\pi(1-|a|^2)}(1-|a|^2)\overline{\Lambda'(a)} - a\beta_{-1}.
\end{aligned}$$

So

$$0 = \beta_{-1} + \overline{a}\beta_{-2} = \sqrt{2\pi(1-|a|^2)}(1-|a|^2)\overline{\Lambda(a) + a\Lambda'(a)},$$

which yields the identity

$$[z\Lambda(z)]'(a) = 0.$$

In order to show for all $l \geq 1$,

$$[z\Lambda(z)]^{(l)}(a) = a\Lambda^{(l)}(a) + l\Lambda^{(l-1)}(a) = 0 \tag{2.11}$$

by induction on l we let $k \geq 1$ and we assume that the identity (2.11) has been proved for $l = 1, 2, \ldots, k$. Then it follows from the Residue theorem and the general Leibniz rule that

$$\begin{aligned}
\beta_{-k-1} &= \langle \psi, E_{-k-1}\rangle = \langle \frac{\overline{\Lambda(z)}}{z}, E_{-k-1}\rangle \\
&= \sqrt{\frac{1-|a|^2}{2\pi}}\,\overline{\int_{bU} \frac{z\Lambda(z)(1-\overline{a}z)^k}{(z-a)^{k+1}} ds} \\
&= \sqrt{\frac{1-|a|^2}{2\pi}}\,\overline{\frac{1}{i}\int_{bU} \frac{\Lambda(z)(1-\overline{a}z)^k}{(z-a)^{k+1}} dz} \\
&= \frac{\sqrt{2\pi(1-|a|^2)}}{k!}\,\overline{[(1-\overline{a}z)^k\Lambda(z)]^{(k)}(a)} \\
&= \frac{\sqrt{2\pi(1-|a|^2)}}{k!}\,\overline{\sum_{j=0}^{k}\binom{k}{j}[(1-\overline{a}z)^k]^{(k-j)}(a)\Lambda^{(j)}(a)} \\
&= \frac{\sqrt{2\pi(1-|a|^2)}}{k!}\,\overline{\sum_{j=0}^{k}\binom{k}{j}\frac{k!}{j!}(1-|a|^2)^j(-\overline{a})^{k-j}\Lambda^{(j)}(a)} \\
&= \sqrt{2\pi(1-|a|^2)}\sum_{j=0}^{k}\binom{k}{j}\frac{1}{j!}(1-|a|^2)^j(-a)^{k-j}\overline{\Lambda^{(j)}(a)}.
\end{aligned} \tag{2.12}$$

Notice also from the inductive hypothesis (2.11) that for $0 \leq j \leq k$

$$\Lambda^{(j)}(a) = (-1)^{k-j}\frac{j!}{k!}a^{k-j}\Lambda^{(k)}(a). \tag{2.13}$$

Thus by using the identities (2.12) and (2.13) we obtain

$$\beta_{-k-1} = \sqrt{2\pi(1-|a|^2)}\ \left[\sum_{j=0}^{k} \frac{1}{j!(k-j)!}(1-|a|^2)^j |a|^{2(k-j)}\right] \overline{\Lambda^{(k)}(a)} \tag{2.14}$$

and

$$\begin{aligned}
\beta_{-k-2} &= \sqrt{2\pi(1-|a|^2)}\ \sum_{j=0}^{k+1} \binom{k+1}{j} \frac{1}{j!}(1-|a|^2)^j (-a)^{k+1-j} \overline{\Lambda^{(j)}(a)} \\
&= \sqrt{2\pi(1-|a|^2)}\ \sum_{j=0}^{k} \binom{k+1}{j} \frac{1}{j!}(1-|a|^2)^j (-a)^{k+1-j} \overline{\Lambda^{(j)}(a)} \\
&\quad + \sqrt{2\pi(1-|a|^2)} \frac{1}{(k+1)!}(1-|a|^2)^{k+1} \overline{\Lambda^{(k+1)}(a)} \\
&= \sqrt{2\pi(1-|a|^2)}(k+1)(-a) \left[\sum_{j=0}^{k} \frac{1}{j!(k+1-j)!}(1-|a|^2)^j |a|^{2(k-j)}\right] \overline{\Lambda^{(k)}(a)} \qquad (2.15) \\
&\quad + \sqrt{2\pi(1-|a|^2)} \frac{1}{(k+1)!}(1-|a|^2)^{k+1} \overline{\Lambda^{(k+1)}(a)}.
\end{aligned}$$

