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ON SPLIT REGULAR δ-HOM-JORDAN-LIE ALGEBRAS

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Abstract. We introduce the class of split regular δ-Hom-Jordan-Lie algebras as the natural generalization of split regular δ-Jordan-Lie algebras. By developing techniques of connections of roots for this kind of algebras, we show that such a split regular δ-Hom-Jordan-Lie L is of the form $L = U + \sum_{[j]\in\Lambda/\sim} I_{[j]}$ with U a subspace of the abelian subalgebra H and any $I_{[j]}$, a well described ideal of L, satisfying $[I_{[j]}, I_{[k]}] = 0$ if $[j] \neq [k]$. Under certain conditions, in the case of L being of maximal length, the simplicity of the algebra is characterized.

1. Introduction

A Hom-algebra is an algebra in which the multiplication is twisted by a linear homomorphism. The notion of Hom-Lie algebras was introduced by Hartwig, Larsson and Silvestrov to describe the q-deformation of the Witt and the Virasoro algebras [11]. Since then, many authors have studied Hom-type algebras [1, 2, 4, 10, 13, 15]. The notion of δ-Jordan-Lie algebras was introduced in [14], which is intimately related to both Lie and Jordan superalgebras.

Recently, the definition of δ-Jordan Lie triple systems and the definition of δ-Hom-Jordan-Lie algebras were introduced in [12, 16], and their representations and T^*-extensions were studied in detail.

As is well-known, the class of the split algebras is specially related to addition quantum numbers, graded contractions, and deformations. For instance, for a physical system which displays a symmetry of L, it is interesting to know in detail the structure of the split decomposition because its roots can be seen as certain eigenvalues which are the additive quantum numbers characterizing the state of such system. Determining the structure of split algebras will become more and more meaningful in the area of research in mathematical physics. Recently, in [3, 5–9], the structure of arbitrary split Lie algebras, arbitrary split

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Leibniz algebras, arbitrary split Lie triple systems, arbitrary split Leibniz triple systems and arbitrary split regular Hom-Lie algebras have been determined by the techniques of connections of roots. The purpose of this paper is to consider the structure of split regular δ-Hom-Jordan-Lie algebras by the techniques of connections of roots based on some work in [3, 5].

Throughout this paper, split regular δ-Hom-Jordan-Lie algebras L are considered of arbitrary dimension and over an arbitrary base field $\mathbb{K}$. This paper is organized as follows. In Section 2, we establish the preliminaries on split regular δ-Hom-Jordan-Lie algebras theory. In Section 3, we develop techniques of connections of roots for δ-Hom-Jordan-Lie algebras. In Section 4, we show that such an arbitrary regular δ-Hom-Jordan-Lie algebras L with a symmetric root system is of the form $L = U + \sum_{[j]\in\Lambda/\sim} I_{[j]}$ with U a subspace of the abelian subalgebra H and any $I_{[j]}$ a well described ideal of L, satisfying $[I_{[j]}, I_{[k]}] = 0$ if $[j] \neq [k]$. In Section 5, we show that under certain conditions, in the case of L being of maximal length, the simplicity of the algebra is characterized.

2. Preliminaries

First we recall the definitions of δ-Jordan-Lie algebras and δ-Hom-Jordan-Lie algebras.

Definition 2.1 ([14])**.** A *δ-Jordan-Lie algebra* L is a vector space over a field $\mathbb{K}$ endowed with a bilinear map $[\cdot,\cdot] : L \times L \to L$ satisfying

$$[x, y] = -\delta[y, x],\ \delta = \pm 1,$$
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0,\ \forall x, y, z \in L.$$

Definition 2.2 ([16])**.** A *δ-Hom-Jordan-Lie algebra* L is a triple $(L, [\cdot,\cdot]_L, \phi)$ consisting of a vector space L, a bilinear map $[\cdot,\cdot]_L : L \times L \to L$ and a map ϕ: $L \to L$ satisfying

$$[x, y] = -\delta[y, x],\ \delta = \pm 1,$$
$$[\phi(x), [y, z]] + [\phi(y), [z, x]] + [\phi(z), [x, y]] = 0,\ \forall x, y, z \in L.$$

When ϕ furthermore is an algebra automorphism it is said that L is a *regular δ-Hom-Jordan-Lie algebra*.

Especially, for $\delta = 1$ one has a Hom-Lie algebra and for $\delta = -1$ a Hom-Jordan-Lie algebra.

Throughout this paper we will consider regular δ-Hom-Jordan-Lie algebras L being of arbitrary dimension and arbitrary base field $\mathbb{K}$. $\mathbb{N}$ denotes the set of all non-negative integers and $\mathbb{Z}$ denotes the set of all integers.

A subalgebra A of L is a linear subspace such that $[A, A] \subset A$ and $\phi(A) = A$. A linear subspace I of L is called an ideal if $[I, L] \subset I$ and $\phi(I) = I$.

