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CLOSED CONVEX SPACELIKE HYPERSURFACES IN LOCALLY SYMMETRIC LORENTZ SPACES

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Abstract. In 1997, H. Li [12] proposed a conjecture: if $M^n (n \geqslant 3)$ is a complete spacelike hypersurface in de Sitter space $S_1^{n+1}(1)$ with constant normalized scalar curvature R satisfying $\frac{n-2}{n} \leqslant R \leqslant 1$, then is M^n totally umbilical? Recently, F. E. C. Camargo et al. ([5]) partially proved the conjecture. In this paper, from a different viewpoint, we study closed convex spacelike hypersurface M^n in locally symmetric Lorentz space L_1^{n+1} and also prove that M^n is totally umbilical if the square of length of second fundamental form of the closed convex spacelike hypersurface M^n is constant, i.e., Theorem 1. On the other hand, we obtain that if the sectional curvature of the closed convex spacelike hypersurface M^n in locally symmetric Lorentz space L_1^{n+1} satisfies $K(M^n) > 0$, then M^n is totally umbilical, i.e., Theorem 2.

1. Introduction and main results

Let L_p^{n+p} be an $(n+p)$-dimensional connected semi-Riemannian manifold of index $p(\geqslant 1)$. In particular, L_1^{n+1} is called a Lorentz space. A hypersurface M^n of a Lorentz space is said to be *spacelike* if the metric on M^n induced from that of the Lorentz space is positive definite. A Lorentz space is said to be *locally symmetric Lorentz space* if the components $\overline{R}_{ABCD,E}$ of the covariant derivative of the Lorentz space L_1^{n+1} curvature tensor $\overline{R}_{ABCD,E} = 0$. When the Lorentz space is of constant curvature c, we call it Lorentz space form. It is well-known that a de Sitter space $S_1^{n+1}(1)$ is a simply connected Lorentz space form with constant sectional curvature 1.

In 1977, Goddard [9] proposed the Conjecture: If M^n is a complete spacelike hypersurface with constant mean curvature H in a de Sitter space $S_1^{n+1}(c)$, then is M^n totally umbilical? In general, the conjecture is false. J. Ramanathan [16] proved Goddard's conjecture for $S_1^3(1)$ and $0 \leqslant H \leqslant 1$. Moreover, when

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$H > 1$, he also showed that the conjecture is false. When $H^2 \leqslant c$ if $n = 2$ or when $n^2H^2 < 4(n-1)c$ if $n \geqslant 3$, K. Akutagawa [1] proved that Goddard's conjecture is true. There are also many results such as [3, 11, 14, 15].

On the other hand, concerning the study of spacelike hypersurface with constant scalar curvature in a de Sitter space, H. Li [12] proposed a conjecture:

Conjecture 1. *If $M^n (n \geqslant 3)$ is a complete spacelike hypersurface in de Sitter space $S_1^{n+1}(1)$ with constant normalized scalar curvature R satisfying $\frac{n-2}{n} \leqslant R \leqslant 1$, then is M^n totally umbilical?*

F. E. C. Camargo et al. [5] proved that Li's question is true if the mean curvature H is bounded. Next J. C. Liu and Z. Y. Sun [13] generalized the partial Conjecture's result of F. E. C. Camargo et al. [5] in locally symmetric Lorentz space L_1^{n+1}. Recently, Z. Y. Sun [18] generalized the partial Conjecture's result of F. E. C. Camargo et al. [5] in Lorentzian manifold. There are also many results such as [4, 6] and [10].

In this paper, first we recall that Choi et al. [8, 17] introduced the class of $(n+1)$-dimensional Lorentz spaces L_1^{n+1} of index 1 which satisfy the following conditions for some constants c_1 and c_2:
(I) for any spacelike vector u and any timelike vector v

$$\overline{K}(u,v) = -\frac{c_1}{n}, \tag{1.1}$$

(II) for any spacelike vectors u and v

$$\overline{K}(u,v) \geqslant c_2, \tag{1.2}$$

where $\overline{K}$ denotes the sectional curvature on L_1^{n+1}.

