

ON COMMUTING ORDINARY DIFFERENTIAL OPERATORS WITH POLYNOMIAL COEFFICIENTS CORRESPONDING TO SPECTRAL CURVES OF GENUS TWO

VALENTINA N. DAVLETSHINA AND ANDREY E. MIRONOV

ABSTRACT. The group of automorphisms of the first Weyl algebra acts on commuting ordinary differential operators with polynomial coefficient. In this paper we prove that for fixed generic spectral curve of genus two the set of orbits is infinite.

1. Introduction and main result

Let us consider a generic equation

$$f(X, Y) = \sum_{i,j} \alpha_{ij} X^i Y^j = 0, \quad \alpha_{ij} \in \mathbb{C}.$$

The group $\text{Aut}(A_1)$, where $A_1 = \mathbb{C}[x][\partial_x]$ is the first Weyl algebra, has a natural action on the set of solutions of this equation. Yu. Berest proposed the following conjecture (cf. [7]): if the genus of the algebraic curve defined by the equation $f(z, w) = 0$ (in the theory of commuting ordinary differential operators the curve defined by this equation is called *the spectral curve*) is one then the set of orbits is infinite, and if the genus is greater than one then for generic α_{ij} the set of orbits is finite. From finiteness of this set for some curve it would be possible to derive the Dixmier conjecture $\text{End}(A_1) = \text{Aut}(A_1)$ for the first Weyl algebra.

In this paper we study the action of the automorphisms group of the first Weyl algebra $A_1 = \mathbb{C}[x][\partial_x]$ on the set of solutions of the equation

$$(1) \quad Y^2 = X^{2g+1} + c_{2g}X^{2g} + \cdots + c_1X + c_0$$

at $g = 2$. It is not difficult to show that any solution X, Y of this equation is a pair of commuting operators. To construct examples of commuting operators from A_1 satisfying (1) is a nontrivial problem. First such examples, for $g =$

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1, were found by Dixmier in [2]. For $g > 1$ the examples of higher rank commuting ordinary differential operators were constructed, using Krichever–Novikov theory (see [3], [4]), in [5]: the operator

$$L_4 = (\partial_x^2 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)^2 + g(g+1)\alpha_3 x$$

commutes with an operator $L_{4g+2} \in A_1$ of order $4g+2$ and L_4, L_{4g+2} satisfy (1) (another examples see in [1]–[10]).

In [7] the set of orbits was studied in the case of genus one spectral curves, i.e., for $g = 1$ in (1). It was shown that the set of orbits is infinite for any such curve. Moreover, for arbitrary $g > 1$ there is a two-parametric family of hyperelliptic spectral curves with infinite set of orbits [7]. More precisely, the operator

$$L_4^\sharp = (\partial_x^2 + \alpha_1 \cosh(x) + \alpha_0)^2 + \alpha_1 g(g+1) \cosh(x), \quad \alpha_1 \neq 0$$

commutes with an operator L_{4g+2}^\sharp of order $4g+2$ (see [6]), and the pair $L_4^\sharp, L_{4g+2}^\sharp$ satisfies the equation

$$(2) \quad (L_{4g+2}^\sharp)^2 = (L_4^\sharp)^{2g+1} + c_{2g}^\sharp (L_4^\sharp)^{2g} + \dots + c_1^\sharp L_4^\sharp + c_0^\sharp$$

for some constants c_i^\sharp . Mokhov [8] noticed that the change of variables

$$x = \ln(y + \sqrt{y^2 - 1})^r, \quad r = \pm 1, \pm 2, \dots,$$

transforms the operators $L_4^\sharp, L_{4g+2}^\sharp$ into the operators with polynomial coefficients in new variable y , e.g.

$$L_4^\sharp = ((1 - y^2)\partial_y^2 - 3y\partial_y + aT_r(y) + b)^2 - ar^2g(g+1)T_r(y), \quad a \neq 0.$$

Here $T_r(y)$ is the Chebyshev polynomial of degree $|r|$. Recall that

$$T_0(y) = 1, \quad T_1(y) = y, \quad T_r(y) = 2yT_{r-1}(y) - T_{r-2}(y), \quad T_{-r}(y) = T_r(y).$$

This gives a family of operators $L_4^\sharp, L_{4g+2}^\sharp \in A_1$ depending on integer r which satisfy (2). It is turn out that for different integers r the pairs $L_4^\sharp, L_{4g+2}^\sharp$ belong to different orbits (see [7]), so for the equation

$$Y^2 = X^{2g+1} + c_{2g}^\sharp X^{2g} + \dots + c_1^\sharp X + c_0^\sharp$$

the set of orbits is infinite.

