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VOLUME MEAN OPERATOR AND DIFFERENTIATION RESULTS ASSOCIATED TO ROOT SYSTEMS

Chaabane Rejeb

Abstract. Let R be a root system in $\mathbb{R}^d$ with Coxeter-Weyl group W and let k be a nonnegative multiplicity function on R. The generalized volume mean of a function $f \in L^1_{loc}(\mathbb{R}^d, m_k)$, with m_k the measure given by $dm_k(x) := \omega_k(x)dx := \prod_{\alpha\in R} |\langle \alpha, x\rangle|^{k(\alpha)}dx$, is defined by: $\forall\ x \in \mathbb{R}^d$, $\forall\ r > 0$, $M^r_B(f)(x) := \frac{1}{m_k[B(0,r)]}\int_{\mathbb{R}^d} f(y)h_k(r, x, y)\omega_k(y)dy$, where $h_k(r, x, \cdot)$ is a compactly supported nonnegative explicit measurable function depending on R and k. In this paper, we prove that for almost every $x \in \mathbb{R}^d$, $\lim_{r\to 0} M^r_B(f)(x) = f(x)$.

1. Introduction and statement of the result

We consider the Euclidean space $\mathbb{R}^d$ equipped with a reduced root system R, i.e., R is a finite subset of $\mathbb{R}^d \setminus \{0\}$ such that for any $\alpha \in R$, $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ et $\sigma_\alpha(R) = R$, where σ_α is the reflection with respect to the hyperplane H_α orthogonal to α (see [8] and [10]). Let us denote by W the Coxeter-Weyl group generated by the reflections σ_α, $\alpha \in R$ and by k a multiplicity function defined on R (i.e., W-invariant) which will be supposed nonnegative throughout this paper.

The Dunkl intertwining operator V_k, associated to the pair (R, k), is the topological isomorphism of $\mathcal{C}^\infty(\mathbb{R}^d)$ onto itself given by

$$V_k(f)(x) := \int_{\mathbb{R}^d} f(y)d\mu^k_x(y), \quad x \in \mathbb{R}^d, \tag{1.1}$$

where μ^k_x is a probability measure with compact support contained in the convex hull of $W.x$, the orbit of x under the W-action (see [3], [12], [13] et [16]). This operator satisfies the following intertwining relation $\Delta_k V_k = V_k\Delta$, where Δ is the usual Laplacian and Δ_k is the Dunkl Laplacian acting on C^2-functions

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as follows

$$\Delta_k f(x) := \Delta f(x) + 2\sum_{\alpha\in R_+} k(\alpha)\Big(\frac{\langle\nabla f(x),\alpha\rangle}{\langle x,\alpha\rangle} - \frac{\|\alpha\|^2}{2}\frac{f(x)-f(\sigma_\alpha.x)}{\langle x,\alpha\rangle^2}\Big),$$

with R_+ a fixed positive subsystem of R.

In order to build a potential theory associated to reflection groups, a type volume mean operator has been introduced in [6] (see also [5]) as a crucial tool. In particular, it allowed us to characterize the notion of Δ_k-harmonicity by the volume mean property (see [5] and [6] for more details).

The volume mean operator of $f \in \mathcal{C}(\mathbb{R}^d)$ has the following form

(1.2)
$$\forall\ x\in\mathbb{R}^d,\ \forall\ r>0,\quad M_B^r(f)(x) := \frac{1}{m_k[B(0,r)]}\int_{\mathbb{R}^d} f(y)h_k(r,x,y)\omega_k(y)dy,$$

where m_k is the measure

$$dm_k(x) := \omega_k(x)dx := \prod_{\alpha\in R_+} |\langle\alpha,x\rangle|^{2k(\alpha)}dx \tag{1.3}$$

and h_k is the so-called harmonic kernel (its properties will be recalled in section 2) defined by

(1.4)
$$\forall\ r>0,\quad \forall\ x,y\in\mathbb{R}^d,\quad h_k(r,x,y) = \int_{\mathbb{R}^d} \mathbf{1}_{[0,r]}\big(\sqrt{\|x\|^2+\|y\|^2-2\langle x,z\rangle}\big)d\mu_y^k(z).$$

Note that if k is the zero function, the measure μ_x^k is equal to δ_x the Dirac measure at x, the Dunkl Laplacian coincides with Δ and as $h_k(r,x,y) = \mathbf{1}_{[0,r]}\big(\|x-y\|\big) = \mathbf{1}_{B(x,r)}(y)$, the generalized volume operator is none more than the classical one.