Let us introduce the class of split algebras in the framework of regular δ-Hom-Jordan-Lie algebras. Denote by H a maximal abelian subalgebra of a

δ-Hom-Jordan-Lie algebra L. For a linear functional

$$\alpha : H \to \mathbb{K},$$

we define the root space of L (with respect to H) associated to α as the subspace

$$L_\alpha = \{v_\alpha \in L : [h, v_\alpha] = \alpha(h)\phi(v_\alpha) \text{ for any } h \in H\}.$$

The elements $\alpha : H \to \mathbb{K}$ satisfying $L_\alpha \neq 0$ are called roots of L with respect to H. We set $\Lambda := \{\alpha \in H^* \setminus \{0\} : L_\alpha \neq 0\}$.

Definition 2.3. We say that L is a *split regular δ-Hom-Jordan-Lie algebra*, with respect to H, if

$$L = H \oplus (\oplus_{\alpha\in\Lambda} L_\alpha).$$

We also say that Λ is the roots system of L.

For convenience, the mappings $\phi|_H$, $\phi|_H^{-1} : H \to H$ will be denoted by ϕ and ϕ^{-1} respectively.

Lemma 2.4. *For any α, $\beta \in \Lambda \cup \{0\}$, the following assertions hold.*
1. *$\phi(L_\alpha) \subset L_{\alpha\phi^{-1}}$ and $\phi^{-1}(L_\alpha) \subset L_{\alpha\phi}$.*
2. *$[L_\alpha, L_\beta] \subset L_{\delta(\alpha\phi^{-1}+\beta\phi^{-1})}$, $\delta = \pm 1$.*

Proof. 1. For $h \in H$ write $h' = \phi(h)$. Then for all $h \in H$ and $v_\alpha \in L_\alpha$, since $[h, v_\alpha] = \alpha(h)\phi(v_\alpha)$, one has

$$[h', \phi(v_\alpha)] = \phi([h, v_\alpha]) = \alpha(h)\phi(\phi(v_\alpha)) = \alpha\phi^{-1}(h')\phi(\phi(v_\alpha)).$$

Therefore we get $\phi(v_\alpha) \in L_{\alpha\phi^{-1}}$ and so $\phi(L_\alpha) \subset L_{\alpha\phi^{-1}}$. In a similar way, one gets $\phi^{-1}(L_\alpha) \subset L_{\alpha\phi}$.

2. For any $h \in H$, $v_\alpha \in L_\alpha$ and $v_\beta \in L_\beta$, by denoting $h' = \phi(h)$, we have that

$$\begin{aligned}
[h', [v_\alpha, v_\beta]] &= [\phi(h), [v_\alpha, v_\beta]] = \delta[[h, v_\alpha], \phi(v_\beta)] + \delta[\phi(v_\alpha), [h, v_\beta]] \\
&= \delta[\alpha(h)\phi(v_\alpha), \phi(v_\beta)] + \delta[\phi(v_\alpha), \beta(h)\phi(v_\beta)] \\
&= \delta(\alpha + \beta)(h)\phi([v_\alpha, v_\beta]) \\
&= \delta(\alpha + \beta)\phi^{-1}(h')\phi([v_\alpha, v_\beta]).
\end{aligned}$$

Therefore we get $[v_\alpha, v_\beta] \in L_{\delta(\alpha\phi^{-1}+\beta\phi^{-1})}$ and so $[L_\alpha, L_\beta] \subset L_{\delta(\alpha\phi^{-1}+\beta\phi^{-1})}$. □

Lemma 2.5. *The following assertions hold.*
1. *If $\alpha \in \Lambda$, then $\alpha\phi^{-z} \in \Lambda$ for any $z \in \mathbb{Z}$.*
2. *$L_0 = H$.*

Proof. 1. It is a consequence of Lemma 2.4-1.

2. It is clear that the root space associated to the zero root satisfies $H \subset L_0$. Conversely, given any $v_0 \in L_0$ we can write

$$v_0 = h \oplus (\oplus_{i=1}^n v_{\alpha_i}),$$

where $h \in H$ and $v_{\alpha_i} \in L_{\alpha_i}$ for $i = 1, \dots, n$, with $\alpha_i \neq \alpha_j$ if $i \neq j$. Hence

$$0 = [h^{'}, h \oplus (\oplus_{i=1}^{n} v_{\alpha_i})] = \oplus_{i=1}^{n} \alpha_i(h^{'})\phi(v_{\alpha_i})$$

for any $h^{'} \in H$. Hence Lemma 2.4-1 and the fact $\alpha_i \neq 0$ give us $v_{\alpha_i} = 0$ for $i = 1, \dots, n$. So $v_0 = h \in H$. □

Definition 2.6. A root system Λ of a split δ-Hom-Jordan-Lie algebra is called *symmetric* if it satisfies that $\alpha \in \Lambda$ implies $-\alpha \in \Lambda$.