The main result of this paper is the following. We give a simple family (probably, the simplest one) of commuting operators from A_1 , satisfying the equation

$$(3) \quad Y^2 = X^5 + c_4 X^4 + c_3 X^3 + c_2 X^2 + c_1 X + c_0, \quad X, Y \in A_1, \quad c_i \in \mathbb{C}$$

with c_i being generic, which belong to different orbits. The proof of the last fact appears to be also extremely simple for this family.

Theorem 1. *The operator*

$$L_4^b = ((\alpha_1 x^2 + 1)\partial_x^2 + (\alpha_2 x + \alpha_3)\partial_x + \alpha_4 x + \alpha_5)^2 + \alpha_1 \alpha_4 g(g+1)x + \alpha_6$$

commutes with an operator L_{10}^b (given by exact formulae in Appendix) at $g = 2$ for any $\alpha_i \in \mathbb{C}$.

The pair L_4^b, L_{10}^b is a solution of (3), where c_i depend polynomially on α_i (see exact formulae in Appendix).

The set of orbits of the group $\text{Aut}(A_1)$ in the space of solutions of the equation (3) with generic c_i is infinite.

We also checked that L_4^b commutes with L_6^b at $g = 1$ and with L_{14}^b at $g = 3$. So we can formulate the conjecture:

The operator L_4^b commutes with an operator of order $4g + 2$.

The operator L_4^b is not self-adjoint hence the methods of [5] are not applicable here. It is an interesting problem to prove this conjecture and develop the methods of [5] to the non-self-adjoint case.

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2. Proof of Theorem 1

The theorem follows from the following observation. Direct calculations show that the operator

$$L_4^b = ((\alpha_1 x^2 + 1)\partial_x^2 + (\alpha_2 x + \alpha_3)\partial_x + \alpha_4 x + \alpha_5)^2 + \alpha_1 \alpha_4 g(g+1)x + \alpha_6$$

commutes with an operator L_{10}^b at $g = 2$. Operators L_4^b, L_{10}^b satisfy (3), where the coefficients c_i depend polynomially on α_i (see exact formulae in Appendix). Thus, we obtain an algebraic morphism $\mathbb{C}^6 \rightarrow \mathbb{C}^5$. Direct calculations show that the differential map (on tangent spaces) at a generic point is surjective, thus our morphism is dominant. Hence, if we fix generic coefficients c_j , we obtain 1-parametric family of solutions of (3). Let's show that operators from this family belong to infinitely many different orbits of the group $\text{Aut}(A_1)$.

Recall that $\text{Aut}(A_1)$ is generated by the following automorphisms (see [2])

$$\varphi_1(x) = x + P_1(\partial_x), \quad \varphi_1(\partial_x) = \partial_x,$$

$$\varphi_2(x) = x, \quad \varphi_2(\partial_x) = \partial_x + P_2(x),$$

$$\varphi_3(x) = \alpha x + \beta \partial_x, \quad \varphi_3(\partial_x) = \gamma \partial_x + \delta x, \quad \alpha\gamma - \beta\delta = 1, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C},$$

where P_1, P_2 are arbitrary polynomials with constant coefficients.

Let $S(L)$ denote the total symbol of an operator $L \in A_1$ (thus, $S(L)$ is a polynomial in two variables; we assume here that operators are written in a canonical form, say, with coefficients in x on the left). Let's remind two simple properties of the total symbol: $S(L_1 + L_2) = S(L_1) + S(L_2)$ and $\deg(S(L_1 L_2)) = \deg(S(L_1)) + \deg(S(L_2))$ for any $L_1, L_2 \in A_1$.