According to [7], for r and x fixed and under the condition $k(\alpha)>0$ for all $\alpha\in R$, we have

$$\text{supp } h_k(r,x,\cdot) = B^W(x,r) := \cup_{g\in W} B(gx,r), \tag{1.5}$$

where $B(\xi,a)$ is the closed Euclidean ball centered at ξ and with radius $a>0$.

When the function k vanishes, the support of the function $h_k(r,x,\cdot)$ is contained in $B^W(x,r)$ (see [6]) and contains $B(x,r)$ (see [7]).

Our aim here is to prove the following result:

Theorem. *Let Ω be a W-invariant open subset of $\mathbb{R}^d$ and let $f\in L^1_{loc}(\Omega,m_k)$. Then for almost every $x\in\Omega$, we have*

$$\lim_{r\downarrow 0^+} M_B^r(f)(x) = f(x). \tag{1.6}$$

Using (1.5), the volume mean of $f\in L^1_{loc}(\Omega,m_k)$ is well defined at (x,r) whenever $B(x,r)\subset\Omega$. Furthermore, by the formula (1.3), we see that negligible sets for the Lebesgue measure coincide with negligible sets for the measure m_k.

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On other hand, a surprising phenomena is that the relation (1.6) means that the function $h_k(r, x, \cdot)$, x is fixed, concentrates only in the neighborhood of x when $r \to 0$ and not on other point gx of the orbit Wx as we could think from (1.5).

2. Some basics from Dunkl analysis

In this section we will recall some results from Dunkl theory which will be useful for the sequel. These concern in particular the Dunkl transform, the Dunkl translation and the harmonic kernel.

2.1. The Dunkl transform and Dunkl's translation operators

• The Dunkl transform of a function $f \in L^1(\mathbb{R}^d, m_k)$ is defined by

$$\mathcal{F}_k(f)(\lambda) := \int_{\mathbb{R}^d} f(x) E_k(-i\lambda, x)\omega_k(x)dx, \quad \lambda \in \mathbb{R}^d, \tag{2.1}$$

where $E_k(x, y) := V_k(e^{\langle x, \cdot\rangle})(y)$, $x, y \in \mathbb{R}^d$, is the Dunkl kernel which is analytically extendable to $\mathbb{C}^d \times \mathbb{C}^d$ and satisfies the following properties (see [2], [4], [9] and [13]): for all $x, y \in \mathbb{R}^d$, all $\lambda \in \mathbb{C}$ and all $g \in W$,

$$E_k(x, y) = E_k(y, x), \quad E_k(x, \lambda y) = E_k(\lambda x, y), \quad E_k(gx, gy) = E_k(x, y). \tag{2.2}$$

Moreover the following inequality holds

$$\forall\ x, y \in \mathbb{R}^d, \quad |E_k(-ix, y)| \leq 1. \tag{2.3}$$

The Dunkl transform shares many properties with the usual Fourier transform (see [9], [13]). In particular, it is an isomorphism of $\mathcal{S}(\mathbb{R}^d)$ (the Schwartz space) onto itself and its inverse is given by

$$\mathcal{F}_k^{-1}(f)(x) = c_k^{-2}\mathcal{F}_k(f)(-x) = c_k^{-2}\int_{\mathbb{R}^d} f(\lambda)E_k(ix, \lambda)\omega_k(\lambda)d\lambda, \quad x \in \mathbb{R}^d,$$

where c_k is the Macdonald-Mehta constant (see [11]) given by

$$c_k := \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}}\omega_k(x)dx. \tag{2.4}$$