3. Connections of roots

In the following, L denotes a split regular δ-Hom-Jordan-Lie algebra with a symmetric root system Λ and $L = H \oplus (\oplus_{\alpha \in \Lambda} L_\alpha)$ the corresponding root decomposition. Given a linear functional $\alpha : H \to \mathbb{K}$, we denote by $-\alpha : H \to \mathbb{K}$ the element in H^* defined by $(-\alpha)(h) := -\alpha(h)$ for all $h \in H$. We begin by developing the techniques of connections of roots in this section.

Definition 3.1. Let α and β be two nonzero roots. We shall say that α is *connected* to β if there exist $\alpha_1, \dots, \alpha_k \in \Lambda$ such that

If $k = 1$, then

1. $\alpha_1 \in \{\alpha\phi^{-n} : n \in \mathbb{N}\} \cap \{\pm\beta\phi^{-m} : m \in \mathbb{N}\}$.

If $k \geq 2$, then

1. $\alpha_1 \in \{\alpha\phi^{-n} : n \in \mathbb{N}\}$.
2. $\delta\alpha_1\phi^{-1} + \delta\alpha_2\phi^{-1} \in \Lambda$,
   $\delta^2\alpha_1\phi^{-2} + \delta^2\alpha_2\phi^{-2} + \delta\alpha_3\phi^{-1} \in \Lambda$,
   $\delta^3\alpha_1\phi^{-3} + \delta^3\alpha_2\phi^{-3} + \delta^2\alpha_3\phi^{-2} + \delta\alpha_4\phi^{-1} \in \Lambda$,
   ....
   $\delta^i\alpha_1\phi^{-i} + \delta^i\alpha_2\phi^{-i} + \delta^{i-1}\alpha_3\phi^{-i+1} + \cdots + \delta\alpha_{i+1}\phi^{-1} \in \Lambda$,
   ....
   $\delta^{k-2}\alpha_1\phi^{-k+2} + \delta^{k-2}\alpha_2\phi^{-k+2} + \delta^{k-3}\alpha_3\phi^{-k+3} + \cdots + \delta^{k-i}\alpha_i\phi^{-k+i} + \cdots$
   $+ \delta\alpha_{k-1}\phi^{-1} \in \Lambda$.
3. $\delta^{k-1}\alpha_1\phi^{-k+1} + \delta^{k-1}\alpha_2\phi^{-k+1} + \delta^{k-2}\alpha_3\phi^{-k+2} + \cdots + \delta^{k-i+1}\alpha_i\phi^{-k+i-1}$
   $+ \cdots + \delta\alpha_k\phi^{-1} \in \{\pm\beta\phi^{-m} : m \in \mathbb{N}\}$.

We shall also say that $\{\alpha_1, \dots, \alpha_k\}$ is a connection from α to β.

Observe that the case $k = 1$ in Definition 3.1 is equivalent to the fact $\beta = \epsilon\alpha\phi^z$ for some $z \in \mathbb{Z}$ and $\epsilon \in \{\pm 1\}$.

Lemma 3.2. *For any $\alpha \in \Lambda$, we have that $\alpha\phi^{z_1}$ is connected to $\alpha\phi^{z_2}$ for every $z_1, z_2 \in \mathbb{Z}$. We also have that $\alpha\phi^{z_1}$ is connected to $-\alpha\phi^{z_2}$ in case $-\alpha\phi^{z_2} \in \Lambda$.*

Proof. By Lemma 2.5-1 we have $\alpha\phi^{z_1}$, $\alpha\phi^{z_2} \in \Lambda$. Set $z = \min\{z_1, z_2\}$. Then $\{\alpha\phi^z\}$ is a connection from $\alpha\phi^{z_1}$ to $\alpha\phi^{z_2}$ and to $-\alpha\phi^{z_2}$ in case $-\alpha\phi^{z_2} \in \Lambda$. □

Lemma 3.3. *Let $\{\alpha_1, \dots, \alpha_k\}$ be a connection from α to β. Then the following assertions hold.*

1. *Suppose $\alpha_1 = \alpha\phi^{-n}$, $n \in \mathbb{N}$. Then for any $r \in \mathbb{N}$ such that $r \geq n$, there exists a connection $\{\overline{\alpha}_1, \dots, \overline{\alpha}_k\}$ from α to β such that $\overline{\alpha}_1 = \alpha\phi^{-r}$.*

2. *Suppose that* $\alpha_1 = \epsilon\beta\phi^{-m}$ *in case* $k = 1$ *or*

$$\delta^{k-1}\alpha_1\phi^{-k+1} + \delta^{k-1}\alpha_2\phi^{-k+1} + \delta^{k-2}\alpha_3\phi^{-k+2} + \cdots + \delta\alpha_k\phi^{-1} = \epsilon\beta\phi^{-m}$$

in case $k \geq 2$*, with* $m \in \mathbb{N}$ *and* $\epsilon \in \{\pm 1\}$*. Then for any* $r \in \mathbb{N}$ *such that* $r \geq m$*, there exists a connection* $\{\overline{\alpha}_1, \dots, \overline{\alpha}_k\}$ *from* α *to* β *such that* $\overline{\alpha}_1 = \epsilon\beta\phi^{-r}$ *in case* $k = 1$ *or*

$$\delta^{k-1}\overline{\alpha}_1\phi^{-k+1} + \delta^{k-1}\overline{\alpha}_2\phi^{-k+1} + \delta^{k-2}\overline{\alpha}_3\phi^{-k+2} + \cdots + \delta\overline{\alpha}_k\phi^{-1} = \epsilon\beta\phi^{-r}$$

in case $k \geq 2$.

