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ON CONTRACTION OF ALGEBRAIC POINTS

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Abstract. We study contraction of points on $\mathbb{P}^1(\bar{\mathbb{Q}})$ with certain control on local ramification indices, with application to the unramified curve correspondence problem initiated by Bogomolov and Tschinkel.

1. Introduction

In this paper we address the following problem: let P be a subset of natural numbers and S_1, S_2 be two subsets of points on $\mathbb{P}^1(\bar{\mathbb{Q}})$. We say S_1 can be P-contracted to S_2 if there is a rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that the image of S_1 under f and all branch points of f are contained in S_2 with all local ramification indices of f belonging to P.

One motivation of our problem is coming from Belyi's theorem. In this language Belyi's theorem states that if P is the set of all natural numbers, then any finite subset $S_1 \subset \mathbb{P}^1(\bar{\mathbb{Q}})$ can be P-contracted to $S_2 = (0, 1, \infty)$ or to any three points in $\mathbb{P}^1(\bar{\mathbb{Q}})$.

Another motivation is coming from the study of unramified correspondences between curves. Following [4], we make the following definition:

Definition 1. By a curve, we mean a smooth projective curve over $\bar{\mathbb{Q}}$. When we write an affine equation for a curve, its smooth projective model is understood. If $C \to C''$ and $C' \to C''$ are surjective morphisms of curves, by a compositum of C and C' over C'', we mean a curve whose function field is a compositum of $\bar{\mathbb{Q}}(C)$ and $\bar{\mathbb{Q}}(C')$ over $\bar{\mathbb{Q}}(C'')$. By an unramified cover of C, we mean a curve $\tilde{C}$ together with an *étale* morphism $\tilde{C} \to C$. Let C, C' be two curves. We call C lies over C' and write $C \Rightarrow C'$ if there exists an unramified cover of C which admits a surjective map to C'. If C lies over C' and C' also lies over C, we call C and C' are equivalent and write $C \Leftrightarrow C'$. Finally, denote by C_n the curve: $y^2 = x^n - 1$.

In the study of such correspondence, an important step which is closely related to our contraction problem is the construction of unramfied covers for

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which we need to find maps from various intermediate curves to $\mathbb{P}^1$ or some elliptic curves with restrictions on local ramification indices and the number of branch points. This method was established by Bogomolov and Tschinkel in [4] where they have showed that any hyperbolic hyperelliptic curve lies over C_6.

Here in Section 2, our main results are:

Theorem 2. *If the only prime divisors of n and m are 2, 3 and 5, then $\mathsf{C}_n \Leftrightarrow \mathsf{C}_m$ and for any $k \geq 5$ we have $\mathsf{C}_k \Rightarrow \mathsf{C}_n$.*

Remark 3. Although Theorem 2 is also established in [4], the proof contains several gaps in the construction of unramified covers. Based on the idea in [4], here we will use a different approach to establish this result.

Theorem 4. *If $n = 2^a 3^b 5^c 7^d$, then $\mathsf{C}_{6\cdot 13^d} \Rightarrow \mathsf{C}_n$.*

In [4], Bogomolov and Tschinkel have made the conjecture that the curve C_6 lies over any other curve. The reason why we are interested in the family of curves $\{\mathsf{C}_n\}$ is that the Bogomolov-Tschinkel conjecture will hold if C_6 lies over C_n for any positive integer n (See Proposition 17). Towards this conjecture, in section 2 and section 3 we introduce the notion of contracting a finite given subset of $\mathbb{P}^1(\bar{\mathbb{Q}})$ into another finite subset of $\mathbb{P}^1(\bar{\mathbb{Q}})$ with restrictions on the local ramification indices (See Definition 15) and the notion of contracting a finite subset of $\mathbb{P}^1(\bar{\mathbb{Q}})$ to a four-point subset of $\mathbb{P}^1(\bar{\mathbb{Q}})$ via elliptic curves (See Definition 31). We have obtained some criterions for a curve C with C_6 lying over C (See Theorem 16, Theorem 32, Corollary 33). In section 4, we will propose a procedure to approach the Bogomolov-Tschinkel conjecture.

2. Unramified correspondences over $\bar{\mathbb{Q}}$

Notations. Let $f : C \to C'$ be a surjective morphism of curves. We denote by $\mathrm{Bran}(f)$ the branch locus of f and denote by $\mathrm{Ram}(f)$ the ramification points of f. For a point $y \in \mathrm{Bran}(f)$, $x \in f^{-1}(y)$, denote by $e(x|y)$ the local ramification index of x at y. For a set of four points $a, b, c, d \in \bar{\mathbb{Q}}$, we denote by $E(a, b, c, d)$ an elliptic curve branched over $\{a, b, c, d\}$.

In this section, we will establish some results about the unramified curve correspondence problem. The key tool is:

Abyhankar's Lemma. *Let $f : C \to C''$ and $g : C' \to C''$ be surjective morphisms of curves. Denote by $\hat{C}$ the compositum of C and C' over C'' with corresponding map h and l:*

$$\begin{array}{ccc} & \hat{C} & \\ {}^{h}\swarrow & & \searrow^{l} \\ C & & C' \\ {}_{f}\searrow & & \swarrow_{g} \\ & C'' & \end{array}$$

Assume $x \in C$ *and* $y \in C'$ *such that* $f(x) = g(y) = z$ *for some point* z *on* C''. *Suppose* $f^{-1}(z) = \{x_1, \ldots, x_s\}$, $g^{-1}(z) = \{y_1, \ldots, y_t\}$ *and denote by* d *the greatest common divisor of* $e(x_i|z)$ *for* $i = 1, \ldots, s$. *If for any* j, *we have*:

$$e(y_j|z) \mid d.$$

Then for any i, x_i *is unramified under* h *and for any* j *and any point* $a \in l^{-1}(y_j)$ *we have*:

$$e(a|y_j) = \frac{e(h(a)|z)}{e(y_j|z)}.$$

In particular, if for all points $x \in C$ *and* $y \in C'$ *with* $f(x) = g(y)$ *we have:*

$$e(y|g(y)) \mid e(x|f(x)).$$

Then a compositum of C *and* C' *over* C'' *is an unramified cover of* C.

Proof. This follows from Theorem 3.9.1 in [7]. □

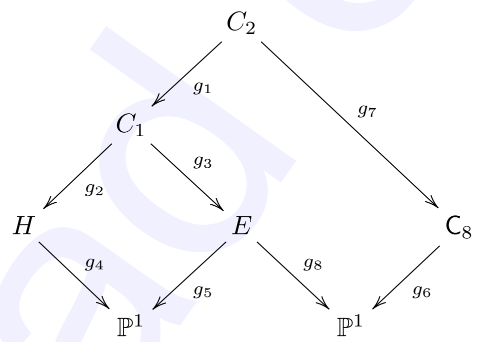
In the following proofs, our main strategy is to construct the unramified covers of curves directly via Abhyankar's lemma. In order to make such constructions using Abhyankar's lemma, we will explicitly contract some cyclotomic roots and also use some special elliptic curves to contract and spread points.

Proposition 5. *Let* H *be a hyperbolic hyperelliptic curve. Then* $H \Rightarrow \mathsf{C}_6$.

Proof. This is one part of Proposition 2.4 in [4]. □

Proposition 6. *Let* H *be a hyperbolic hyperelliptic curve. Then* $H \Rightarrow \mathsf{C}_8$.