Furthermore, the following Plancherel theorem holds: The transformation $c_k^{-1}\mathcal{F}_k$ extends uniquely to an isometric isomorphism of $L^2(\mathbb{R}^d, m_k)$ and we have the Plancherel formula:

$$\forall\ f \in L^2(\mathbb{R}^d, m_k), \quad \|c_k^{-1}\mathcal{F}_k(f)\|_{L^2(\mathbb{R}^d,m_k)} = \|f\|_{L^2(\mathbb{R}^d,m_k)}. \tag{2.5}$$

• The Dunkl translation operators $\tau_x, x \in \mathbb{R}^d$, are defined on $\mathcal{C}^\infty(\mathbb{R}^d)$ by (see [17])

$$\forall\ y \in \mathbb{R}^d, \quad \tau_x f(y) = \int_{\mathbb{R}^d} V_k \circ T_z \circ V_k^{-1}(f)(y)d\mu_x(z), \tag{2.6}$$

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where T_x is the classical translation operator given by $T_xf(y) = f(x+y)$. The Dunkl translation operators have the following properties:

$$\tau_0 f = f, \quad \tau_x f(y) = \tau_y f(x),$$
$$\tau_x(\Delta_k f) = \Delta_k(\tau_x f), \quad \tau_x E_k(\cdot, z)(y) = E_k(x,z)E_k(y,z).$$

Note that if $f \in \mathcal{S}(\mathbb{R}^d)$, the function $\tau_x f \in \mathcal{S}(\mathbb{R}^d)$ and it can be defined by means of the Dunkl transform as follows (see [17]):

$$\begin{aligned} \tau_x f(y) &= \mathcal{F}_k^{-1}[E_k(ix,\cdot)\mathcal{F}_k(f)](y) \\ &= c_k^{-2}\int_{\mathbb{R}^d} \mathcal{F}_k(f)(\lambda)E_k(ix,\lambda)E_k(iy,\lambda)\omega_k(\lambda)d\lambda, \quad y \in \mathbb{R}^d. \end{aligned} \tag{2.7}$$

For $x \in \mathbb{R}^d$, the operator τ_x can be extended to the space $L^2(\mathbb{R}^d, m_k)$ as follows: Fix $f \in L^2(\mathbb{R}^d, m_k)$. Since $|E_k(-ix,\xi)| \leq 1$, the function $\xi \mapsto E_k(ix,\xi)\mathcal{F}_k(f)(\xi)$ belongs to $L^2(\mathbb{R}^d, m_k)$. Hence, by Plancherel theorem, there exists a unique $L^2(\mathbb{R}^d, m_k)$-function denoted by $\tau_x f$ and called the x-Dunkl translate function of f such that

$$\mathcal{F}_k(\tau_x f)(\xi) = E_k(ix,\xi)\mathcal{F}_k(f)(\xi). \tag{2.8}$$

For more properties on Dunkl translation operators when they act on $L^2(\mathbb{R}^d, m_k)$ we can see ([15]).

2.2. The harmonic kernel

In this section we recall some results of [6].

Let $(r,x,y) \mapsto h_k(r,x,y)$ be the harmonic kernel defined by (1.4). It satisfies the following properties (see [6]):

(1) For all $r > 0$ and $x, y \in \mathbb{R}^d$, $0 \leq h_k(r,x,y) \leq 1$.
(2) For all fixed $x, y \in \mathbb{R}^d$, the function $r \mapsto h_k(r,x,y)$ is right-continuous and non decreasing on $]0,+\infty[$.
(3) For all $r > 0$, $x, y \in \mathbb{R}^d$ and $g \in W$, we have

$$h_k(r,x,y) = h_k(r,y,x) \quad \text{and} \quad h_k(r,gx,y) = h_k(r,x,g^{-1}y). \tag{2.9}$$

(4) For all $r > 0$ and $x \in \mathbb{R}^d$, we have

$$\|h_k(r,x,\cdot)\|_{k,1} := \int_{\mathbb{R}^d} h_k(r,x,y)\omega_k(y)dy = m_k(B(0,r)) = \frac{d_k r^{d+2\gamma}}{d+2\gamma}, \tag{2.10}$$

where d_k is the constant

$$d_k := \int_{S^{d-1}} \omega_k(\xi)d\sigma(\xi) = \frac{c_k}{2^{d/2+\gamma-1}\Gamma(d/2+\gamma)}.$$

Here $d\sigma(\xi)$ is the surface measure of the unit sphere S^{d-1} of $\mathbb{R}^d$ and c_k is the Macdonald-Mehta constant (2.4).