Since by assumption (2.9), $1/\sqrt{2\pi(1-|a|^2)}\left[\overline{\beta_{-k-1}} + a\ \overline{\beta_{-k-2}}\right] = 0$, it is easy to see from (2.14) and (2.15) that

$$\begin{aligned}
0 &= \Lambda^{(k)}(a) \Bigg[\sum_{j=0}^{k} \frac{1}{j!(k-j)!}(1-|a|^2)^j |a|^{2(k-j)} \\
&\quad - (k+1) \sum_{j=0}^{k} \frac{1}{j!(k+1-j)!}(1-|a|^2)^j |a|^{2(k+1-j)} \Bigg] \\
&\quad + \frac{a}{(k+1)!}(1-|a|^2)^{k+1} \Lambda^{(k+1)}(a) \\
&= \Lambda^{(k)}(a) \sum_{j=0}^{k} \left[\left(\frac{1}{j!(k-j)!} - \frac{k+1}{j!(k+1-j)!}|a|^2 \right) (1-|a|^2)^j |a|^{2(k-j)} \right] \\
&\quad + \frac{a}{(k+1)!}(1-|a|^2)^{k+1} \Lambda^{(k+1)}(a) \\
&= \Lambda^{(k)}(a) \Bigg\{ \frac{1}{k!}(1-|a|^2)|a|^{2k} \\
&\quad + \sum_{j=1}^{k} \left[\left(\frac{1}{j!(k-j)!} - \frac{k+1}{j!(k+1-j)!}|a|^2 \right) (1-|a|^2)^j |a|^{2(k-j)} \right] \Bigg\} \\
&\quad + \frac{a}{(k+1)!}(1-|a|^2)^{k+1} \Lambda^{(k+1)}(a)
\end{aligned}$$

$$
\begin{aligned}
&= \Lambda^{(k)}(a)\Bigg\{\frac{1}{1!(k-1)!}(1-|a|^2)^2|a|^{2(k-1)} \\
&\quad + \sum_{j=2}^{k}\Bigg[\left(\frac{1}{j!(k-j)!}-\frac{k+1}{j!(k+1-j)!}|a|^2\right)\cdot \\
&\quad (1-|a|^2)^j|a|^{2(k-j)}\Bigg]\Bigg\} + \frac{a}{(k+1)!}(1-|a|^2)^{k+1}\Lambda^{(k+1)}(a) \\
&= \cdots \\
&= \Lambda^{(k)}(a)\Bigg\{\frac{1}{r!(k-r)!}(1-|a|^2)^{r+1}|a|^{2(k-r)} \\
&\quad + \sum_{j=r+1}^{k}\left[\left(\frac{1}{j!(k-j)!}-\frac{k+1}{j!(k+1-j)!}|a|^2\right)(1-|a|^2)^j|a|^{2(k-j)}\right]\Bigg\} \\
&\quad + \frac{a}{(k+1)!}(1-|a|^2)^{k+1}\Lambda^{(k+1)}(a) \\
&= \cdots \\
&= \Lambda^{(k)}(a)\left[\frac{1}{k!}(1-|a|^2)^{k+1}\right] + \frac{a}{(k+1)!}(1-|a|^2)^{k+1}\Lambda^{(k+1)}(a) \\
&= \frac{(1-|a|^2)^{k+1}}{k!}\left[\frac{a}{k+1}\Lambda^{(k+1)}(a)+\Lambda^{(k)}(a)\right] \\
&= \frac{(1-|a|^2)^{k+1}}{k!}\frac{1}{k+1}\left[a\Lambda^{(k+1)}(a)+(k+1)\Lambda^{(k)}(a)\right] \\
&= \frac{(1-|a|^2)^{k+1}}{k!}\frac{1}{k+1}[z\Lambda(z)]^{(k+1)}(a)
\end{aligned}
$$

which implies that the equation (2.11) holds for $l = k+1$ and hence we have proved that for all $l \geq 1$