Proof. 1. By Lemma 2.5-1 we have $\{\alpha_1\phi^{n-r}, \dots, \alpha_k\phi^{n-r}\} \subset \Lambda$. Define $\overline{\alpha}_i := \alpha_i\phi^{n-r}$, $i = 1, \dots, k$. Then Lemma 2.5-1 allows us to verify that $\{\overline{\alpha}_1, \dots, \overline{\alpha}_k\}$ is a connection from α to β. This connection clearly satisfies $\overline{\alpha}_1 = (\alpha\phi^{-n})\phi^{n-r} = \alpha\phi^{-r}$.

2. Lemma 2.5-1 allows us to assert that $\{\alpha_1\phi^{m-r}, \dots, \alpha_k\phi^{m-r}\} \subset \Lambda$. Define now $\overline{\alpha}_i := \alpha_i\phi^{m-r}$, $i = 1, \dots, k$. Then Lemma 2.5-1 gives us that $\{\overline{\alpha}_1, \dots, \overline{\alpha}_k\}$ is a connection from α to β. It is clear that $\overline{\alpha}_1 = \epsilon\beta\phi^{-r}$ in case $k = 1$, and also

$$\begin{aligned}
&\delta^{k-1}\overline{\alpha}_1\phi^{-k+1} + \delta^{k-1}\overline{\alpha}_2\phi^{-k+1} + \delta^{k-2}\overline{\alpha}_3\phi^{-k+2} + \cdots + \delta\overline{\alpha}_k\phi^{-1} \\
&= (\delta^{k-1}\alpha_1\phi^{-k+1} + \delta^{k-1}\alpha_2\phi^{-k+1} + \delta^{k-2}\alpha_3\phi^{-k+2} + \cdots + \delta\alpha_k\phi^{-1})\phi^{m-r} \\
&= \epsilon\beta\phi^{-r}
\end{aligned}$$

in case $k \geq 2$. □

Proposition 3.4. *The relation* $\sim$ *in* Λ*, defined by* $\alpha \sim \beta$ *if and only if* α *is connected to* β*, is of equivalence.*

Proof. Lemma 3.2 gives us $\alpha \sim \alpha$ for any $\alpha \in \Lambda$. That is, the relation $\sim$ is reflexive.

To verify the symmetric character of $\sim$, suppose $\alpha \sim \beta$. Then there exists a connection

$$\{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{k-1}, \alpha_k\} \subset \Lambda \tag{3.1}$$

from α to β.

If $k = 1$, then $\alpha_1 = \alpha\phi^{-n}$ and $\alpha_1 = \epsilon\beta\phi^{-m}$ with $n, m \in \mathbb{N}$ and $\epsilon \in \{\pm 1\}$. Hence, $\{\epsilon\alpha_1\}$ is a connection from β to α.

If $k \geq 2$, observe that condition 3 in Definition 3.1 let us distinguish two possibilities. In the first one

$$\begin{aligned}
&\delta^{k-1}\alpha_1\phi^{-k+1} + \delta^{k-1}\alpha_2\phi^{-k+1} + \delta^{k-2}\alpha_3\phi^{-k+2} + \cdots \\
&+ \delta^i\alpha_{k-i+1}\phi^{-i} + \cdots + \delta\alpha_k\phi^{-1} = \beta\phi^{-m}
\end{aligned} \tag{3.2}$$

while in the second one

$$\begin{aligned}
&\delta^{k-1}\alpha_1\phi^{-k+1} + \delta^{k-1}\alpha_2\phi^{-k+1} + \cdots \\
&+ \delta^i\alpha_{k-i+1}\phi^{-i} + \cdots + \delta\alpha_k\phi^{-1} = -\beta\phi^{-m}
\end{aligned} \tag{3.3}$$

for some $m \in \mathbb{N}$.