When L_1^{n+1} satisfies conditions (1.1) and (1.2), we will say that L_1^{n+1} satisfies condition $(*)$.

Remark 1. The Lorentz space form $\overline{M}_1^{n+1}(c)$ satisfies condition $(*)$, where $-\frac{c_1}{n} = c_2 = c$.

There are several examples of Lorentz spaces which are not Lorentz space forms and satisfies condition $(*)$. For instance, semi-Riemannian product manifold $H_1^k(-\frac{c_1}{n}) \times N^{n+1-k}(c_2)$, $c_1 > 0$, and $\mathbf{R}_1^k \times S^{n+1-k}(1)$. In particular, $\mathbf{R}_1^1 \times S^n(1)$ is a so-called *Einstein Static Universe*. Also the *Robertson-Walker spacetime* $N(c,f) = I \times_f N^3(c)$ is another general example of Lorentz space, where I denotes an open interval of $\mathbf{R}_1^1$ and $f > 0$ a smooth function defined on the interval I, $N^3(c)$ a 3-dimensional Riemannian manifold of constant curvature c. $N(c,f)$ also satisfy $(*)$ if we choose an appropriate function f . For more details, we refer the readers to [8, 17].

In order to prove our theorems, first we introduce some basic facts and notations. Let $\overline{R}_{CD}$ be the components of the Ricci tensor of L_1^{n+1} satisfying

$(*)$, then the scalar curvature $\overline{R}$ of L_1^{n+1} is given by

$$\overline{R} = \sum_{A=1}^{n+1} \epsilon_A \overline{R}_{AA} = -2\sum_{i=1}^{n} \overline{R}_{(n+1)ii(n+1)} + \sum_{i,j=1}^{n} \overline{R}_{ijji} = 2c_1 + \sum_{i,j=1}^{n} \overline{R}_{ijji}.$$

It is well known that $\overline{R}$ is constant when the Lorentz space L_1^{n+1} is locally symmetric. Hence $\sum_{i,j=1}^{n} \overline{R}_{ijji}$ is constant. From (2.1) in Section 2, we can define a P such that

$$n(n-1)P = n^2H^2 - S = \sum_{i,j=1}^{n} \overline{R}_{ijji} - n(n-1)R. \tag{1.3}$$

Hence, when M^n is a spacelike hypersurface in locally symmetric Lorentz space L_1^{n+1} satisfying $(*)$, we conclude from (1.3) that the normalized scalar curvature R of M^n is constant if and only if P is constant.

With the appearance of Simons integral inequality, there are many research results on the classification of the submanifolds with constant square of length of second fundamental form. Chern-do Carmo-Koboyashi [7] gave some results on the classification of the submanifolds with constant square of length of second fundamental form in sphere spaces. Q. L. Wang [19] studied spacelike hypersurfaces with constant square of length of second fundamental form in sphere spaces and obtained two sufficient conditions $H^2 \geqslant \frac{n-1}{n^2}S$ or $\mathrm{Ric}(M^n) \geqslant n-1$ to be a totally umbilical for this hypersurfaces.

In this paper, we study closed convex spacelike hypersurfaces with constant square of length of second fundamental form in locally symmetric Lorentz spaces satisfying $(*)$ and obtain the following results.

Theorem 1. *Let $M^n (n \geqslant 3)$ be a closed convex spacelike hypersurface with constant square of length of second fundamental form in an $(n+1)$-dimensional locally symmetric Lorentz space L_1^{n+1} satisfying $(*)$. Suppose that the P defined by (1.3) satisfies $P \leqslant \frac{2c}{n}$ and $c = 2c_2 + \frac{c_1}{n} > 0$, where c_1, c_2 given as in $(*)$. Then $S = nH^2$ and M^n is totally umbilical.*

In particular, when we take $L_1^{n+1} = S_1^{n+1}(1)$ in Theorem 1, then $\frac{c_1}{n} = c_2 = c = 1$ and $P = 1 - R$. Meanwhile, the condition $P \leqslant \frac{2c}{n}$ reduces to $R \geqslant \frac{n-2}{n}$ in Theorem 1. We obtain the following.