Proof. Consider the following diagrams:



and

$$\mathsf{C}_8 \xrightarrow{f_1} \mathbb{P}^1 \xrightarrow{f_2} \mathbb{P}^1 \xrightarrow{f_3} \mathbb{P}^1 .$$

Denote $f_3 \circ f_2 \circ f_1$ by f. In these diagrams:

(i) The map f_1 is the standard degree 2 projection with $\mathrm{Bran}(f_1)$ containing all 8th roots of unity with local ramification indices being 2;

(ii) The map f_2 is x^4;

(iii) The map f_3 is $(\frac{x-1}{x+1})^2$;

(iv) The map g_4 is the standard degree 2 projection which has 6 branch points;

(v) The map g_6 is f. Bran(g_6)=$\{0,1,\infty\}$ with all local ramification indices being 4;

(vi) E is an elliptic curve branched at 4 points of Bran(g_4);

(vii) The map g_8 is the standard degree 2 projection combined with an automorphism of $\mathbb{P}^1$ such that Bran(g_8) contains $\{0,1,\infty\}$;

(viii) The map g_5 is a composition of a multiplication-by-2 map, a translation-by-R map and the standard degree 2 projection such that the image of R under the standard degree 2 projection is a point in Bran(g_4) which is different from the 4 points in (vi);

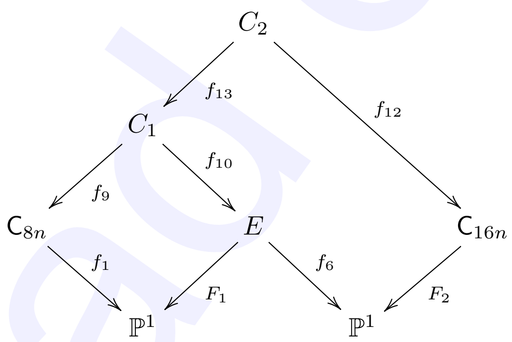
(ix) The curve C_1 is a compositum of H and E over $\mathbb{P}^1$. Since Bran(g_5) consists of the image of two-torsion points of E under the standard projection, by (iv) and (v) we see that C_1 is an unramified cover of H;

(x) The curve C_2 is a compositum of C_1 and C_8 over $\mathbb{P}^1$. Note that in (viii) all two torsion points of E are mapped to a point in Bran(g_4) which is different from the 4 points in (vi). By Abyhankar's lemma, these points are in the branch locus of g_3 with local ramification indices being 2. Thus, Bran($g_8 \circ g_3$) contains 0,1 and ∞ with local ramification indices being 4. By Abhyankar's lemma, we have: C_2 is an unramified cover of C_1. Combined with (ix), we see that C_2 is an unramified cover of H which maps surjectively onto C_8. □

Proposition 7. *$\mathsf{C}_{8n} \Rightarrow \mathsf{C}_{16n}$ and $\mathsf{C}_{16n} \Rightarrow \mathsf{C}_{24n}$ for $n \geq 1$.*

Proof. First, let us show: $\mathsf{C}_{8n} \Rightarrow \mathsf{C}_{16n}$ for $n \geq 1$:

Consider the following diagrams:



$$E \xrightarrow{f_5} E \xrightarrow{f_4} E \xrightarrow{f_3} \mathbb{P}^1 \xrightarrow{f_2} \mathbb{P}^1$$

$$\mathsf{C}_{16n} \xrightarrow{f_{11}} \mathbb{P}^1 \xrightarrow{f_8} \mathbb{P}^1 \xrightarrow{f_7} \mathbb{P}^1$$

In these diagrams:

(i) The map f_2 is x^2;

(ii) The curve E is defined by: $y^2 = x^3 - x$ and f_3 is the standard projection;

(iii) The map f_4 is the translation-by-R map where $R = (1,0)$;

(iv) The map f_5 is the multiplication-by-2 map;
(v) The map F_1 is $f_2 \circ f_3 \circ f_4 \circ f_5$;
(vi) The map f_6 is the standard degree 2 projection;
(vii) The map f_7 is $(\frac{x-1}{x+1})^2$;
(viii) The map f_8 is x^{8n};
(ix) The map f_{11} is the standard degree 2 projection;
(x) The map F_2 is $f_7 \circ f_8 \circ f_{11}$. $\mathrm{Bran}(F_2) = \{0, 1, \infty\}$ with corresponding local ramification indices being 4, $8n$, 4;

(xi) The map f_1 is the projection y composed with an automorphism of $\mathbb{P}^1$ which maps three branch points to 0,1 and ∞ such that $f_1^{-1}(1)$ and $f_1^{-1}(\infty)$ each contains one point with ramification index 8n and $f_1^{-1}(0)$ contains two points with ramification indices 4n;

(xii) The curve C_1 is a compositum of C_{8n} and E over $\mathbb{P}^1$ (via map f_1 and F_1);

(xiii) The curve C_2 is a compositum of C_1 and C_{16n} over $\mathbb{P}^1$ (via map $f_6 \circ f_{10}$ and F_2).

We see that:

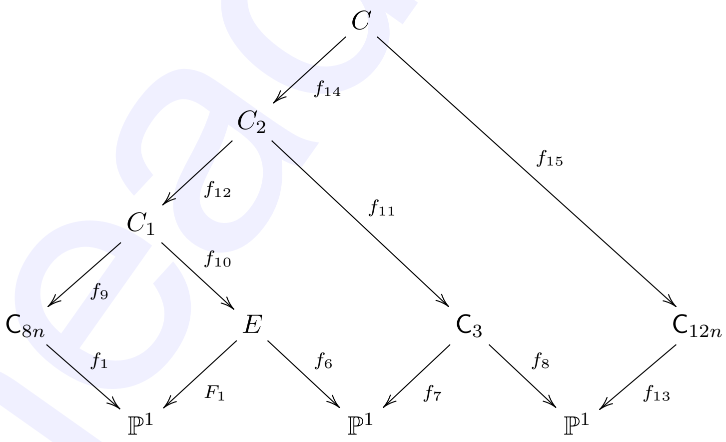
(1) Since $\mathrm{Bran}(F_1)$=$(0, 1, \infty)$ with local ramification indices: $2, 4, 4$ (over $1, 0, \infty$ respectively), combined with (xi) we get: f_9 is unramified and each point of $F_1^{-1}(1)$ has ramification index $4n$ under f_{10}. Note that: $E[2]$ is contained in $F_1^{-1}(1)$;

(2) By (1), $\mathrm{Bran}(f_6 \circ f_{10})$=$(0, 1, -1, \infty)$ with all local ramification indices being $8n$;

(3) By (2) and (x), f_{13} is unramified. Combined with (1) we have:

$$\mathsf{C}_{8n} \Rightarrow \mathsf{C}_{\mathsf{16n}}.$$

Next let us show: $\mathsf{C}_{8n} \Rightarrow \mathsf{C}_{12n}$ for $n \geq 1$ and n even:
(which is the same as $\mathsf{C}_{16n} \Rightarrow \mathsf{C}_{24n}$ for $n \geq 1$)
Consider the following diagrams:



$$E \xrightarrow{f_5} E \xrightarrow{f_4} E \xrightarrow{f_3} \mathbb{P}^1 \xrightarrow{f_2} \mathbb{P}^1$$

In these diagrams:

(i) The map f_2 is x^2;