Suppose first that (3.2) holds. Then Lemma 2.5-1 allows us to take the set

$$\{\beta\phi^{-m}, -\delta\alpha_k\phi^{-1}, -\delta\alpha_{k-1}\phi^{-3}, -\delta\alpha_{k-2}\phi^{-5}, \dots, \\ -\delta\alpha_{k-i}\phi^{-2i-1}, \dots, -\delta\alpha_2\phi^{-2k+3}\} \subset \Lambda.$$

We are going to show that this set is a connection from β to α. It is clear that it satisfies condition 1 of Definition 3.1, so let us check that it also satisfies condition 2. We have

$$\begin{aligned} &(\beta\phi^{-m})\phi^{-1} - \delta(\alpha_k\phi^{-1})\phi^{-1} \\ &= (\beta\phi^{-m} - \delta\alpha_k\phi^{-1})\phi^{-1} \\ &= (\delta^{k-1}\alpha_1\phi^{-k+1} + \delta^{k-1}\alpha_2\phi^{-k+1} + \cdots + \delta^{k-i+1}\alpha_i\phi^{-k+i-1} \\ &\quad + \cdots + \delta^2\alpha_{k-1}\phi^{-2})\phi^{-1}, \end{aligned}$$

the last equality being a consequence of equation (3.2). So

$$\begin{aligned} &(\beta\phi^{-m})\phi^{-1} - \delta(\alpha_k\phi^{-1})\phi^{-1} \\ &= \delta(\delta^{k-2}\alpha_1\phi^{-k+2} + \delta^{k-2}\alpha_2\phi^{-k+2} + \cdots + \delta^{k-i}\alpha_i\phi^{-k+i} + \cdots + \delta\alpha_{k-1}\phi^{-1})\phi^{-2}. \end{aligned}$$

Since

$$\delta^{k-2}\alpha_1\phi^{-k+2} + \delta^{k-2}\alpha_2\phi^{-k+2} + \cdots + \delta^{k-i}\alpha_i\phi^{-k+i} + \cdots + \delta\alpha_{k-1}\phi^{-1} \in \Lambda,$$

by condition 2 of Definition 3.1, Lemma 2.5-1 let us assert $(\beta\phi^{-m})\phi^{-1} - \delta(\alpha_k\phi^{-1})\phi^{-1} \in \Lambda$.

For any $1 \le i \le k-2$ we also have that

$$\begin{aligned} &(\beta\phi^{-m})\phi^{-i} - \delta(\alpha_k\phi^{-1})\phi^{-i} - \delta^2(\alpha_{k-1}\phi^{-3})\phi^{-i+1} \\ &- \cdots - \delta^i(\alpha_{k-(i-1)}\phi^{-2i+1})\phi^{-1} \\ &= (\beta\phi^{-m} - \delta\alpha_k\phi^{-1} - \delta^2\alpha_{k-1}\phi^{-2} - \cdots - \delta^i\alpha_{k-(i-1)}\phi^{-i})\phi^{-i} \\ &= (\delta^{k-1}\alpha_1\phi^{-k+1} + \delta^{k-1}\alpha_2\phi^{-k+1} + \cdots + \delta^{i+1}\alpha_{k-i}\phi^{-i-1})\phi^{-i}, \end{aligned}$$

the last equality being a consequence of equation (3.2). Hence,

$$\begin{aligned} &(\beta\phi^{-m})\phi^{-i} - \delta(\alpha_k\phi^{-1})\phi^{-i} - \delta^2(\alpha_{k-1}\phi^{-3})\phi^{-i+1} \\ &- \cdots - \delta^i(\alpha_{k-(i-1)}\phi^{-2i+1})\phi^{-1} \\ &= \delta^i(\delta^{k-i-1}\alpha_1\phi^{-k+i+1} + \delta^{k-i-1}\alpha_2\phi^{-k+i+1} + \cdots + \delta\alpha_{k-i}\phi^{-1})\phi^{-2i}. \end{aligned}$$

Taking now into account that, by condition 2 of Definition 3.1 applied to (3.1),

$$\delta^{k-i-1}\alpha_1\phi^{-k+i+1} + \delta^{k-i-1}\alpha_2\phi^{-k+i+1} + \cdots + \delta\alpha_{k-i}\phi^{-1} \in \Lambda,$$

we get as consequence of Lemma 2.5-1 that

$$\begin{aligned} &(\beta\phi^{-m})\phi^{-i} - \delta(\alpha_k\phi^{-1})\phi^{-i} - \delta^2(\alpha_{k-1}\phi^{-3})\phi^{-i+1} \\ &- \cdots - \delta^i(\alpha_{k-(i-1)}\phi^{-2i+1})\phi^{-1} \in \Lambda. \end{aligned}$$

We have showed that our set satisfies condition 2 of Definition 3.1. It just remains to prove that this set also satisfies condition 3 of this definition. We have, as above,

$$\begin{aligned}&(\beta\phi^{-m})\phi^{-k+1}-\delta(\alpha_k\phi^{-1})\phi^{-k+1}-\delta^2(\alpha_{k-1}\phi^{-3})\phi^{-k+2}\\&-\cdots-\delta^{k-1}(\alpha_2\phi^{-2k+3})\phi^{-1}\\=&\,(\beta\phi^{-m}-\delta\alpha_k\phi^{-1}-\delta^2\alpha_{k-1}\phi^{-2}-\cdots-\delta^{k-1}\alpha_2\phi^{-k+1})\phi^{-k+1}\\=&\,\delta^{k-1}(\alpha_1\phi^{-k+1})\phi^{-k+1}.\end{aligned}$$