$$[z\Lambda(z)]^{(l)}(a) = 0.$$

Now since then for all $l \geq 0$

$$[z\Lambda(z) - a\Lambda(a)]^{(l)}(a) = 0,$$

the holomorphic function $h(z) = z\Lambda(z) = zP\left(\overline{\psi(\zeta)}\big/\zeta\right)(z)$ is identically equal to the constant $aP\left(\overline{\psi(\zeta)}\big/\zeta\right)(a)$ which is, by (2.10)

$$aP\left(\overline{\psi(\zeta)}\big/\zeta\right)(a) = \frac{a\overline{\beta_{-1}}}{\sqrt{2\pi(1-|a|^2)}}.$$

Observe that the orthogonal projection of ψ equals the constant

$$P^{\perp}(\psi)(z) = \overline{zP\left(\overline{\psi(\zeta)}\big/\zeta\right)(z)} = \overline{aP\left(\overline{\psi(\zeta)}\big/\zeta\right)(a)}.$$

Therefore finally the symbol ψ is exactly equal to

$$\psi = P(\psi) + \overline{aP\left(\overline{\psi(\zeta)}\big/\zeta\right)(a)} = \frac{\overline{a}\beta_{-1}}{\sqrt{2\pi(1-|a|^2)}} + \sum_{q=0}^{\infty} \beta_q E_q$$

which finishes the proof of Theorem 2.11. ☐

It was proved in [6] that the following commuting property of Toeplitz operators holds for arbitrary C^∞ smooth bounded simply connected domains provided analyticity(co-analyticity) of symbols is given with respect to the basis $\mathcal{L}_0$ by using the transformation rule of Toeplitz operators under biholomorphic mappings. It is not that difficult to prove the same result for the case of the unit disc with the generalized definition of analyticity(co-analyticity) of symbols in terms of the orthonormal basis $\mathcal{L}$ corresponding to an arbitrary point $a \in U$.

Theorem 2.12. *Suppose that φ and ψ are symbols in $L^\infty(bU)$. Then the Toeplitz operators T_φ and T_ψ on the Hardy space $H^2(bU)$ commute if and only if either both φ and ψ are analytic or both φ and ψ are co-analytic with respect to $\mathcal{L}$ or $\alpha\varphi + \beta\psi$ is constant for some constants α and β not both* 0.

Proof. Two operators T_φ and T_ψ obviously commute if the sufficient condition holds. Let $\varphi = \sum_{p=-\infty}^{\infty} \alpha_p E_p$ and $\psi = \sum_{q=-\infty}^{\infty} \beta_q E_q$ be given. Suppose that $T_\varphi T_\psi = T_\psi T_\varphi$. It then follows from the identity (2.4) that for all $m \geq 0$ and $l \geq 0$

$$(\alpha_{m+1} + \overline{a}\alpha_m)(\beta_{-l-1} + \overline{a}\beta_{-l-2}) = (\beta_{m+1} + \overline{a}\beta_m)(\alpha_{-l-l} + \overline{a}\alpha_{-l-2}). \tag{2.16}$$

If for all $m \geq 0$, $\alpha_{m+1} + \overline{a}\alpha_m = \beta_{m+1} + \overline{a}\beta_m = 0$, by the proof of Theorem 2.11, φ and ψ are both co-analytic with respect to $\mathcal{L}$. Similarly if for all $l \geq 0$, $\alpha_{-l-l} + \overline{a}\alpha_{-l-2} = \beta_{-l-1} + \overline{a}\beta_{-l-2} = 0$, by the proof of Theorem 2.11, φ and ψ are both analytic with respect to $\mathcal{L}$.