Without loss of generality we can assume that $\alpha_1 \neq 0$, for if $\alpha_1 = 0$, then the coefficients c_i form an algebraic subset of dimension one in \mathbb{C}^5 (as it follows from the exact formulae in Appendix).

Now let $\varphi \in \text{Aut}(A_1)$. Consider two possibilities: either $S(\varphi(x))$ or $S(\varphi(\partial_x))$ has degree > 1 , or $\deg S(\varphi(x)) = \deg S(\varphi(\partial_x)) = 1$.

In the first case, as it obviously follows from the two simple properties of the total symbol and the standard property of the polynomial degree:

$$\deg(S(\varphi(L_4^b))) = 4(\deg S(\varphi(x)) + \deg S(\varphi(\partial_x))) > 8 = \deg(S(L_4^b)),$$

a contradiction.

In the second case

$$\varphi(x) = \alpha x + \beta \partial_x + C_1, \quad \varphi(\partial_x) = \gamma \partial_x + \delta x + C_2,$$

where $\alpha\gamma - \beta\delta = 1$, $\alpha, \beta, \gamma, \delta, C_1, C_2 \in \mathbb{C}$, and an easy direct calculation shows that such an automorphism preserves the form of the operator L_4^b only if $C_1 = C_2 = \beta = \delta = 0$, $\alpha = \gamma = 1$.

Thus, the operators L_4^b belong to different orbits for different values of parameters.

3. Appendix

The operator

$$L_4^b = ((\alpha_1 x^2 + 1)\partial_x^2 + (\alpha_2 x + \alpha_3)\partial_x + \alpha_4 x + \alpha_5)^2 + \alpha_1 \alpha_4 g(g+1)x + \alpha_6$$

commutes with an operator

$$\begin{aligned} L_{10}^b = & P^5 + \left(\frac{5}{2}\alpha_1(6x\alpha_4 + \alpha_2 + 2\alpha_5) + 3\alpha_1^2 - \frac{5\alpha_2^2}{4} \right) P^3 + 45\alpha_1\alpha_4(x^2\alpha_1 + 1)P^2 \\ & + \frac{15}{2}\alpha_1\alpha_4 \left(34x\alpha_1 + 3(x\alpha_2 + \alpha_3) \right) P^2 \partial_x \\ & - 30\alpha_1\alpha_4 \left(2x^2\alpha_1^2 + \alpha_1(3x^2\alpha_2 + 6x\alpha_3 - 10) - 3\alpha_2 \right) P \partial_x \\ & + q_2 P + q_1 \partial_x + q_0, \end{aligned}$$

where

$$\begin{aligned} P = & (\alpha_1 x^2 + 1)\partial_x^2 + (\alpha_2 x + \alpha_3)\partial_x + \alpha_4 x + \alpha_5, \\ q_0 = & 15x^2(\alpha_1 + 3\alpha_2)\alpha_4^2\alpha_1^2 \\ & + \frac{3}{2}x\alpha_4 \left(6\alpha_1^2 + (13\alpha_2 - 64\alpha_5)\alpha_1 - 60\alpha_3\alpha_4 + 6\alpha_2(\alpha_2 + 20\alpha_5) \right) \alpha_1^2 \\ & + \frac{3}{2}\alpha_4 \left(9\alpha_3\alpha_1^2 + (8\alpha_2\alpha_3 + 60\alpha_5\alpha_3 + 58\alpha_4)\alpha_1 + 30\alpha_2\alpha_4 \right) \alpha_1, \\ q_1 = & 15x^2\alpha_4 \left(5\alpha_1^2 + 4(\alpha_2 + 3\alpha_5)\alpha_1 + 3\alpha_2^2 \right) \alpha_1^2 \\ & - 30x\alpha_4 \left((2\alpha_1 - 3\alpha_2)\alpha_3 + 12\alpha_4 \right) \alpha_1^2 \end{aligned}$$