Corollary 1. *Let $M^n (n \geqslant 3)$ be a closed convex spacelike hypersurface with normalized scalar curvature R satisfying $R \geqslant \frac{n-2}{n}$ and constant square of length of second fundamental form in an $(n+1)$-dimensional de Sitter space $S_1^{n+1}(1)$. Then $S = nH^2$ and M^n is totally umbilical.*

Remark 2. The condition $R \geqslant \frac{n-2}{n}$ in Corollary 1 is better than the condition $\frac{n-2}{n} \leqslant R \leqslant 1$ in Conjecture 1 and we don't need to assume that the normalized scalar curvature R is constant in Corollary 1.

If the sectional curvature of the hypersurface M^n satisfies $K(M^n) > 0$, then we get

Theorem 2. *Let $M^n (n \geqslant 3)$ be a closed convex spacelike hypersurface with constant square of length of second fundamental form in an $(n+1)$-dimensional locally symmetric Lorentz space L_1^{n+1} satisfying $(*)$. Suppose that the sectional curvature of the hypersurface M^n satisfies $K(M^n) > 0$. Then M^n is totally umbilical.*

When we take $L_1^{n+1} = S_1^{n+1}(1)$ in Theorem 2, we have:

Corollary 2. *Let $M^n (n \geqslant 3)$ be a closed convex spacelike hypersurface with constant square of length of second fundamental form in an $(n+1)$-dimensional de Sitter space $S_1^{n+1}(1)$. Suppose that the sectional curvature of the hypersurface M^n satisfies $K(M^n) > 0$. Then M^n is totally umbilical.*

Since the condition $R \geqslant \frac{n-2}{n}$ is essential in Conjecture 1, it is not known whether does the conditions $R \geqslant \frac{n-2}{n}$ and $P \leqslant \frac{2c}{n}$ are essential in Corollary 1 and Theorem 1? Hence, we propose the following.

Problem. *Let $M^n (n \geqslant 3)$ be a closed spacelike hypersurface in an $(n+1)$-dimensional de Sitter space $S_1^{n+1}(1)$. Then M^n is totally umbilical?*

2. Preliminaries

In this section, we review some fundamental facts and give estimate the Laplacian ΔS of the squared length S of the second fundamental form for spacelike hypersurfaces in locally symmetric Lorentz spaces L_1^{n+1} satisfying $(*)$. We shall make use of the following convention on the ranges of indices: $1 \leqslant A, B, C, \ldots \leqslant n+1$; $1 \leqslant i, j, k, \ldots \leqslant n$.

We assume that M^n is a spacelike hypersurface in a Lorentz space L_1^{n+1}. Choose a local field of pseudo-Riemannian orthonormal frames $\{e_1, \ldots, e_{n+1}\}$ in L_1^{n+1} such that, restricted to M^n, $\{e_1, \ldots, e_n\}$ are tangent to M^n and e_{n+1} is normal to M^n. That is, $\{e_1, \ldots, e_n\}$ are spacelike vectors and e_{n+1} is a timelike vector. Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the fields of dual frames and the connection forms of L_1^{n+1}, respectively. Let $\epsilon_i = 1, \epsilon_{n+1} = -1$, then the structure equations of L_1^{n+1} are given by

$$d\omega_A = -\sum_B \epsilon_B \omega_{AB} \wedge \omega_B,\ \omega_{AB} + \omega_{BA} = 0,$$
$$d\omega_{AB} = -\sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \epsilon_C \epsilon_D \overline{R}_{ABCD} \omega_C \wedge \omega_D.$$