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(5) The harmonic kernel satisfies the following geometric inequality: if $\|a-b\| \leq 2r$ with $r>0$, then

$$\forall\ \xi \in \mathbb{R}^d, \quad h_k(r,a,\xi) \leq h_k(4r,b,\xi)$$

(see [6], Lemma 4.1). In the classical case (i.e., $k=0$), this inequality says that if $\|a-b\| \leq 2r$, then $B(a,r) \subset B(b,4r)$.

(6) Let $r>0$ and $x \in \mathbb{R}^d$. Then the function $h_k(r,x,\cdot)$ is upper semi-continuous on $\mathbb{R}^d$.

Note that to prove the upper semi-continuity of $h_k(r,x,\cdot)$, with $r>0$ and $x \in \mathbb{R}^d$, the authors of [6] have constructed a decreasing sequence $(\varphi_\varepsilon) \subset \mathcal{D}(\mathbb{R}^d)$ of radial functions such that for every $\varepsilon>0$, $0 \leq \varphi_\varepsilon \leq 1$, $\varphi_\varepsilon = 1$ on $B(0,r)$ and for all $y \in \mathbb{R}$

$$\lim_{\varepsilon \to 0} \varphi_\varepsilon(y) = \mathbf{1}_{B(0,r)}(y), \quad h_k(r,x,y) = \lim_{\varepsilon \to 0} \tau_{-x}\varphi_\varepsilon(y), \tag{2.11}$$

where τ_x is the x-Dunkl translation operator.

In order to prove our result, we will need two lemmata. But, at first, we start by the following remark:

Remark 2.1. Let $r>0$. The function $\mathbf{1}_{B(0,r)}$ is in $L^2(\mathbb{R}^d,m_k)$. For $x \in \mathbb{R}^d$, we can then define $\tau_{-x}(\mathbf{1}_{B(0,r)})$ as being the $L^2(\mathbb{R}^d,m_k)$-function whose Dunkl transform is equal to

$$\mathcal{F}_k\left(\tau_{-x}(\mathbf{1}_{B(0,r)})\right)(\xi) = E_k(-ix,\xi)\mathcal{F}_k\left(\mathbf{1}_{B(0,r)}\right)(\xi) \tag{2.12}$$

(see (2.8)). This $L^2(\mathbb{R}^d,m_k)$-function (which coincides also with $\mathbf{1}_{B(x,r)}$ when $k=0$) has been used formally in ([15] and [1]) for studying the $L^p(\mathbb{R}^d,m_k)$-boundedness of the Dunkl-Hardy-Littlewood maximal operator. In the next result, we will show that this function coincides almost everywhere with $h_k(r,x,\cdot)$. But, in contrast to the harmonic kernel, the L^2-definition (2.12) of the function $\tau_{-x}(\mathbf{1}_{B(0,r)})$ does not give any precision neither on its support nor on some geometric properties like the geometric inequality mentioned above.

Lemma 2.1. *Let $r>0$ and $x \in \mathbb{R}^d$. Then, for almost every $y \in \mathbb{R}^d$, we have*

$$h_k(r,x,y) = \tau_{-x}(\mathbf{1}_{B(0,r)})(y). \tag{2.13}$$

Proof. We consider the sequence (φ_ε) as in (2.11). By the monotone convergence theorem, we can see that $\tau_{-x}\varphi_\varepsilon \to h_k(r,x,\cdot)$ in $L^2(\mathbb{R}^d,m_k)$.