$$\begin{aligned}
& + 15\alpha_4 \left(5\alpha_1^2 + 2(3\alpha_3^2 + 4\alpha_2 - 6\alpha_5)\alpha_1 + 3\alpha_2^2 \right) \alpha_1, \\
q_2 = & - 135x^2\alpha_1^2\alpha_4^2 + \frac{15}{2}x\alpha_1 \left(18\alpha_1^2 - 2(9\alpha_2 + 14\alpha_5)\alpha_1 + \alpha_2^2 \right) \alpha_4 \\
& + \frac{1}{4} \left(\alpha_2^4 + 6\alpha_1^3(\alpha_2 + 2\alpha_5) \right. \\
& + \alpha_1(-4\alpha_2^3 - 8\alpha_5\alpha_2^2 + 54\alpha_3\alpha_4\alpha_2 + 252\alpha_4^2) \\
& \left. + \alpha_1^2(\alpha_2^2 + 16\alpha_5\alpha_2 + 16\alpha_5^2 - 228\alpha_3\alpha_4) \right).
\end{aligned}$$

The spectral curve of L_4^b, L_{10}^b is

$$w^2 = z^5 + c_4z^4 + c_3z^3 + c_2z^2 + c_1z + c_0,$$

where

$$\begin{aligned}
c_4 &= b_4 - 5\alpha_6, \quad c_3 = b_3 - 4b_4\alpha_6 + 10\alpha_6^2, \quad c_2 = b_2 - 3b_3\alpha_6 + 6b_4\alpha_6^2 - 10\alpha_6^3, \\
c_1 &= b_1 - 2b_2\alpha_6 + 3b_3\alpha_6^2 - 4b_4\alpha_6^3 + 5\alpha_6^4, \\
c_0 &= b_0 - \alpha_6(b_1 + \alpha_6(\alpha_6(b_3 - b_4\alpha_6 + \alpha_6^2) - b_2)), \\
b_4 &= 6\alpha_1^2 + 5(\alpha_2 + 2\alpha_5)\alpha_1 - \frac{5\alpha_2^2}{2}, \\
b_3 &= \frac{3}{16} \left(48\alpha_1^4 + 96(\alpha_2 + 2\alpha_5)\alpha_1^3 + 4(-\alpha_2^2 + 44\alpha_5\alpha_2 + 44\alpha_5^2 + 28\alpha_3\alpha_4)\alpha_1^2 \right. \\
& \quad \left. - 4(11\alpha_2^3 + 22\alpha_5\alpha_2^2 + 14\alpha_3\alpha_4\alpha_2 - 28\alpha_4^2)\alpha_1 + 11\alpha_2^4 \right), \\
b_2 &= \frac{1}{8} \left(-5\alpha_2^6 + 3\alpha_1(10\alpha_2^3 + 20\alpha_5\alpha_2^2 + 39\alpha_3\alpha_4\alpha_2 - 78\alpha_4^2)\alpha_2^2 \right. \\
& \quad + 72\alpha_1^5(\alpha_2 + 2\alpha_5) + 72\alpha_1^4(\alpha_2^2 + 6\alpha_5\alpha_2 + 6\alpha_5^2 + 8\alpha_3\alpha_4) \\
& \quad + 4\alpha_1^3 \left(-17\alpha_2^3 + 120\alpha_5^2\alpha_2 \right. \\
& \quad + 45\alpha_3\alpha_4\alpha_2 + 80\alpha_5^3 + 234\alpha_4^2 + 6(\alpha_2^2 + 39\alpha_3\alpha_4)\alpha_5 \left. \right) \\
& \quad \left. - 3\alpha_1^2 \left(11\alpha_2^4 + 80\alpha_5\alpha_2^3 + 4(20\alpha_5^2 + 39\alpha_3\alpha_4)\alpha_2^2 \right. \right. \\
& \quad \left. \left. + 12\alpha_4(13\alpha_3\alpha_5 - 10\alpha_4)\alpha_2 + 6\alpha_4^2(3\alpha_3^2 - 64\alpha_5) \right) \right), \\
b_1 &= \frac{1}{16} \left(\alpha_2^8 - 8\alpha_1(\alpha_2^3 + 2\alpha_5\alpha_2^2 + 9\alpha_3\alpha_4\alpha_2 - 18\alpha_4^2)\alpha_2^4 \right. \\
& \quad + 12\alpha_1^5(\alpha_2^3 + 48\alpha_5^2\alpha_2 + 78\alpha_3\alpha_4\alpha_2 + 32\alpha_5^3 + 180\alpha_4^2) \\
& \quad + 6(3\alpha_2^2 + 32\alpha_3\alpha_4)\alpha_5 \left. \right) + 36\alpha_1^6 \left((\alpha_2 + 2\alpha_5)^2 + 12\alpha_3\alpha_4 \right) \\
& \quad + 6\alpha_1^2 \left(3\alpha_2^6 + 16\alpha_5\alpha_2^5 + 8(2\alpha_5^2 + 9\alpha_3\alpha_4)\alpha_2^4 + 12\alpha_4(8\alpha_3\alpha_5 - 5\alpha_4)\alpha_2^3 \right. \\
& \quad \left. + 6\alpha_4^2(15\alpha_3^2 - 44\alpha_5)\alpha_2^2 - 288\alpha_3\alpha_4^3\alpha_2 + 288\alpha_4^4 \right)
\end{aligned}$$