(ii) The curve E is defined by: $y^2 = x^3 - x$ and f_3 is the standard projection;

(iii) The map f_4 is the translation-by-R map where $R = (1, 0)$;

(iv) The map f_5 is the multiplication-by-3 map;

(v) The map f_6 is the standard projection combined with an automorphism of $\mathbb{P}^1$ such that: $f_6(E[3])$ is the union of one point (this point is denoted by a) from Bran(f_6) and $(1, \zeta_3, \zeta_3^2, \infty)$;

(vi) The map f_7 is the multiplication-by-3 map combined with the standard projection;

(vii) The map f_8 is y combined with an automorphism of $\mathbb{P}^1$ which maps the three branch points to $0, 1, \infty$;

(viii) The map f_1 is the projection y composed with an automorphism of $\mathbb{P}^1$ which maps three branch points to 0,1 and ∞ such that $f_1^{-1}(1)$ and $f_1^{-1}(\infty)$ each contains one point with ramification index $8n$ and $f_1^{-1}(0)$ contains two points with ramification indices 4n;

(ix) The curve C_1 is a compositum of C_{8n} and E over $\mathbb{P}^1$ (via f_1 and $f_2 \circ f_3 \circ f_4 \circ f_5$);

(x) The curve C_2 is a compostitum of C_1 and C_3 over $\mathbb{P}^1$ (via $f_6 \circ f_{10}$ and f_7);

(xi) The curve C is a compositum of C_2 and C_{12n} (via $f_8 \circ f_{11}$ and f_{13});

(xii) The map f_{13} is the standard projection to $\mathbb{P}^1$ composed with x^{6n} and $(\frac{x-1}{x+1})^2$. Bran(f_{13})=$(0, 1, \infty)$ and the corresponding ramification indices are $(4, 6n, 4)$.

We see that:

(1) Since Bran($f_2 \circ f_3 \circ f_4 \circ f_5$)=$(0, 1, \infty)$ with ramification indices: $2, 4, 4$ (over $1, 0, \infty$ respectively), combined with (viii) we get: f_9 is unramified and each point of $(f_2 \circ f_3 \circ f_4 \circ f_5)^{-1}(1)$ has ramification index $4n$ under f_{10}. Note that: $E[3]$ is contained in $(f_2 \circ f_3 \circ f_4 \circ f_5)^{-1}(1)$;

(2) By (v) and (1), Bran($f_6 \circ f_{10}$)=$(a, 1, \zeta_3, \zeta_3^2, \infty)$ with local ramification indices being $4n$(over $1, \zeta_3, \zeta_3^2, \infty$) and $8n$(over a);

(3) By (vi) and (2), f_{12} is unramified and $\mathsf{C}_3[3] \subseteq f_7^{-1}(1, \zeta_3, \zeta_3^2, \infty)$ which has ramification indices $2n$ under f_{11};

(4) By (vii) and (3), $(0, 1, \infty) \subset$ Bran($f_8 \circ f_{11}$) and they have local ramification indices $6n$;

(5) By (4) and (xii), we know that C is an unramified cover of C_2 and hence we have:

$$\mathsf{C}_{8n} \Rightarrow \mathsf{C}_{12n}$$

for $n \geq 1$ and n is even which is the same as

$$\mathsf{C}_{16n} \Rightarrow \mathsf{C}_{24n}$$

for $n \geq 1$. □

Corollary 8. *If* $n \geq 6$ *and the only prime divisors of* n *are 2 and 3, then* $\mathsf{C}_6 \Rightarrow \mathsf{C}_n$.

Proof. Write $n = 2^s 3^t$, we have: (repeat applying Proposition 7)

$$\mathsf{C}_6 \Rightarrow \mathsf{C}_8 \Rightarrow \mathsf{C}_{16} \Rightarrow \mathsf{C}_{16\cdot 2} \Rightarrow \mathsf{C}_{16\cdot 2^2} \Rightarrow \cdots \Rightarrow \mathsf{C}_{16\cdot 2^{s+t}} \Rightarrow \mathsf{C}_{16\cdot 2^{s+t-1}\cdot 3} \Rightarrow$$

$$\mathsf{C}_{16\cdot 2^{s+t-2}\cdot 3^2} \Rightarrow \cdots \Rightarrow \mathsf{C}_{16\cdot 2^s\cdot 3^t} \Rightarrow \mathsf{C}_{2^s\cdot 3^t} = \mathsf{C}_n.$$ □

Proposition 9. $\mathsf{C}_6 \Rightarrow \mathsf{C}_5$.

Proof. By Abhyankar's Lemma and Corollary 8, we only need to exhibit a map from C_5 to $\mathbb{P}^1$ such that the branch points are exactly $(0,1,\infty)$ and all local ramification indices have only prime divisors 2 or 3.
Consider the following maps:

$$\mathsf{C}_5 \xrightarrow{f_1} \mathbb{P}^1 \xrightarrow{f_2} \mathbb{P}^1 \xrightarrow{f_3} \mathbb{P}^1 \xrightarrow{f_4} \mathbb{P}^1 \xrightarrow{f_5} \mathbb{P}^1 \xrightarrow{f_6} \mathbb{P}^1 \xrightarrow{f_7} \mathbb{P}^1 \xrightarrow{f_8} \mathbb{P}^1 \xrightarrow{f_9} \mathbb{P}^1$$

Here: (ζ_5 is denoted by t)
(i) The map f_1 is the degree 2 projection.
Bran(f_1)=$(1, t, t^2, t^3, t^4, \infty)$ and all ramification indices are 2;
(ii) The map f_2 is $z + \frac{1}{z}$.
Ram(f_2)=$(1, -1)$ with all ramification indices 2 and
Bran(f_2)$\cup f_1$(Bran(f_1))=$(2, -2, t+t^4, t^2+t^3, \infty)$. This set is denoted by B_2;
(iii) The map f_3 is $-\frac{1}{z}$.
f_3 is clearly unramified and $f_3(B_2) = (-\frac{1}{2}, \frac{1}{2}, t^2+t^3, t+t^4, 0)$. This set is denoted by B_3;
(Note that $(t+t^4)(t^2+t^3) = t^3+t^4+t+t^2 = -1$.)
(iv) The map f_4 is $z^2 + z - 1$.
Ram(f_4)=$(-\frac{1}{2}, \infty)$ with all ramification indices 2 and
Bran(f_4) $\cup f_4(B_3) = (-\frac{5}{4}, \infty, -\frac{1}{4}, 0, -1)$. This set is denoted by B_4;
(v) The map f_5 is $-4z$.
Clearly it is unramified and $f_5(B_4) = (0, 1, 4, 5, \infty)$. This set is denoted by B_5;
(vi) The map f_6 is $4(z - \frac{5}{2})^2$.
Ram(f_6)=$(\frac{5}{2}, \infty)$ with all ramification indices 2 and
Bran(f_6) $\cup f_6(B_5) = (0, \infty, 25, 9)$. This set is denoted by B_6;
(vii) The map f_7 is $\frac{1}{2}(\frac{1}{2}(z + \frac{225}{z}) + 15)$.
Ram(f_7)=$(15, -15)$ with all ramification indices 2 and
Bran(f_7) $\cup f_7(B_6) = (0, 15, 16, \infty)$. This set is denoted by B_7;
(viii) The map f_8 is $\frac{z}{z-15}$.
Clearly it is unramified and $f_8(B_7) = (0, 1, 16, \infty)$. This set is denoted by B_8;
(ix) The map f_9 is $\frac{(z-1)^{32}\cdot(z-16)^3}{(z-10)^8\cdot z^{27}}$.
Ram(f_9)=$(0, 1, 10, 16, \infty)$ with corresponding ramification indices 3^3, 2^5, 2^3, 3, 3 and

(Note that $\frac{df_9}{f_9} = \frac{4320}{z(z-1)(z-10)(z-16)}$ and the computation for ramification index of ∞ follows from the Riemann-Hurwitz Formula.)