On the other hand, since $\mathbf{1}_{B(0,r)} \in L^2_k(\mathbb{R}^d)$, we have

$$\begin{aligned}
&\left\|\tau_{-x}\varphi_\varepsilon - \tau_{-x}(\mathbf{1}_{B(0,r)})\right\|_{L^2_k(\mathbb{R}^d)} \\
&= c_k^{-1}\left\|\mathcal{F}_k\left[\tau_{-x}\varphi_\varepsilon\right] - \mathcal{F}_k\left[\tau_{-x}(\mathbf{1}_{B(0,r)})\right]\right\|_{L^2_k(\mathbb{R}^d)} \\
&= c_k^{-1}\left\|E_k(-ix,\cdot)\mathcal{F}_k\left[\varphi_\varepsilon\right] - E_k(-ix,\cdot)\mathcal{F}_k\left[\mathbf{1}_{B(0,r)}\right]\right\|_{L^2_k(\mathbb{R}^d)} \\
&\leq c_k^{-1}\left\|\mathcal{F}_k\left[\varphi_\varepsilon\right] - \mathcal{F}_k\left[\mathbf{1}_{B(0,r)}\right]\right\|_{L^2_k(\mathbb{R}^d)} \\
&= \left\|\varphi_\varepsilon - \mathbf{1}_{B(0,r)}\right\|_{L^2_k(\mathbb{R}^d)} \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0,
\end{aligned}$$

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where we have used Plancherel formula (2.5) for Dunkl's transform in the first and the last lines, the relations (2.7) and (2.12) in the second line, the inequality $|E_k(-ix,\xi)| \le 1$ in the third line and the monotone convergence theorem in the last line.

Thus, we get $(\tau_{-x}\varphi_\varepsilon)$ converges also to $\tau_{-x}(\mathbf{1}_{B(0,r)})$ in $L^2(\mathbb{R}^d, m_k)$. This proves the desired equality. □

Our second lemma, proved in [6], says that:

Lemma 2.2. *Let $x \in \mathbb{R}^d$. Then the family of probability measures*

$$d\eta^k_{x,r}(y) = \frac{1}{m_k[B(0,r)]} h_k(r,x,y)\omega_k(y)dy$$

is an approximation of the Dirac measure δ_x as $r \to 0$. More precisely

1. *For all $\alpha > 0$, $\lim_{r\to 0} \int_{\|x-y\|>\alpha} d\eta^k_{x,r}(y) = 0$.*
2. *Let $\Omega \subset \mathbb{R}^d$ be a W-invariant open set, f a locally bounded measurable function defined on Ω and $x \in \Omega$. If f is continuous at x, then*

$$\lim_{r\to 0} \int_{\mathbb{R}^d} f(y) d\eta^k_{x,r} = \lim_{r\to 0} M^r_B(f)(x) = f(x). \tag{2.14}$$

3. Proof of the result

Firstly, we will show the result when the W-invariant open set Ω is the whole space $\mathbb{R}^d$.

Theorem 3.1. *Let $f \in L^1_{loc}(\mathbb{R}^d, m_k)$. Then, for almost every $x \in \mathbb{R}^d$, the relation* (1.6) *holds.*

To prove this theorem, we will use the weak-$L^1(\mathbb{R}^d, m_k)$ estimates of the Dunkl-Hardy-Littlewood maximal function (see [1] and [15]). This idea has been taken from the classical case (see [14]).

Proof. `Step 1:` Suppose that f is a continuous function on $\mathbb{R}^d$. In this case, the result follows immediately from the relation (2.14).

`Step 2:` We will prove the result when $f \in L^1(\mathbb{R}^d, m_k)$. To do this, it suffices to show that

$$f^*(x) := \limsup_{r\to 0} M^r_B(|f - f(x)|)(x) = 0$$

for almost every $x \in \mathbb{R}^d$.