Here the components $\overline{R}_{CD}$ of the Ricci tensor and the scalar curvature $\overline{R}$ of Lorentz spaces L_1^{n+1} are given, respectively, by

$$\overline{R}_{CD} = \sum_B \epsilon_B \overline{R}_{BCDB},\ \overline{R} = \sum_A \epsilon_A \overline{R}_{AA}.$$

The components $\overline{R}_{ABCD;E}$ of the covariant derivative of the Riemannian curvature tensor $\overline{R}$ are defined by

$$\sum_E \epsilon_E \overline{R}_{ABCD;E}\omega_E = d\overline{R}_{ABCD} - \sum_E \epsilon_E(\overline{R}_{EBCD}\omega_{EA} + \overline{R}_{AECD}\omega_{EB} + \overline{R}_{ABED}\omega_{EC} + \overline{R}_{ABCE}\omega_{ED}).$$

We restrict these forms to M^n in L_1^{n+1}, then $\omega_{n+1} = 0$. Hence, we have $\sum_i \omega_{(n+1)i} \wedge \omega_i = 0$. Using Cartan's lemma, we know that there are h_{ij} such that $\omega_{(n+1)i} = \sum_j h_{ij}\omega_j$ and $h_{ij} = h_{ji}$, where the h_{ij} are the coefficients of the second fundamental form of M^n. This gives the second fundamental form of M^n, $h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j$.

The Gauss equation, components R_{ij} of the Ricci tensor and the normalized scalar curvature R of M^n are given, respectively, by

$$R_{ijkl} = \overline{R}_{ijkl} - (h_{il}h_{jk} - h_{ik}h_{jl}),$$

$$R_{ij} = \sum_k \overline{R}_{kijk} - nHh_{ij} + \sum_k h_{ik}h_{kj},$$

$$n(n-1)R = \sum_{i,j} \overline{R}_{ijji} - n^2H^2 + S, \tag{2.1}$$

where $H = \frac{1}{n}\sum_j h_{jj}$ and $S = \sum_{i,j} h_{ij}^2$ are the mean curvature and the squared length of the second fundamental form of M^n, respectively.

Let h_{ijk} and h_{ijkl} be the first and the second covariant derivatives of h_{ij}, respectively, so that

$$\sum_k h_{ijk}\omega_k = dh_{ij} - \sum_k (h_{ik}\omega_{kj} + h_{kj}\omega_{ki}),$$

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} - \sum_l (h_{ljk}\omega_{li} + h_{ilk}\omega_{lj} + h_{ijl}\omega_{lk}).$$

Thus, we have the Codazzi equation and the Ricci identity

$$h_{ijk} - h_{ikj} = \overline{R}_{(n+1)ijk}, \tag{2.2}$$

$$h_{ijkl} - h_{ijlk} = -\sum_m (h_{im}R_{mjkl} + h_{jm}R_{mikl}). \tag{2.3}$$

Let $\overline{R}_{ABCD;E}$ be the covariant derivative of $\overline{R}_{ABCD}$. Thus, restricted on M^n, $\overline{R}_{(n+1)ijk;l}$ is given by

$$\overline{R}_{(n+1)ijk;l} = \overline{R}_{(n+1)ijkl} + \overline{R}_{(n+1)i(n+1)k}h_{jl} + \overline{R}_{(n+1)ij(n+1)}h_{kl} + \sum_m \overline{R}_{mijk}h_{ml}, \tag{2.4}$$

where $\overline{R}_{(n+1)ijk;l}$ denotes the covariant derivative of $\overline{R}_{(n+1)ijk}$ as a tensor on M^n so that

$$\sum_l \overline{R}_{(n+1)ijk;l}\omega_l = d\overline{R}_{(n+1)ijk} - \sum_l \overline{R}_{(n+1)ljk}\omega_{li} - \sum_l \overline{R}_{(n+1)ilk}\omega_{lj} - \sum_l \overline{R}_{(n+1)ijl}\omega_{lk}.$$