Bran$(f_9) \cup f_9(B_8) = (0, 1, \infty)$.

By the computations in (i)-(ix), we see that Bran $(f_9 \circ f_8 \circ f_7 \circ f_6 \circ f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1) = \text{Bran}(f_9) \cup f_9(B_8) = (0, 1, \infty)$ with all local ramification indices only having prime divisors 2 or 3 (Note that in each step, the local ramification indices only have prime divisors 2 or 3). □

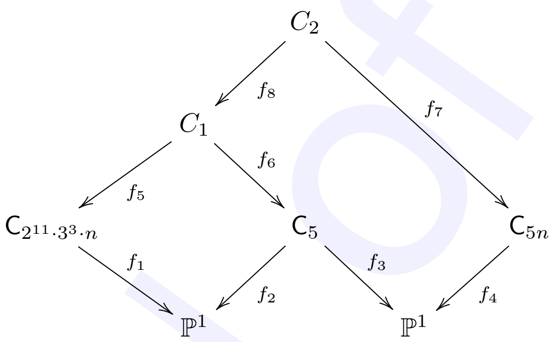
Proposition 10. $\mathsf{C}_{2^{11}\cdot 3^3\cdot n} \Rightarrow \mathsf{C}_{5n}$ *for* $n \geq 1$.

Proof. Let us still use this diagram:

$$\mathsf{C}_5 \xrightarrow{f_1} \mathbb{P}^1 \xrightarrow{f_2} \mathbb{P}^1 \xrightarrow{f_3} \mathbb{P}^1 \xrightarrow{f_4} \mathbb{P}^1 \xrightarrow{f_5} \mathbb{P}^1 \xrightarrow{f_6} \mathbb{P}^1 \xrightarrow{f_7} \mathbb{P}^1 \xrightarrow{f_8} \mathbb{P}^1 \xrightarrow{f_9} \mathbb{P}^1$$

Here f_i are the maps as in the last proposition and let $f = f_9 \circ f_8 \circ f_7 \circ f_6 \circ f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$. Note that f is a Belyi map from C_5 to $\mathbb{P}^1$ with all local ramification indices divides $2^{10} \cdot 3^3$.

Now let us consider the following diagram:



In this diagram:

(i) The map f_1 is the projection y composed with an automorphism of $\mathbb{P}^1$ which maps three branch points to 0,1 and ∞ such that $f_1^{-1}(0)$ and $f_1^{-1}(1)$ each contains one point with ramification index $2^{11} \cdot 3^3 \cdot n$ and $f_1^{-1}(\infty)$ contains two points with ramification indices $2^{10} \cdot 3^3 \cdot n$.

(ii) The map f_2 is the map f above.

(iii) The curve C_1 is a compositum of $\mathsf{C}_{2^{11}\cdot 3^3\cdot n}$ and C_5 over $\mathbb{P}^1$.

(iv) By (i) and (ii), f_5 is unramified and each point in $f_2^{-1}(0, 1, \infty)$ has ramification index a multiple of n under f_6.

(v) The map f_3 is the projection y composed with an automorphism of $\mathbb{P}^1$ which maps three branch points to 0,1 and ∞ with ramification indices 5.

(vi) The map f_4 is the projection y composed with an automorphism of $\mathbb{P}^1$ which maps three branch points to 0,1 and ∞ with ramification indices 5n.

(vii) The curve C_2 is a compositum of C_1 and C_{5n} over $\mathbb{P}^1$ (via map $f_3 \circ f_6$ and f_4).

(viii) From (iv), (v) and (vi) and Abhyankar's lemma, we see that f_8 is unramified.

(ix) By (iv) and (viii), C_2 is an unramified cover of $\mathsf{C}_{2^{11}.3^3.n}$ which maps surjectively onto C_{5n}. □

Corollary 11. *If $n \geq 5$ and the only prime divisors of n are 2, 3 or 5, then: $\mathsf{C}_6 \Rightarrow \mathsf{C}_n$.*

Proof. Write n as $2^r 3^s 5^t$ and m as $2^r 3^s$, we have:

If $t = 0$, this follows from Corollary 8.

If $t \neq 0$, then: (repeat using Proposition 10)

$$\mathsf{C}_6 \Rightarrow \mathsf{C}_{2^{11t}.3^{3t}.m} \Rightarrow \mathsf{C}_{2^{11(t-1)}.3^{3(t-1)}.5m} \Rightarrow \mathsf{C}_{2^{11(t-2)}.3^{3(t-2)}.5^2m}$$
$$\Rightarrow \cdots \Rightarrow \mathsf{C}_{2^{11(t-t)}.3^{3(t-t)}.5^tm} = \mathsf{C}_n.$$
□

Proof of Theorem 2. Assume the only prime divisors of n and m are 2, 3 or 5. By Proposition 5, C_n lies over C_6. By Corollary 11, C_6 also lies over C_m and consequently C_n lies over C_m. Similarly C_m also lies over C_n. Hence C_n and C_m are equilvalent. For the second part, just note that for $k \geq 5$, C_k is a hyperbolic hyperelliptic curve. □

Proposition 12. $\mathsf{C}_{6 \cdot 13} \Rightarrow \mathsf{C}_7$.

Proof. Consider the following maps:

$$\mathsf{C}_7 \xrightarrow{h_1} \mathbb{P}^1 \xrightarrow{h_2} \mathbb{P}^1 \xrightarrow{h_3} \mathbb{P}^1 \xrightarrow{h_4} \mathbb{P}^1 \xrightarrow{h_5} \mathbb{P}^1 \xrightarrow{h_6} \mathbb{P}^1$$

Here:

(i) The map h_1 is the degree 2 projection.

Bran(h_1)=$(1, t, t^2, t^3, t^4, t^5, t^6\infty)$ and all local ramification indices are 2;

(ii) The map h_2 is $z + \frac{1}{z}$.

Ram(h_2)=$(1, -1)$ with all ramification indices 2 and

Bran(h_2)$\cup f_1$(Bran(h_1))=$(2, -2, t+t^6, t^2+t^5, t^3+t^4, \infty)$. This set is denoted by D_2;

(iii) The map h_3 is $\frac{z+2}{z-2}$.

h_3 is unramified and $h_3(D_2) = (\infty, 0, t_1, t_2, t_3, 1)$. This set is denoted by D_3.

Here t_i are roots of $7z^3 + 35z^2 + 21z + 1 = 0$;

(iv) The map h_4 is $7z^3 + 35z^2 + 21z + 1$.

Ram(h_4)=(-$\frac{1}{3}$, $-3, \infty$) with all ramification indices 2 or 3 and

Bran(h_4) $\cup h_4(D_3) = (0, 1, 64, -\frac{64}{27}, \infty)$. This set is denoted by D_4;

(v) The map h_5 is $256 \cdot \frac{z-1}{z-64}$.