$$\begin{aligned}
& + 4\alpha_1^3 \left(\alpha_2^5 - 30\alpha_5\alpha_2^4 - 48(2\alpha_5^2 + 3\alpha_3\alpha_4)\alpha_2^3 \right. \\
& - 2(32\alpha_5^3 + 288\alpha_3\alpha_4\alpha_5 + 153\alpha_4^2)\alpha_2^2 \\
& - 18\alpha_4(27\alpha_4\alpha_3^2 + 16\alpha_5^2\alpha_3 - 32\alpha_4\alpha_5)\alpha_2 \\
& + 36\alpha_4^2(-3\alpha_5\alpha_3^2 + 24\alpha_4\alpha_3 + 28\alpha_5^2) \left. \right) + \alpha_1^4 \left(-47\alpha_2^4 - 160\alpha_5\alpha_2^3 \right. \\
& + 96(\alpha_5^2 - 6\alpha_3\alpha_4)\alpha_2^2 + 128(4\alpha_5^3 + 9\alpha_3\alpha_4\alpha_5 + 18\alpha_4^2)\alpha_2 \\
& \left. + 8(32\alpha_5^4 + 288\alpha_3\alpha_4\alpha_5^2 + 792\alpha_4^2\alpha_5 + 189\alpha_3^2\alpha_4^2) \right), \\
b_0 = & \frac{3}{8}\alpha_1\alpha_4 \left(36(\alpha_3(\alpha_2 + 2\alpha_5) - 6\alpha_4)\alpha_1^6 \right. \\
& + 6 \left(2\alpha_3\alpha_2^2 + 2(6\alpha_4 + 13\alpha_3\alpha_5)\alpha_2 + 32\alpha_3\alpha_5^2 + 45\alpha_3^2\alpha_4 - 48\alpha_4\alpha_5 \right) \alpha_1^5 \\
& + \left(-47\alpha_3\alpha_2^3 + 186\alpha_4\alpha_2^2 - 36\alpha_3^2\alpha_4\alpha_2 + 128\alpha_3\alpha_5^3 + 1728\alpha_3\alpha_4^2 \right. \\
& + 96(\alpha_2\alpha_3 + \alpha_4)\alpha_5^2 + 24 \left((15\alpha_3^2 + 16\alpha_2)\alpha_4 - 4\alpha_2^2\alpha_3 \right) \alpha_5 \left. \right) \alpha_1^4 \\
& + 4 \left(\alpha_3\alpha_2^4 - (29\alpha_4 + 24\alpha_3\alpha_5)\alpha_2^3 - 3(15\alpha_4\alpha_3^2 + 16\alpha_5^2\alpha_3 + 2\alpha_4\alpha_5)\alpha_2^2 \right. \\
& - 4 \left(18\alpha_4\alpha_5\alpha_2^2 + (4\alpha_5^3 + 63\alpha_4^2)\alpha_3 - 12\alpha_4\alpha_5^2 \right) \alpha_2 \\
& \left. + 2\alpha_4 \left(16\alpha_5^3 + 180\alpha_3\alpha_4\alpha_5 - 27(\alpha_3^3 - 8\alpha_4)\alpha_4 \right) \right) \alpha_1^3 \\
& + 6 \left(3\alpha_3\alpha_2^5 + 12\alpha_3\alpha_5\alpha_2^4 + 2(9\alpha_4\alpha_3^2 + 4\alpha_5^2\alpha_3 - 8\alpha_4\alpha_5)\alpha_2^3 \right. \\
& - 4\alpha_4(-3\alpha_5\alpha_2^3 + 3\alpha_4\alpha_3 + 4\alpha_5^2)\alpha_2^2 - 6\alpha_4^2(-3\alpha_3^3 + 20\alpha_5\alpha_3 + 8\alpha_4)\alpha_2 \\
& + 12\alpha_4^3(16\alpha_5 - 3\alpha_3^2) \left. \right) \alpha_1^2 - 2\alpha_2^2 \left(4\alpha_3\alpha_2^4 - 6(\alpha_4 - \alpha_3\alpha_5)\alpha_2^3 \right. \\
& \left. + 3\alpha_4(3\alpha_3^2 - 4\alpha_5)\alpha_2^2 - 36\alpha_3\alpha_4^2\alpha_2 + 36\alpha_4^3 \right) \alpha_1 + \alpha_2^6(\alpha_2\alpha_3 - 2\alpha_4).
\end{aligned}$$