Condition 1 of Definition 3.1 applied to the connection (3.1) gives us now that $\alpha_1=\alpha\phi^{-n}$ for some $n\in\mathbb{N}$ and so

$$\begin{aligned}&(\beta\phi^{-m})\phi^{-k+1}-\delta(\alpha_k\phi^{-1})\phi^{-k+1}-\delta^2(\alpha_{k-1}\phi^{-3})\phi^{-k+2}\\&-\cdots-\delta^{k-1}(\alpha_2\phi^{-2k+3})\phi^{-1}\\=&\,\delta^{k-1}\alpha\phi^{-2k-n+2}\in\{\delta^{k-1}\alpha\phi^{-h}:h\in\mathbb{N}\}.\end{aligned}$$

We have therefore showed that our set is actually a connection from β to α.

Suppose now we are in the second possibility, given by equation (3.3). Then we can prove, as in the first possibility given by equation (3.2), that

$$\{\beta\phi^{-m},\delta\alpha_k\phi^{-1},\delta\alpha_{k-1}\phi^{-3},\delta\alpha_{k-2}\phi^{-5},\dots,\delta\alpha_{k-i}\phi^{-2i-1},\dots,\delta\alpha_2\phi^{-2k+3}\}$$

is a connection from β to α. We conclude $\beta\sim\alpha$, and thus the relation $\sim$ is symmetric.

Finally, let us verify that $\sim$ is transitive. Suppose $\alpha\sim\beta$ and $\beta\sim\gamma$, and write $\{\alpha_1,\dots,\alpha_k\}$ for a connection from α to β which satisfies

$$\alpha_1=\epsilon\beta\phi^{-m}\text{ if }k=1 \tag{3.4}$$

or

$$\begin{aligned}&\delta^{k-1}\alpha_1\phi^{-k+1}+\delta^{k-1}\alpha_2\phi^{-k+1}+\delta^{k-2}\alpha_3\phi^{-k+2}+\cdots+\delta\alpha_k\phi^{-1}\\=&\,\epsilon\beta\phi^{-m}\text{ if }k\geq2\end{aligned} \tag{3.5}$$

for some $m\in\mathbb{N}$, $\epsilon\in\{\pm1\}$; and $\{h_1,\dots,h_p\}$ for a connection from β to γ, being then

$$h_1=\beta\phi^{-r} \tag{3.6}$$

for some $r\in\mathbb{N}$. Note that Lemma 3.3 let us suppose $m=r$.

If $p=1$, then $h_1=\tau\gamma\phi^{-t}$ with $t\in\mathbb{N}$ and $\tau\in\{\pm1\}$. Since $m=r$, we have $\alpha_1=\epsilon\beta\phi^{-m}=\epsilon h_1=\epsilon\tau\gamma\phi^{-t}$ if $k=1$, and

$$\begin{aligned}&\delta^{k-1}\alpha_1\phi^{-k+1}+\delta^{k-1}\alpha_2\phi^{-k+1}+\delta^{k-2}\alpha_3\phi^{-k+2}+\cdots+\delta\alpha_k\phi^{-1}\\=&\,\epsilon\beta\phi^{-m}=\epsilon h_1=\epsilon\tau\gamma\phi^{-t}\end{aligned}$$

if $k\geq2$. Hence, we get that $\{\alpha_1,\dots,\alpha_k\}$ is also a connection from α to γ.

If $p\geq2$, then, taking into account the equations (3.4), (3.5) and (3.6) and that $m=r$, it is easy to show that a connection from α to γ is $\{\alpha_1,\dots,\alpha_k,h_2,$

$\ldots, h_p\}$ if $\epsilon = 1$, and $\{\alpha_1, \ldots, \alpha_k, -h_2, \ldots, -h_p\}$ if $\epsilon = -1$. Summarizing, the connection relation is also transitive and so it is an equivalence relation. □

4. Decompositions

Proposition 3.4 tells us the connection relation $\sim$ in Λ is an equivalence relation. So we denote by

$$\Lambda/\sim := \{[\alpha] : \alpha \in \Lambda\},$$

where $[\alpha]$ denotes the set of nonzero roots of L which are connected to α. Our next goal is to associate an adequate ideal $I_{[\alpha]}$ to any $[\alpha]$. For a fixed $\alpha \in \Lambda$, we define

$$I_{0,[\alpha]} := \text{span}_{\mathbb{K}}\{[L_\beta, L_{-\beta}] : \beta \in [\alpha]\} \subset H$$

and

$$V_{[\alpha]} := \oplus_{\beta \in [\alpha]} L_\beta.$$

Then we denote by $I_{[\alpha]}$ the direct sum of the two subspaces above, that is,

$$I_{[\alpha]} := I_{0,[\alpha]} \oplus V_{[\alpha]}.$$

Proposition 4.1. *For any $\alpha \in \Lambda$, the linear subspace $I_{[\alpha]}$ is a subalgebra of L.*