Clearly it is unramified and $h_5(D_4) = (0, 4, 13, 256, \infty)$. This set is denoted by D_5;

(vi) The map h_6 is

$$\frac{z^{12301875} \cdot (z-6)^{32752512} \cdot (z-256)^{13}}{(z-4)^{42120000} \cdot (z-13)^{2560000} \cdot (z+14)^{374400}}.$$

(This map is coming from a search using Belyi's formula (See Definition 21 and the proof of Proposition 23).)

Ram(h_6)=$(0, 4, 6, 13, -14, 256, \infty)$ with corresponding ramification indices $3^9 5^4, 2^6 3^4 5^4 13, 2^7 3^9 13, 2^{12} 5^4, 2^7 3^2 5^2 13, 13, 5$ and Bran$(h_6) \cup h_6(D_5) = (0, 1, \infty)$.

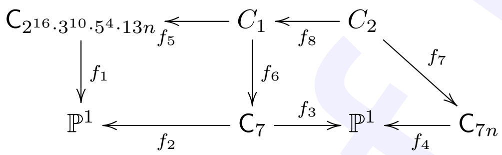
By (i)-(vi), $h_6 \circ h_5 \circ h_4 \circ h_3 \circ h_2 \circ h_1$ is a Belyi map with all local ramification indices dividing $2^{15} 3^{10} 5^4 13$. By Abhyankar's Lemma, Proposition 7 and Proposition 10, we have:

$$\mathsf{C}_{6 \cdot 13} \Rightarrow \mathsf{C}_{2^{15} 3^{10} 5^4 13} \Rightarrow \mathsf{C}_7$$ □

Proposition 13. $\mathsf{C}_{2^{16} \cdot 3^{10} \cdot 5^4 \cdot 13n} \Rightarrow \mathsf{C}_{7n}$ *for* $n \geq 1$.

Proof. Let $h = h_6 \circ h_5 \circ h_4 \circ h_3 \circ h_2 \circ h_1$ where h_i are the maps in the last proposition. Note that h is a Belyi map from C_7 to $\mathbb{P}^1$ with all local ramification indices divides $2^{15} 3^{10} 5^4 13$.

Now consider the following diagram:



In this diagram:

(i) The map f_1 is the projection y composed with an automorphism of $\mathbb{P}^1$ which maps three branch points to 0,1 and ∞ such that $f_1^{-1}(0)$ and $f_1^{-1}(1)$ each contains one point with ramification index $2^{16} \cdot 3^{10} \cdot 5^4 \cdot 13n$ and $f_1^{-1}(\infty)$ contains two points with ramification indices $2^{15} \cdot 3^{10} \cdot 5^4 \cdot 13n$;

(ii) The map f_2 is the map h above;

(iii) The curve C_1 is a compositum of $\mathsf{C}_{2^{16} \cdot 3^{10} \cdot 5^4 \cdot 13n}$ and C_7 over $\mathbb{P}^1$;

(iv) By (i) and (ii), f_5 is unramified and each point in $f_2^{-1}(0, 1, \infty)$ has ramification index a multiple of n under f_6;

(v) The map f_3 is the projection y composed with an automorphism of $\mathbb{P}^1$ which maps three branch points to 0,1 and ∞ with ramification indices 7;

(vi) The map f_4 is the projection y composed with an automorphism of $\mathbb{P}^1$ which maps three branch points to 0,1 and ∞ with ramification indices 7n.;

(vii) The curve C_2 is a compositum of C_1 and C_{7n} over $\mathbb{P}^1$ (via map $f_3 \circ f_6$ and f_4);

(viii) By the computations in (iv), (v), (vi) and Abhyankar's lemma, f_8 is unramified;

(ix) By (iv) and (viii), C_2 is an unramified cover of $\mathsf{C}_{2^{16} \cdot 3^{10} \cdot 5^4 \cdot 13n}$ which maps subjectively onto C_{7n}. □

Proof of Theorem 4. Set $m = 2^a 3^b 5^c$.

If $d = 0$, this follows from Theorem 2.

If $d \neq 0$, then:

$$\mathsf{C}_{6 \cdot 13^d} \Rightarrow \mathsf{C}_{2^{16d} \cdot 3^{10d} \cdot 5^{4d} \cdot 13^d \cdot m} \Rightarrow \mathsf{C}_{2^{16(d-1)} \cdot 3^{10(d-1)} \cdot 5^{4(d-1)} \cdot 13^{d-1} \cdot 7 \cdot m}$$

$$\Rightarrow \mathsf{C}_{2^{16(d-2)}\cdot 3^{10(d-2)}\cdot 5^{4(d-2)}\cdot 13^{d-2}\cdot 7^2\cdot m} \Rightarrow \cdots$$
$$\Rightarrow \mathsf{C}_{2^{16(d-d)}\cdot 3^{10(d-d)}\cdot 5^{4(d-d)}\cdot 13^{d-d}\cdot 7^d\cdot m} = \mathsf{C}_n. \qquad \square$$

By similar construction as in Proposition 12, we can also have:

Proposition 14. $\mathsf{C}_{6\cdot 11\cdot 43} \Rightarrow \mathsf{C}_7$.

Proof. We consider the same maps as in Proposition 12 except that we replace h_6 by:

$$\frac{z^{8620425}\cdot(z-13)^{7208960}\cdot(z-56)^{1539648}}{(z-4)^{14860800}\cdot(z-48)^{2507760}\cdot(z-256)^{473}}$$

we have:

Ram(h_6)=$(0,4,13,48,56,256,\infty)$ with corresponding ramification indices $3^6\cdot 5^2\cdot 11\cdot 43, 2^9\cdot 3^3\cdot 5^2\cdot 43, 2^{17}\cdot 5\cdot 11, 2^4\cdot 3^6\cdot 5\cdot 43, 2^6\cdot 3^7\cdot 11, 11\cdot 43, 5$ and $\mathrm{Bran}(h_6)\cup h_6(D_5)=(0,1,\infty)$.

Thus $h_6\circ h_5\circ h_4\circ h_3\circ h_2\circ h_1$ is a Belyi map with all local ramification indices dividing $2^{18}\cdot 3^8\cdot 5^2\cdot 11\cdot 43$. By Abhyankar's lemma, Proposition 7 and Proposition 10, we have:

$$\mathsf{C}_{6\cdot 11\cdot 43} \Rightarrow \mathsf{C}_{2^{18}\cdot 3^8\cdot 5^2\cdot 11\cdot 43} \Rightarrow \mathsf{C}_7 \qquad \square$$

Definition 15. Let k be a field. Let P be a subset of natural numbers and S be a subset of points on $\mathbb{P}^1(\bar{k})$. We call a curve C is P-ramified over S if there exists a morphism from C to $\mathbb{P}^1$ such that all branch points are contained in S and all local ramification indices are contained in P. Given two subsets S_1 and S_2 of points on $\mathbb{P}^1(\bar{k})$, we say S_1 can be P-contracted to S_2, if there exists a morphism $f:\mathbb{P}^1\to\mathbb{P}^1$ such that $f(S_1)$ and $\mathrm{Bran}(f)$ is contained in S_2 and all local ramification indices are contained in P.