Next we compute the Laplacian $\Delta h_{ij} = \sum_k h_{ijkk}$. Combining Gauss equation, Codazzi equation, Ricci identity and (2.4), we have (see [13])

$$\begin{aligned}\Delta h_{ij} = (nH)_{ij} &+ \sum_k \left(\overline{R}_{(n+1)ijk;k} + \overline{R}_{(n+1)kik;j}\right)\\ &- \sum_k \left(h_{kk}\overline{R}_{(n+1)ij(n+1)} + h_{ij}\overline{R}_{(n+1)k(n+1)k}\right)\\ &- \sum_{k,l}(2h_{kl}\overline{R}_{lijk} + h_{jl}\overline{R}_{lkik} + h_{il}\overline{R}_{lkjk})\\ &- nH\sum_l h_{il}h_{lj} + Sh_{ij}.\end{aligned}$$

According to the above equation, the Laplacian ΔS of the squared length S of the second fundamental form h_{ij} of M^n is obtained

$$\begin{aligned}\frac{1}{2}\Delta S =& \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij}\Delta h_{ij}\\ =& \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j}(nH)_{ij}h_{ij} + \sum_{i,j,k}\left(\overline{R}_{(n+1)ijk;k} + \overline{R}_{(n+1)kik;j}\right)h_{ij}\\ &- \left(\sum_{i,j} nHh_{ij}\overline{R}_{(n+1)ij(n+1)} + S\sum_k \overline{R}_{(n+1)k(n+1)k}\right)\\ &- 2\sum_{i,j,k,l}(h_{kl}h_{ij}\overline{R}_{lijk} + h_{il}h_{ij}\overline{R}_{lkjk}) - nH\sum_{i,j,l} h_{il}h_{lj}h_{ij} + S^2.\end{aligned} \tag{2.5}$$

Choose a local orthonormal frame field $\{e_1, \dots, e_n\}$ such that $h_{ij} = \lambda_i\delta_{ij}$, where λ_i, $1 \leqslant i \leqslant n$, are principal curvatures of M^n. Estimating the right-hand side of (2.5) by using the curvature conditions $(*)$, the following lemma was obtained by Liu-Sun ([13]).

Lemma 2.1 ([13, Lemma 2.1])**.** *Let M^n be a spacelike hypersurface in a locally symmetric Lorentz space L_1^{n+1} satisfying $(*)$. Then*

$$\frac{1}{2}\Delta S \geqslant \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i(nH)_{ii} + nc(S - nH^2) + \left(S^2 - nH\sum_i \lambda_i^3\right), \tag{2.6}$$

where $c = 2c_2 + \frac{c_1}{n}$ and c_1, c_2 are given as in $(*)$.

In the following, we will continue to calculate ΔS for spacelike hypersurfaces in locally symmetry Lorentz spaces satisfying $(*)$. Thus, we need the following algebraic Lemma.

Lemma 2.2 ([2]). *Let $\mu_1, \ldots, \mu_n$ be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = B^2$, where $B \geqslant 0$ is constant. Then*

$$\left|\sum_i \mu_i^3\right| \leqslant \frac{n-2}{\sqrt{n(n-1)}} B^3$$

and equality holds if and only if at least $n-1$ of the μ_i's are equal.

Let $\phi = \sum_{i,j} \phi_{ij} \omega_i \otimes \omega_j$ be a symmetric tensor defined on M^n, where $\phi_{ij} = h_{ij} - H\delta_{ij}$. It is easy to check that ϕ is traceless. Choose a local orthonormal frame field $\{e_1, \ldots, e_n\}$ such that $h_{ij} = \lambda_i \delta_{ij}$ and $\phi_{ij} = \mu_i \delta_{ij}$. Let $|\phi|^2 = \sum_i \mu_i^2$. A direct computation gets

$$|\phi|^2 = S - nH^2 = \frac{1}{2n} \sum_{i,j} (\lambda_i - \lambda_j)^2. \tag{2.7}$$