Proof. First, it is sufficient to check that $I_{[\alpha]}$ satisfies $[I_{[\alpha]}, I_{[\alpha]}] \subset I_{[\alpha]}$. By $I_{0,[\alpha]} \subset H$, it is clear that $[I_{0,[\alpha]}, I_{0,[\alpha]}]$=0 and we have

$$[I_{0,[\alpha]} \oplus V_{[\alpha]}, I_{0,[\alpha]} \oplus V_{[\alpha]}] \subset [I_{0,[\alpha]}, V_{[\alpha]}] + [V_{[\alpha]}, I_{0,[\alpha]}] + [V_{[\alpha]}, V_{[\alpha]}]. \tag{4.7}$$

Let us consider the first summand in (4.7). For $\beta \in [\alpha]$, by Lemmas 2.4 and 3.2, one gets $[I_{0,[\alpha]}, L_\beta] \subset L_{\delta\beta\phi^{-1}}$, where $\delta\beta\phi^{-1} \in [\alpha]$. Hence

$$[I_{0,[\alpha]}, V_{[\alpha]}] \subset V_{[\alpha]}. \tag{4.8}$$

Similarly, we can also get

$$[V_{[\alpha]}, I_{0,[\alpha]}] \subset V_{[\alpha]}. \tag{4.9}$$

Next, we consider the third summand in (4.7). Given β, $\gamma \in [\alpha]$ such that $[L_\beta, L_\gamma] \neq 0$, if $\gamma = -\beta$, we have $[L_\beta, L_\gamma] = [L_\beta, L_{-\beta}] \subset I_{0,[\alpha]}$. Suppose $\gamma \neq -\beta$, by Lemma 2.4-2, one gets $\delta\beta\phi^{-1} + \delta\gamma\phi^{-1} \in \Lambda$. Therfore, we get $\{\beta, \gamma\}$ is a connection from β to $\beta\phi^{-1} + \gamma\phi^{-1}$. The transitivity of $\sim$ gives that $\beta\phi^{-1} + \gamma\phi^{-1} \in [\alpha]$ and so $[L_\beta, L_\gamma] \subset L_{\delta\beta\phi^{-1} + \delta\gamma\phi^{-1}} \subset V_{[\alpha]}$. Hence

$$[\oplus_{\beta \in [\alpha]} L_\beta, \oplus_{\beta \in [\alpha]} L_\beta] \subset I_{0,[\alpha]} \oplus V_{[\alpha]}.$$

That is,

$$[V_{[\alpha]}, V_{[\alpha]}] \subset I_{[\alpha]}. \tag{4.10}$$

From (4.7), (4.8), (4.9) and (4.10), we get $[I_{[\alpha]}, I_{[\alpha]}] \subset I_{[\alpha]}$.

Second, we have to verify that $\phi(I_{[\alpha]}) = I_{[\alpha]}$. It is a direct consequence of Lemmas 2.4-1 and 3.2. □

Proposition 4.2. *If $[\alpha] \neq [\beta]$, then $[I_{[\alpha]}, I_{[\beta]}] = 0$.*

Proof. We have

$$[I_{0,[\alpha]} \oplus V_{[\alpha]}, I_{0,[\beta]} \oplus V_{[\beta]}] \subset [I_{0,[\alpha]}, V_{[\beta]}] + [V_{[\alpha]}, I_{0,[\beta]}] + [V_{[\alpha]}, V_{[\beta]}]. \quad (4.11)$$

Let us consider the third summand $[V_{[\alpha]}, V_{[\beta]}]$ in (4.11) and suppose there exist $\alpha_1 \in [\alpha]$ and $\alpha_2 \in [\beta]$ such that $[L_{\alpha_1}, L_{\alpha_2}] \neq 0$. By known condition $[\alpha] \neq [\beta]$, one gets $\alpha_1 \neq -\alpha_2$. So $\alpha_1\phi^{-1} + \alpha_2\phi^{-1} \in \Lambda$. Hence $\{\alpha_1, \alpha_2, -\delta\alpha_1\phi^{-1}\}$ is a connection from α_1 to α_2. By the transitivity of the connection relation, we have $\alpha \in [\beta]$, a contradiction. Hence $[L_{\alpha_1}, L_{\alpha_2}] = 0$ and so

$$[V_{[\alpha]}, V_{[\beta]}] = 0. \quad (4.12)$$

Next we consider the first summand $[I_{0,[\alpha]}, V_{[\beta]}]$ in (4.11). Let us take $\alpha_1 \in [\alpha]$ and $\alpha_2 \in [\beta]$ and conclude that

$$\alpha_2([L_{\alpha_1}, L_{-\alpha_1}]) = 0. \quad (4.13)$$

Indeed, by applying Hom-Jacobi identity and (4.12), one gets

$$[[L_{\alpha_1}, L_{-\alpha_1}], \phi(L_{\alpha_2})] = 0. \quad (4.14)$$

By ϕ is an algebra automorphism and (4.14), one gets

$$\phi[\phi^{-1}[L_{\alpha_1}, L_{-\alpha_1}], (L_{\alpha_2})] = 0, \quad (4.15)$$