References

- [1] V. N. Davletshina, *Commuting differential operators of rank two with trigonometric coefficients*, Sib. Math. J. **56** (2015), no. 3, 405–410.
- [2] J. Dixmier, *Sur les algèbres de Weyl*, Bull. Soc. Math. France **96** (1968), 209–242.
- [3] I. M. Krichever, *Commutative rings of ordinary linear differential operators*, Functional Anal. Appl. **12** (1978), no. 3, 175–185.
- [4] I. M. Krichever and S. P. Novikov, *Holomorphic bundles over algebraic curves and nonlinear equations*, Russian Math. Surveys **35** (1980), no. 6, 47–68.
- [5] A. E. Mironov, *Self-adjoint commuting ordinary differential operators*, Invent. Math. **197** (2014), no. 2, 417–431.
- [6] ———, *Periodic and rapid decay rank two self-adjoint commuting differential operators*, Topology, geometry, integrable systems, and mathematical physics, 309–321, Amer. Math. Soc. Transl. Ser. 2, 234, Amer. Math. Soc., Providence, RI, 2014.
- [7] A. E. Mironov and A. B. Zheglov, *Commuting ordinary differential operators with polynomial coefficients and automorphisms of the first Weyl algebra*, Int. Math. Res. Not. **2016** (2016), no. 10, 2974–2993.

- [8] O. I. Mokhov, *Commuting ordinary differential operators of arbitrary genus and arbitrary rank with polynomial coefficients*, *Topology, geometry, integrable systems, and mathematical physics*, 323–336, Amer. Math. Soc. Transl. Ser. 2, 234, Amer. Math. Soc., Providence, RI, 2014.
- [9] V. S. Oganessian, *Commuting differential operators of rank 2 and arbitrary genus g with polynomial coefficients*, *Russian Math. Surveys* **70** (2015), no. 1, 165–167.
- [10] ———, *Commuting Differential Operators of Rank 2 with Polynomial Coefficients*, *Functional Anal. Appl.* **50** (2016), no. 1, 54–61.

VALENTINA N. DAVLETSHINA
SOBOLEV INSTITUTE OF MATHEMATICS
NOVOSIBIRSK STATE UNIVERSITY
RUSSIA
E-mail address: v.davletshina@gmail.com

ANDREY E. MIRONOV
SOBOLEV INSTITUTE OF MATHEMATICS
NOVOSIBIRSK STATE UNIVERSITY
RUSSIA
E-mail address: mironov@math.nsc.ru