Theorem 16. *Let k=$\mathbb{Q}$. If a curve C is P-ramified over S which can be P-contracted to $(0,1,\infty)$ such that all numbers in P only have prime divisors 2, 3 or 5, then $\mathsf{C}_6\Rightarrow C$. If we further allow 7 appearing as prime divisors of numbers in P, then there exists a positive integer n such that $\mathsf{C}_{6\cdot 13^n}\Rightarrow C$.*

Proof. This follows from Theorem 2 and Theorem 3. $\square$

Proposition 17. *If $\mathsf{C}_6\Rightarrow\mathsf{C}_n$ holds for any positive integer n, then for any curve C, we have $\mathsf{C}_6\Rightarrow C$.*

Proof. By Belyi's theorem C is P-ramified over $(0,1,\infty)$ for some finite set P. Let n be the least common multiple of numbers in P. Then we have: $\mathsf{C}_6\Rightarrow\mathsf{C}_n\Rightarrow C$. $\square$

In [4], we have the following conjecture:

Conjecture 18. *Let C be any curve over $\bar{\mathbb{Q}}$. Then $\mathsf{C}_6\Rightarrow C$.*

We will describe a possible way to approach this conjecture in the last section.

3. Contraction of points on $\mathbb{P}^1(\bar{\mathbb{Q}})$

In this section, we discuss the problem of contraction of points on $\mathbb{P}^1(\bar{\mathbb{Q}})$ with certain control on local ramification indices.

The first result is from [4, Theorem 4.4]:

Theorem 19. *Let S be a finite set of points on $\mathbb{P}^1(\bar{\mathbb{Q}})$. Then there exists a map*

$$f : \mathbb{P}^1 \to \mathbb{P}^1$$

which is defined over $\mathbb{Q}$ such that:

$$f(S) \cup Ram(f) \subset \mathbb{P}^1(\mathbb{Q})$$

and moreover, all local ramification indices are powers of 2.

Proof. Denote $m = \max(\deg(s))$ for $s \in S$ and assume $x \in S$ has degree m.

Assume $2^{k-1} \leq m < 2^k$ for some positive integer k. Let $r = 2^k - m$ and consider polynomials $f \cdot g_r$ where f is the minimal polynomial of x and g_r runs over all monic polynomials of degree r with rational coefficients. Let us denote by L_r the space of such polynomials.

We claim that there is a polynomial $g \in L_r$ such that all finite ramification points of $F = fg$ are simple (order 2) and there are at least r rational ramification points. Indeed, given $x_1, \ldots, x_r \in \mathbb{Q}$, the condition that $x_1, \ldots, x_r$ are ramification points of F yields a system of r linear equations on the coefficients of g_r in terms of x_i and the coefficients of f. The corresponding system of linear equations is nondegenerate if $\{x_1, \ldots, x_r\}$ does not intersect the common roots of f' and f. Thus we obtain a rational map defined over $\mathbb{Q}$ from $\mathbb{A}^r$ to L_r and clearly each point in L_r only has a finite number of preimages. Note that a condition that for $h \in L_r$ the derivative h' has multiple roots defines a divisor D in L_r. Therefore the preimage of D can not be the whole domain of our rational map and thus we can pick some $(x_1, \ldots, x_r)$ such that the corresponding F satisfying our condition.

Now by our claim we can pick one such polynomial g and look at the map $F : \mathbb{P}^1 \mapsto \mathbb{P}^1$ given by $F = fg$. Note that the set of ramification points of F consists of r rational points, and some other points with algebraic degree less than m and the point ∞. Also all ramification points except ∞ are simple and the ramification index at ∞ is 2^k. Thus, every point in the set $S \cup F(S) \cup \mathrm{Ram}(F)$ has algebraic degree at most m, and the number of points with degree m in $F(S) \cup \mathrm{Ram}(F)$ is strictly less than that for S. Repeating this construction, we see that the composition of all these maps is a desired map. □

Since every curve admits a map to $\mathbb{P}^1$ with simple ramification points, we have an immediate corollary:

Corollary 20. *Let C be a curve over $\bar{\mathbb{Q}}$. Then C is P-ramified over a finite set of points on $\mathbb{P}^1(\mathbb{Q})$ with P being the subset of natural numbers containing all powers of 2.*

This theorem and its corollary is a generalization of the first step in the proof of Belyi's theorem in [1]. It is natural to consider whether in the second step in the proof of Belyi's theorem, one can also impose some restriction on local ramification indices. Let us consider the case of using Belyi's functions.

Definition 21. We call a morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a Belyi function with respect to a k-tuple $(n_1, \ldots, n_k)$ if:

$$f(x) = \prod_{i=1}^{k} (x - n_i)^{r_i} \tag{3.1}$$

with

$$\mathrm{Ram}(f) = (n_1, \ldots, n_k, \infty) \ \text{ and } \ f(\infty) = 1.$$

Remark 22. These maps are those appearing in Belyi's second proof of his theorem in [2]. Note that for $k \geq 3$, ∞ is a ramification point with index $k-1$.

A simple observation is:

Proposition 23. *Let P be the subset of natural numbers whose prime divisors are contained in a finite set of primes $\{p_1, \ldots, p_s\}$. Let S be a finite set of integers $\{n_1, \ldots, n_k\}$ plus ∞ such that:*
(i) *for any pair (i, j), $n_i - n_j \in P$;*
(ii) $k - 1 \in P$.
Then S can be P-contracted to $(0, 1, \infty)$.

Proof. As in [2], in (3.1) let us take:

$$r_i = (-1)^{i-1} V(n_1, \ldots, \hat{n_i}, \ldots, n_k),$$

where the term with a hat is to be omitted and V denotes the Vandermonde determinant. □

Conversely, if we use Belyi's functions to contract points, then the converse of the above proposition is true for $k = 3$:

Proposition 24. *Let P be the subset of natural numbers whose prime divisors are contained in primes $\{2, p_2, \ldots, p_s\}$. Let $S = \{n_1, n_2, n_3, \infty\}$. If S is P-contracted to $(0, 1, \infty)$ by some Belyi function, then for any pair (i, j), we have: $n_i - n_j \in P$.*

Moreover, there are only finitely many such sets S modulo translation and multiplication.

Proof. Let f be a Belyi function with respect to (n_1, n_2, n_3):

$$f(x) = \prod_{i=1}^{3} (x - n_i)^{r_i}.$$

We have:

$$r_1 + r_2 + r_3 = 0 \tag{3.2}$$

and

$$(n_2+n_3)r_1+(n_1+n_3)r_2+(n_1+n_2)r_3=0 \tag{3.3}$$

with

$$r_i \in P.$$

Modulo translation and multiplication, we may assume $n_1 = 0$ and $(n_2, n_3) = 1$. From (3.2) and (3.3), we have:

$$n_2 = r_3, n_3 = -r_2, r_1 = n_3 - n_2 \text{ and } (r_2, r_3) = 1.$$

Hence, n_2, n_3 and $n_2 - n_3$ are all in P.

Moreover, since (3.2) can be transformed into a unit equation in $\{2, p_2, \ldots, p_s\}$-units, it only has finitely many coprime solutions which means such S are finite modulo translation and multiplication (See Theorem 7.4.2 in [6]). □

Remark 25. From this proposition, we see that in the case of $k = 3$ if we use Belyi functions to contract points on $\mathbb{P}^1(\mathbb{Z})$, then the prime divisors of local ramification indices are depended on the prime divisors of pairwise differences between these points.