Hence, $|\phi|^2 = 0$ if and only if M^n is totally umbilical. We also get

$$\sum_i \lambda_i^3 = nH^3 + 3H \sum_i \mu_i^2 + \sum_i \mu_i^3.$$

By applying Lemma 2.2 to the real numbers $\mu_1, \ldots, \mu_n$, we obtain

$$\begin{aligned} -nH \sum_i \lambda_i^3 =& -n^2H^4 - 3nH^2 \sum_i \mu_i^2 - nH \sum_i \mu_i^3 \\ \geqslant& 2n^2H^4 - 3nSH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} |H| (S - nH^2)^{\frac{3}{2}}. \end{aligned} \tag{2.8}$$

Substituting (2.7) and (2.8) into (2.6), we obtain the following.

Lemma 2.3. *Let M^n be a spacelike hypersurface in a locally symmetric Lorentz space L_1^{n+1} satisfying $(*)$, then*

$$\frac{1}{2}\Delta S \geqslant \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{ii} + |\phi|^2 L_{|H|}(|\phi|), \tag{2.9}$$

where $|\phi|^2 = S - nH^2$, $L_{|H|}(|\phi|) = |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} |H||\phi| + nc - nH^2$, $c = 2c_2 + \frac{c_1}{n}$ and c_1, c_2 are given as in $(*)$.

3. The proofs of main theorems

The proof of Theorem 1. Since S is constant, from Lemma 2.3 we have

$$0 = \frac{1}{2}\Delta S \geqslant \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{ii} + |\phi|^2 \left(|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} |H||\phi| + nc - nH^2 \right). \tag{3.1}$$

Since M^n is closed convex hypersurface, we take a point $p \in M^n$ such that $(nH)(p) = \min_{x \in M^n}(nH)(x)$, where $nH = \sum_i \lambda_i$. By the maximum principle, we get

$$(nH)_{ii}(p) \geqslant 0, \quad i = 1, \dots, n. \tag{3.2}$$

Substituting (3.2) into (3.1), note that M^n is convex hypersurface, i.e., $\lambda_i \geqslant 0$, we have

$$0 \geqslant |\phi|^2(p)\left(|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi| + nc - nH^2\right)(p). \tag{3.3}$$

Combining (1.3) and (2.7), we have

$$|\phi|^2 = n(n-1)(H^2 - P). \tag{3.4}$$

Next, we will consider the following polynomial given by

$$L_H(x) = x^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|x + nc - nH^2.$$

We claim that

$$L_H(|\phi|) > 0.$$

Case 1, $H^2 < \frac{4(n-1)}{n^2}c$. Then the discriminant of $L_{|H|}(x)$ is negative. Therefore, we have $L_{|H|}(|\phi|) > 0$.

Case 2, $H^2 \geqslant \frac{4(n-1)}{n^2}c$. Let α be the biggest root of the equation $L_{|H|}(x) = 0$, which is positive. We know that α is the only one root of $L_{|H|}(x)$ if $H^2 = \frac{4(n-1)}{n^2}c$.

If we can prove that $(|\phi|)^2 = |\phi|^2 > \alpha^2$, then we have $|\phi| > \alpha$. Hence, $L_{|H|}(|\phi|) > 0$. Since $P \leqslant \frac{2c}{n}$ and $c > 0$, it follows from (3.4) that

$$|\phi|^2 \geqslant (n-1)(nH^2 - 2c). \tag{3.5}$$

By virtue of (3.5), it is straightforward to verify that

$$|\phi|^2 - \alpha^2 \geqslant \frac{n-2}{2(n-1)}\left(n^2H^2 - nH\sqrt{n^2H^2 - 4(n-1)c} - 2(n-1)c\right).$$

Thus, $|\phi|^2 - \alpha^2 > 0$ if and only if

$$n^2H^2 - nH\sqrt{n^2H^2 - 4(n-1)c} - 2(n-1)c > 0. \tag{3.6}$$

Taking into account that the inequality (3.6) is equivalent to $4(n-1)^2c^2 > 0$, which is true because of $c > 0$. Hence, $|\phi|^2 - \alpha^2 > 0$ which proves our claim.