If for all $m \geq 0$ and $l \geq 0$, $\alpha_{m+1} + \overline{a}\alpha_m = \alpha_{-l-l} + \overline{a}\alpha_{-l-2} = 0$, then for some constant c, $\sum_{p=-\infty}^{-1} \alpha_p E_p = c + \sum_{p=0}^{\infty} \alpha_p E_p$ which is in both $H^2(bU)$ and $H^2(bU)^\perp$ and so $\sum_{p=-\infty}^{-1} \alpha_p E_p = 0$ and hence $\varphi = -c = 1 \cdot (-c) + 0 \cdot \psi$. Similarly if for all $m \geq 0$ and $l \geq 0$, $\beta_{m+1} + \overline{a}\beta_m = \beta_{-l-l} + \overline{a}\beta_{-l-2} = 0$, then for some nonzero vector (α, β) in $\mathbb{C}^2$, $\alpha\varphi + \beta\psi$ is a constant. As the remaining case, suppose that there exist $m_0 \geq 0$ and $l_0 \geq 0$ such that $\alpha_{m_0+1} + \overline{a}\alpha_{m_0} \neq 0$ and $\alpha_{-l_0-1} + \overline{a}\alpha_{-l_0-2} \neq 0$. Since

$$\frac{\beta_{m_0+1} + \overline{a}\beta_{m_0}}{\alpha_{m_0+1} + \overline{a}\alpha_{m_0}} = \frac{\beta_{-l_0-1} + \overline{a}\beta_{-l_0-2}}{\alpha_{-l_0-l} + \overline{a}\alpha_{-l_0-2}},$$

it follows from (2.16) and (2.3) that for $m, l \geq 0$ with $m \neq l$,

$$\begin{aligned}
[T_\psi]_{m,l} &= \frac{1}{\sqrt{2\pi(1-|a|^2)}}(\beta_{m-l} + \overline{a}\beta_{m-l-1}) \\
&= \frac{1}{\sqrt{2\pi(1-|a|^2)}} \frac{\beta_{m_0+1} + \overline{a}\beta_{m_0}}{\alpha_{m_0+1} + \overline{a}\alpha_{m_0}}(\alpha_{m-l} + \overline{a}\alpha_{m-l-l})
\end{aligned}$$

$$= \frac{\beta_{m_0+1} + \overline{a}\beta_{m_0}}{\alpha_{m_0+1} + \overline{a}\alpha_{m_0}} [T_\varphi]_{m,l}.$$

Thus for $m, l \geq 0$

$$\left[T_\psi - \frac{1}{\sqrt{2\pi(1-|a|^2)}} (\beta_0 + \overline{a}\beta_{-1}) I \right]_{m,l}$$

$$= \frac{\beta_{m_0+1} + \overline{a}\beta_{m_0}}{\alpha_{m_0+1} + \overline{a}\alpha_{m_0}} \left[T_\varphi - \frac{1}{\sqrt{2\pi(1-|a|^2)}} (\alpha_0 + \overline{a}\alpha_{-1}) I \right]_{m,l}$$

which yields

$$\psi - \frac{\beta_{m_0+1} + \overline{a}\beta_{m_0}}{\alpha_{m_0+1} + \overline{a}\alpha_{m_0}} \varphi$$

$$= \frac{1}{\sqrt{2\pi(1-|a|^2)}} \left[\beta_0 + \overline{a}\beta_{-1} - \frac{\beta_{m_0+1} + \overline{a}\beta_{m_0}}{\alpha_{m_0+1} + \overline{a}\alpha_{m_0}} (\alpha_0 + \overline{a}\alpha_{-1}) \right].$$

Therefore the proof of the theorem is finished. □

References

[1] S. Bell, *Solving the Dirichlet problem in the plane by means of the Cauchy integral*, Indiana Univ. Math. J. **39** (1990), no. 4, 1355–1371.

[2] ______, *The Szegő projection and the classical objects of potential theory in the plane*, Duke Math. J. **64** (1991), no. 1, 1–26.

[3] A. Brown and P. R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213** (1963/1964), 89–102.

[4] Y.-B. Chung, *Classification of Toeplitz operators on Hardy spaces of bounded domains in the plane*, Math. Notes **101** (2017), no. 3, 529–541.

[5] ______, *Matrices of Toeplitz operators on Hardy spaces over bounded domains*, Bull. Korean Math. Soc., to appear.

[6] ______, *Toeplitz operators on Hardy and Bergman spaces over bounded domains in the plane*, Honam Math. J., to appear.

Young-Bok Chung
Department of Mathematics
Chonnam National University
Gwangju 61186, Korea
E-mail address: ybchung@jnu.ac.kr