However, starting with $k = 4$, we have exceptional examples. Let us see one example:

Example 26. Let P be the subset of natural numbers whose prime divisors are contained in $\{2, 3\}$ and $S = \{0, 1, 5, 6\}$. Then we have the following Belyi function with respect to this 4-tuple:

$$f(x) = \frac{(x-1)^3(x-6)^2}{x^2(x-5)^3}.$$

Hence, S can be $P-$contracted to $(0, 1, \infty)$ but 5, which is the difference between 5 and 0, is not in P.

Although for $k \geq 4$ there are some exceptional examples, we have the following:

Theorem 27. *Let P be a subset of natural numbers containing prime divisors $3, p_2, \ldots, p_s$. Then the set of collections of 4-tuples (n_1, n_2, n_3, n_4) plus ∞ which can be P-contracted to $(0, 1, \infty)$ by some Belyi's functions are contained in some finite union of hyperplanes in $\mathbb{A}^4(\mathbb{Z})$. (Modulo translation and multiplication, it's contained in some finite union of lines in $\mathbb{A}^2(\mathbb{Q})$.) Moreover, the number of such 4-tuples which do not satisfy condition* (i) *in Proposition 23 is infinite modulo translation and multiplication.*

Proof. Let f be a Belyi function with respect to the 4-tuple (n_1, n_2, n_3, n_4):

$$f(x) = \prod_{i=1}^{4}(x-n_i)^{r_i}.$$

Then we have:

$$r_1 + r_2 + r_3 + r_4 = 0 \tag{3.4}$$

and

$$(n_2+n_3+n_4)r_1+(n_1+n_3+n_4)r_2+(n_1+n_2+n_4)r_3+(n_1+n_2+n_3)r_4 = 0 \tag{3.5}$$

and

$$(n_2n_3 + n_2n_4 + n_3n_4)r_1 + \cdots + (n_1n_2 + n_1n_3 + n_2n_3)r_4 = 0 \tag{3.6}$$

with

$$r_i \in P.$$

Since (3.4) can be transformed into a unit equation in $\{3, p_2, \ldots, p_s\}$-units, we have:

Either some proper subsum of $r_1 + r_2 + r_3 + r_4$ vanishes or it will only have finitely many coprime solutions. For each solution in the second case, the corresponding 4-tuple is contained in the hyperplane defined by (3.5) (Although such corresponding 4-tuple may not exist). The remaining case is either $r_1 + r_2, r_1 + r_3$ or $r_1 + r_4$ vanishes. Without loss of generality, assume $r_1 + r_2 = 0$ which implies $r_3 + r_4 = 0$. Thus, (3.5) and (3.6) are reduced to:

$$(n_2 - n_1)r_1 + (n_4 - n_3)r_3 = 0 \tag{3.7}$$

and

$$(n_2 - n_1)(n_3 + n_4)r_1 + (n_4 - n_3)(n_1 + n_2)r_3 = 0. \tag{3.8}$$

Substitute (3.7) into (3.8) yields:

$$(n_2 - n_1)(-n_1 - n_2 + n_3 + n_4) = 0$$

which means our 4-tuple is contained in the hyperplane defined by the equation:

$$-x_1 - x_2 + x_3 + x_4 = 0.$$

Moreover, from (3.7) and (3.8), if we translate n_1 to 0, all solutions of (3.7) and (3.8) are: (modulo translation and multiplication)

$$n_1 = 0, n_2 = 2r_3, n_3 = r_1 + r_3, n_4 = r_3 - r_1.$$

with $(r_1, r_3) = 1$ and all prime divisors of them are in $\{3, p_2, \ldots, p_s\}$.

Therefore, we have infinitely many such 4-tuples which do not satisfy condition (i) of Proposition 23 since the unit equation:

$$\frac{n_3}{n_2} + \frac{n_4}{n_2} = 1$$

in $\{3, p_2, \ldots, p_s\}$-units only have finitely many coprime solutions. □

Remark 28. By similar argument, we can get similar results for $k \geq 5$. Thus, most k-tuples plus ∞ can not be P-contracted to $(0, 1, \infty)$ by using Belyi's functions if we let P be a subset of natural numbers whose prime divisors lie in a finite set of primes. This suggests that Question 1.4 in [3] may not have an affirmative answer.

Now let us discuss using elliptic curves to contract points and their relation to our unramified curve correspondence problem.

Following [4]:

Notation 29. Let E and E' be two elliptic curves and π and π' be the standard projection to $\mathbb{P}^1$. Write:

$$E \rightharpoondown E'$$

if $\mathrm{Bran}(\pi')$ is projectively equivalent to a set of four points in $\pi(E[\infty])$. Here, $E[\infty]$ is the set of torsion points on E.

One of the reasons why we study such relations comes from:

Theorem 30. *Let C' be a hyperbolic curve and $g : C' \to \mathbb{P}^1$ be a morphism with*

$$Bran(g) \subset \pi(E_n[\infty])$$

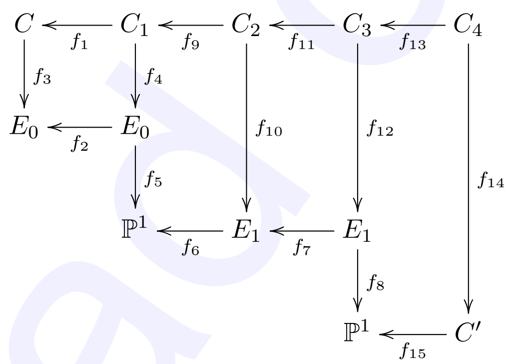
for some elliptic curve E_n. Denote by L the least common multiple of all local ramification indices of g. Assume we have:

$$E_0 \rightharpoondown E_1 \rightharpoondown \cdots \rightharpoondown E_n$$

and let C be a hyperbolic curve which admits a map onto E_0 such that there exists one branch point whose all local ramification indices are divisible by $2^n L$. Then we have:

$$C \Rightarrow C'.$$

Proof. Let us prove for the case $n = 1$. For $n > 1$, the proof is similar. Consider the following diagram:



In this diagram:

(i) The map f_{15} is g;

(ii) The maps f_5, f_6 and f_8 are the degree 2 projections such that: $\mathrm{Bran}(f_{15}) \subset f_8(E_1[\infty])$, $\mathrm{Bran}(f_6) \subset f_5(E_0[\infty])$;

(iii) The map f_7 is multiplication-by-m map with $f_8^{-1}(\mathrm{Bran}(f_{15})) \subset E_1[m]$;

(iv) The map f_2 is multiplication-by-n map with $f_5^{-1}(\mathrm{Bran}(f_6)) \subset E_0[n]$.;

(v) The map f_3 is a map onto E_0 branched at the identity element of E_0 with all local ramification indices being divisible by $2L$;

(vi) The curve C_1 is a compositum of C and E_0. By (iv) and (v), f_1 is unramified and points in $f_5^{-1}(\mathrm{Bran}(f_6))$ have local ramification indices $2L$ under f_4;

(vii) The curve C_2 is a compositum of C_1 and E_1. By (vi), f_9 is unramified and the local ramification index of the identity element of E_1 under f_{10} is divisible by L;

(viii) The curve C_3 is a compositum of C_2 and E_1. Clearly f_{11} is unramified and by (vii) the local ramification indices of points in $f_8^{-1}(\mathrm{Bran}(f_{15}))$ under f_{12} are divisible by L;

(ix) The curve C_4 is a compositum of C_3 and C'. From the computation in (viii), we see that f_{13} is unramified.