From $L_H(|\phi|) > 0$ and (3.3), we have

$$|\phi|^2(p)\left(|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi| + nc - nH^2\right)(p) = 0. \tag{3.8}$$

Combining $L_H(|\phi|) > 0$ and (3.8), we have

$$|\phi|^2(p) = 0. \tag{3.9}$$

For any point $q \in M^n$, combining $S(q) = S(p)$, (2.7) and (3.9), note that $(nH)(p) = \min_{x \in M^n}(nH)(x) \geqslant 0$, we have

$$\begin{aligned}|\phi|^2(q) =& S(q) - nH^2(q)\\ =& S(p) - nH^2(p)\\ =& |\phi|^2(p) = 0.\end{aligned}$$

Therefore, for any point $q \in M^n$, we have $|\phi|^2(q) = 0$. That is, $|\phi|^2 = 0$ which shows M^n is totally umbilical. This completes the proof of Theorem 1. □

The proof of Theorem 2. Combining (2.2) and (2.3), we get

$$\begin{aligned}\Delta h_{ij} =& \sum_k h_{ikjk} + \overline{R}_{(n+1)jkk}\\ =& \sum_k \left(h_{kikj} - \sum_l (h_{kl}R_{lijk} + h_{il}R_{lkjk}) + \overline{R}_{(n+1)ijkk} \right).\end{aligned}$$

Since $h_{kikj} = h_{kkij} + \overline{R}_{(n+1)kikj}$, we obtain

$$\Delta h_{ij} = (nH)_{ij} + \sum_k \left(\overline{R}_{(n+1)ijk,k} + \overline{R}_{(n+1)kik,j}\right) - \sum_{k,l}(h_{kl}R_{lijk} + h_{il}R_{lkjk}). \tag{3.10}$$

From (3.10), we have

$$\begin{aligned}\frac{1}{2}\Delta S =& \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij}\Delta h_{ij}\\ =& \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{ii} + \sum_k \left(\overline{R}_{(n+1)ijk,k} + \overline{R}_{(n+1)kik,j}\right) h_{ij}\\ &- \sum_{i,j,k,l} h_{ij}(h_{kl}R_{lijk} + h_{il}R_{lkjk})\\ =& \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{ii} + \sum_k \left(\overline{R}_{(n+1)ijk,k} + \overline{R}_{(n+1)kik,j}\right) h_{ij}\\ &+ \sum_{i<k}(h_{kk} - h_{ii})^2 R_{kiik}.\end{aligned} \tag{3.11}$$

Since L_1^{n+1} is locally symmetric, we have $\sum_{i,j,k}\left(\overline{R}_{(n+1)ijk,k} + \overline{R}_{(n+1)kik,j}\right) h_{ij} = 0$. Combining (3.11), we have

$$\frac{1}{2}\Delta S = \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{ii} + \sum_{i<k}(h_{kk} - h_{ii})^2 R_{kiik}. \tag{3.12}$$

Since S is constant, from (3.12) we have

$$0 \geqslant \sum_i \lambda_i (nH)_{ii} + \sum_{i<k}(h_{kk} - h_{ii})^2 R_{kiik}. \tag{3.13}$$

By taking the similar processing as in the proof of Theorem 1 on the inequality $(nH)_{ii}(p) \geqslant 0$, we can arrive to $(nH)_{ii}(p) \geqslant 0, \quad i = 1, \ldots, n$.

Combining $(nH)_{ii}(p) \geqslant 0$ and (3.13), note that M^n is convex hypersurface, i.e., $\lambda_i \geqslant 0$, we have

$$0 \geqslant \sum_{i<k} (h_{kk} - h_{ii})^2 R_{kiik}. \tag{3.14}$$

Since $R_{kiik} > 0$, from (3.14) we have $h_{ii} = h_{jj}$, i.e., M^n is totally umbilical. This completes the proof of Theorem 2. □

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