• At first, we claim that there exists a constant $C > 0$ such that

$$\forall \ t > 0, \ \ m_k\{f^* > t\} := m_k\{x \in \mathbb{R}^d, \ \ f^*(x) > t\} \le \frac{C}{t}\|f\|_{L^1_k(\mathbb{R}^d)}. \tag{3.1}$$

Indeed, we have

$$f^*(x) \le \sup_{r>0} M^r_B(|f - f(x)|)(x) \le M_k(|f|)(x) + |f(x)|, \tag{3.2}$$

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where $M_k(g)$ is the maximal function of $g \in L^1(\mathbb{R}^d, m_k)$ defined by

$$M_k(g)(x) := \sup_{r>0} \frac{1}{m_k\left(B(0,r)\right)} \left| \int_{\mathbb{R}^d} g(y)\tau_{-x}\left(1_{B(0,r)}\right)(y)\omega_k(y)dy \right|$$

(see [1] and [15]). We notice that from (2.13) and (1.2), we have

$$M_k(|f|)(x) = \sup_{r>0} M_B^r(|f|)(x),$$

which justifies (3.2). Consequently,

$$\{f^* > t\} \subset \{M_k(|f|) + |f| > t\} \subset \{M_k(|f|) > t/2\} \cup \{|f| > t/2\}.$$

This implies that

$$m_k\{f^* > t\} \le m_k\{M_k(|f|) > t/2\} + m_k\{|f| > t/2\}. \tag{3.3}$$

From ([15] or [1]), there exists a constant $C_1 > 0$ such that

$$m_k\{M_k(|f|) > t/2\} \le \frac{2C_1}{t}\|f\|_{L^1(\mathbb{R}^d, m_k)} \tag{3.4}$$

and from Markov's inequality, we have

$$m_k\{|f| > t/2\} \le \frac{2}{t}\|f\|_{L^1(\mathbb{R}^d, m_k)}. \tag{3.5}$$

Then we deduce (3.1) from (3.3), (3.4) and (3.5) with $C = 2C_1 + 2$.

• Let $\varepsilon > 0$ and let $g \in \mathcal{D}(\mathbb{R}^d)$ such that $\|f - g\|_{L^1(\mathbb{R}^d, m_k)} \le \varepsilon$. For every $x \in \mathbb{R}^d$, Step 1 applied to the function $y \mapsto |g(y) - g(x)|$ shows that $g^*(x) = 0$. This implies that $(f - g)^* \le f^* + g^* = f^*$. Since $f^* = (f - g + g)^* \le (f - g)^* + g^* = (f - g)^*$, we get $f^* = (f - g)^*$. Consequently, by (3.1) we obtain

$$m_k\{f^* > t\} = m_k\{(f - g)^* > t\} \le \frac{C}{t}\|f - g\|_{L^1(\mathbb{R}^d, m_k)} = \frac{C}{t}\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, this proves that

$$\forall\, t > 0, \quad m_k\{f^* > t\} = 0.$$

Finally, since

$$\{f^* > 0\} = \cup_{n \ge 1}\{f^* > 1/n\},$$

we deduce that $m_k\{f^* > 0\} = 0$. That is $f^* = 0$ a.e. as desired.

`Step 3:` Let $f \in L^1_{loc}(\mathbb{R}^d, m_k)$. For every $n \in \mathbb{N} \setminus \{0\}$, the function $f_n = f\mathbf{1}_{B(0,n)}$ is in $L^1(\mathbb{R}^d, m_k)$. By Step 2, we have ${f_n}^*(x) = 0$ for all $x \in \mathbb{R}^d \setminus E_n$, where E_n is a measurable set such that $m_k(E_n) = 0$.

We will prove that $\{f^* > 0\} \subset \cup_{n \ge 1} E_n$ which will imply the desired result.

Let $x \in \mathbb{R}^d$ such that $f^*(x) > 0$. There is an integer $n = n_x \in \mathbb{N} \setminus \{0\}$ such that supp $h_k(r, x, \cdot) \subset B(0, n)$ for every $r \le 1$. This implies that $f^*(x) = {f_n}^*(x) > 0$. That is $x \in E_n$. This completes the proof. □

Now, we will prove our result that we recall below:

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Corollary 3.1. *Let $\Omega \subset \mathbb{R}^d$ be a W-invariant open set. If $f \in L^1_{loc}(\Omega, m_k)$, then (1.6) holds for almost every $x \in \Omega$.*

Proof. Let $n \in \mathbb{N}$ large enough such that

$$\Omega_{\frac{1}{n}} := \{x \in \Omega,\ dist(x, \partial\Omega) > 1/n\} = \{x \in \Omega,\ B(x, 1/n) \subset \Omega\}$$

is a nonempty set. For such n, we consider $O_n := B_o(0, n) \cap \Omega_{\frac{1}{n}}$ and $K_n = \overline{O_n}$, where $B_o(a, r)$ is the open ball centered at $a \in \mathbb{R}^d$ and with radius $r > 0$.