By (vi)-(ix), we see that C_4 is an unramified cover of C which maps onto C' and consequently we have:

$$C \Rightarrow C'.$$

□

Definition 31. Given a finite set S of points on $\mathbb{P}^1(\bar{\mathbb{Q}})$, we call S can be contracted to (a, b, c, d) if there exist some elliptic curves $E_0 = E(a, b, c, d)$, $E_1, \ldots, E_n$ with:

$$E_0 \rightharpoondown E_1 \rightharpoondown \cdots \rightharpoondown E_n$$

such that S is projectively equilvalent to a subset in $\pi(E_n[\infty])$. Here, π is the standard projection of E_n to $\mathbb{P}^1$.

Theorem 32. *Let C be a curve, p be an odd prime and P be the subset of natural numbers whose prime divisors are less than p. If C can be P-contracted to a finite set of points S which can be contracted to (a, b, c, d) which can be P-contracted to $(0, 1, \infty)$, then there exists $n \in P$ such that*:

$$\mathsf{C}_n \Rightarrow C.$$

Proof. By assumption, there exists $L \in P$ and a map:

$$f : C \to \mathbb{P}^1 \ \text{ with } \ \mathrm{Bran}(f) \subset S$$

such that all local ramification indices divide L.

Also there exists $M \in P$ and a map:

$$g : E(a, b, c, d) = E \to \mathbb{P}^1 \ \text{ with } \ \mathrm{Bran}(g) \subset (0, 1, \infty)$$

such that all local ramification indices of g divide M.

Now let us consider the following diagram:

$$\begin{array}{ccc} \mathsf{C}_{2LM} & \xleftarrow{f_2} & C_1 \\ \big\downarrow{\scriptstyle f_1} & & \big\downarrow{\scriptstyle f_3} \\ \mathbb{P}^1 & \xleftarrow{f_4} & E \end{array}$$

In this diagram:

(i) The map f_1 is the standard degree 2 projection combined with an automorphism of $\mathbb{P}^1$ such that $(0, 1, \infty)$ are contained in the branch locus of f_1;

(ii) The map f_4 is the map g;

(iii) The curve C_1 is a compositum of C_{2LM} and E via f_1 and f_4. From (i) and (ii) we see that f_2 is unramified and f_3 is a map from C_1 onto E such that at least one branch point have all local ramification indices $2L$. By Theorem 30, we are done. □

A direct corollary is:

Corollary 33. *Let C be a curve and P be the subset of natural numbers whose prime divisors lie in $\{2, 3, 5\}$. If C can be P-contracted to a finite set of points S which can be contracted to (a, b, c, d) which can be P-contracted to $(0, 1, \infty)$, then we have:*

$$\mathsf{C}_6 \Rightarrow C.$$

Remark 34. From this corollary, we see that if we want to use elliptic curves to attack the unramified curve correspondence problem, one important thing is the intersection of the image under the standard projection of the torsion points for two different elliptic curves on $\mathbb{P}^1$ as well as the intersection of the image under the standard projection of the torsion points for one elliptic curve and the set of roots of unity on $\mathbb{P}^1$. In general, the intersection number is always finite (see [5]), but we only need to find some special elliptic curves to approach our problem.

4. A possible procedure to approach conjecture 18

From Proposition 17 and our proof of Theorem 2, Theorem 4 and Theorem 32, we propose a possible way to approach Conjecture 18:

Step 0: We already know (by Theorem 2) that if $n \geq 5$ is a positive integer whose only prime divisors are 2, 3 or 5, then we have:

$$\mathsf{C}_6 \Leftrightarrow \mathsf{C}_n.$$

Step I: Start with $p = 7$.

Step II: Let us show that C_p is P-ramified over some points S which can be contracted to (a, b, c, d) which can be P-contracted to $(0, 1, \infty)$ (or more intermediate steps like these) such that all numbers in P only have prime divisors less than p and deduce that:

$$\mathsf{C}_6 \Rightarrow \mathsf{C}_p.$$

Step III: Use the construction in last step (which is a combination of diagrams in Proposition 10, Theorem 30 and Theorem 32) to show:

$$\mathsf{C}_{mn} \Rightarrow \mathsf{C}_{pn}$$

for some m whose prime divisors are less than p and for any $n \geq 1$.

Step IV: Use the result in last step to show:

$$\mathsf{C}_6 \Rightarrow \mathsf{C}_n$$

for all n whose prime divisors are less than or equal to p. By Proposition 5 we can conclude that C_n and C_m are equivalent for any n and m whose prime divisors are less than or equal to p.

Step V: Consider the next prime and go back to Step II.

If eventually we can finish the above procedure for all primes, then by Proposition 17, Conjecture 18 will be true.

Actually the only hard part of the above procedure is Step II. Step III and Step IV can be done in a similar fashion as we did in Proposition 10, Corollary 11 Theorem 30 and Theorem 32.

Proposition 35. *Suppose C_p is P-ramified over some points S which can be contracted to some 4-tuple (a, b, c, d) which can be P-contracted to $(0, 1, \infty)$ such that all numbers in P only have prime divisors less than p. Then there exists some m whose prime divisors are less than p such that for any $n \geq 1$, we have*:

$$\mathsf{C}_{mn} \Rightarrow \mathsf{C}_{pn}.$$

Proposition 36. *Assume C_n and C_m are equilvalent for any n and m whose prime divisors are less than p. Suppose there exists some m whose prime divisors are less than p such that for any $n \geq 1$, $\mathsf{C}_{mn} \Rightarrow \mathsf{C}_{pn}$. Then we have*:

$$\mathsf{C}_6 \Rightarrow \mathsf{C}_n$$

for any n whose prime divisors are less than or equal to p.

Proof. (Sketch) As mentioned, it is similar as the proof of Proposition 10, Corollary 11, Theorem 30 and Theorem 32. For the proof of Proposition 35, we will use a diagram similar as in the proof of Proposition 10. Replace C_5 by C_p and C_{5n} by C_{pn}. The maps f_3 and f_4 are still the projection y composed with an automorphism of $\mathbb{P}^1$ such that both of them have branch points $\{0, 1, \infty\}$. The difference is in Proposition 10, f_2 is a map from C_5 to $\mathbb{P}^1$. Here we do not have such a map. Instead under our assumption, f_2 will be replaced by a diagram which is a combination of the diagrams in Theorem 30 and Theorem 32. Also we can find one desired positive integer m as in the proof of Theorem 32. Now Proposition 35 will be established if we do the similar computation as in Theorem 30 and Theorem 32. For Proposition 36, we can prove it in the same way as the proof of Corollary 11 (Instead of repeating using Proposition 10, this time we repeat using Proposition 35). □

Remark 37. Finally, let us describe a directed graph structure between all hyperbolic curves. We regard each hyperbolic curve as a point in our graph. If C_1 and C_2 are two hyperbolic curves such that C_1 implies C_2, then we associate a directed edge from C_1 to C_2. If they are equilvalent, then we associate a simple edge between C_1 and C_2. In this way, Conjecture 18 can be formulated as: This graph is strongly connected. Even if Conjecture 18 does not hold, it is still interesting to investigate the structure of subsets of coprime number m

and n with different domination areas of C_m over C_n and also modular curves $X(n)$. Propositions 12 and 14 are two examples of this.

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