As $\Omega_{\frac{1}{n}}$ is W-invariant, we can see that O_n (resp. K_n) is a W-invariant open (resp. W-invariant compact) subset of Ω. Moreover, we have for every n large enough

$$K_n \subset O_{n+1} \subset K_{n+1} \quad \text{and} \quad \Omega = \cup_n K_n = \cup_n O_n.$$

Now, let f_n be the function given by $f_n(x) = f(x)$ if $x \in K_n$ and $f_n(x) = 0$ if $x \in \mathbb{R}^d \backslash K_n$. Clearly f_n belongs to $L^1_{loc}(\mathbb{R}^d, m_k)$ and by Theorem 3.1 we have $f_n(x) = \lim_{r\to 0} M^r_B(f_n)(x)$ for almost every $x \in \mathbb{R}^d$.

Let

$$E_n := \Big\{x \in \mathbb{R}^d,\ f_n(x) \neq \lim_{r\to 0} M^r_B(f_n)(x)\Big\} \quad \text{and}$$
$$E := \Big\{x \in \Omega,\ f(x) \neq \lim_{r\to 0} M^r_B(f)(x)\Big\}.$$

Since f_n is continuous on the open set $\mathbb{R}^d \backslash K_n$, by (2.14) we deduce that $E_n \subset K_n \subset \Omega$. Let us now take $x \in E$. There exist $R > 0$ and $N \in \mathbb{N}$ such that $B(x, R) \subset O_N \subset K_{N+1} \subset \Omega$. We will show that $x \in E_{N+1}$. As O_N and K_{N+1} are invariant under the action of the Coxeter-Weyl group W and thanks to the support property of the function $h_k(r, x, \cdot)$ we have

$$\forall\ r \in]0, R], \quad \text{supp} h_k(r, x, \cdot) \subset O_N \subset K_{N+1}.$$

But $f = f_{N+1}$ on O_N. Therefore, if $x \notin E_{N+1}$, i.e.,

$$f_{N+1}(x) = \lim_{r\to 0} M^r_B(f_{N+1})(x),$$

then $f(x) = \lim_{r\to 0} M^r_B(f)(x)$ and $x \notin E$, a contradiction.

Thus $x \in E_{N+1}$. This proves that $E \subset \cup_n E_n$ and E is a negligible set as desired. □

By taking f the characteristic function of a measurable set $E \subset \mathbb{R}^d$, we obtain

Corollary 3.2. *Let $E \subset \mathbb{R}^d$ be a measurable set and $x \in \mathbb{R}^d$. Then for almost every $x \in E$ we have*

$$\frac{\|h_k(r, x, \cdot)\|_{L^1(E, m_k)}}{\|h_k(r, x, \cdot)\|_{L^1(\mathbb{R}^d, m_k)}} \longrightarrow 1 \quad as \quad r \to 0.$$

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When $k = 0$, the previous result takes the following form: for almost every $x \in E$ we have

$$\frac{m_0(E \cap B(x,r))}{m_0(B(x,r))} \longrightarrow 1 \quad \text{as} \quad r \to 0.$$

where m_0 is the Lebesgue measure on $\mathbb{R}^d$. In this case, the point x is called of Lebesgue density of E (see [14]).

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Chaabane Rejeb
Université de Tunis El Manar
Faculté des Sciences de Tunis
Laboratoire d'Analyse Mathématiques et Applications LR11ES11
2092 El Manar I, Tunis, Tunisie
and
Laboratoire de Mathématiques et Physique Théorique CNRS-UMR 7350
Université de Tours
Campus de Grandmont, 37200 Tours, France
E-mail address: chaabane.rejeb@gmail.com

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