that is

$$[\phi^{-1}[L_{\alpha_1}, L_{-\alpha_1}], (L_{\alpha_2})] = 0, \quad (4.16)$$

where $\phi^{-1}[L_{\alpha_1}, L_{-\alpha_1}] \subset H$. Hence (4.16) gives

$$\alpha_2\phi^{-1}([L_{\alpha_1}, L_{-\alpha_1}]) = 0 \quad (4.17)$$

for any $\alpha_1 \in [\alpha]$ and $\alpha_2 \in [\beta]$. By Lemma 2.4-1 and ϕ is an algebra automorphism, we get

$$\phi([L_{\alpha_1}, L_{-\alpha_1}]) \subset [L_{\alpha_1\phi^{-1}}, L_{-\alpha_1\phi^{-1}}],$$

that is

$$[L_{\alpha_1}, L_{-\alpha_1}] \subset \phi^{-1}([L_{\alpha_1\phi^{-1}}, L_{-\alpha_1\phi^{-1}}])$$

and by (4.17), one gets

$$\alpha_2([L_{\alpha_1}, L_{-\alpha_1}]) = 0.$$

From $[[L_{\alpha_1}, L_{-\alpha_1}], L_{\alpha_2}] \subset \alpha_2([L_{\alpha_1}, L_{-\alpha_1}])\phi(L_{\alpha_2}) = 0$, we prove that $[I_{0,[\alpha]}, V_{[\beta]}] = 0$. In a similar way, we get $[V_{[\alpha]}, I_{0,[\beta]}] = 0$ and we conclude, together with (4.11) and (4.12), that $[I_{[\alpha]}, I_{[\beta]}] = 0$. □

Definition 4.3. A δ-Hom-Jordan-Lie algebra L is said to be *simple* if its product is nonzero and its only ideals are $\{0\}$ and L.

Theorem 4.4. *The following assertions hold.*

1. *For any $\alpha \in \Lambda$, the subalgebra*

$$I_{[\alpha]} = I_{0,[\alpha]} \oplus V_{[\alpha]}$$

of L associated to $[\alpha]$ is an ideal of L.

2. *If L is simple, then there exists a connection from α to β for any $\alpha, \beta \in \Lambda$ and $H = \sum_{\alpha\in\Lambda}[L_\alpha, L_{-\alpha}]$.*

Proof. 1. Since $[I_{[\alpha]}, H] = [I_{[\alpha]}, L_0] \subset V_{[\alpha]}$, taking into account Propositions 4.1 and 4.2, we have

$$[I_{[\alpha]}, L] = [I_{[\alpha]}, H \oplus (\oplus_{\beta \in [\alpha]} L_\beta) \oplus (\oplus_{\gamma \notin [\alpha]} L_\gamma)] \subset I_{[\alpha]}.$$

As we also have by Lemmas 2.4-1 and 3.2 that $\phi(I_{[\alpha]}) = I_{[\alpha]}$, we conclude that $I_{[\alpha]}$ is an ideal of I.

2. The simplicity of L implies $I_{[\alpha]} = L$. From here, it is clear that $[\alpha] = \Lambda$ and $H = \sum_{\alpha \in \Lambda} [L_\alpha, L_{-\alpha}]$. □

Theorem 4.5. *For a vector space complement U of $span_{\mathbb{K}}\{[L_\alpha, L_{-\alpha}] : \alpha \in \Lambda\}$ in H, we have*

$$L = U + \sum_{[\alpha] \in \Lambda/\sim} I_{[\alpha]},$$

where any $I_{[\alpha]}$ is one of the ideas of L described in Theorem 4.4-1, satisfying $[I_{[\alpha]}, I_{[\beta]}] = 0$, whenever $[\alpha] \neq [\beta]$.

Proof. Each $I_{[\alpha]}$ is well defined and, by Theorem 4.4-1, an ideal of L. It is clear that

$$L = H \oplus (\oplus_{\alpha \in \Lambda} L_\alpha) = U + \sum_{[\alpha] \in \Lambda/\sim} I_{[\alpha]}.$$

Finally Proposition 4.2 gives us $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$. □

Definition 4.6. The *annihilator* of a δ-Hom-Jordan-Lie algebra L is the set $\mathrm{Z}(L) = \{x \in L : [x, L] = 0\}$.

Corollary 4.7. *If $Z(L) = 0$ and $[L, L] = L$, then L is the direct sum of the ideals given in Theorem 4.4,*

$$L = \oplus_{[\alpha] \in \Lambda/\sim} I_{[\alpha]}.$$

Proof. From $[L, L] = L$, it is clear that $L = \oplus_{[\alpha] \in \Lambda/\sim} I_{[\alpha]}$. Finally, the sum is direct because $Z(L) = 0$ and $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$. □

5. The simple components

In this section we focus on the simplicity of split regular δ-Hom-Jordan-Lie algebras by centering our attention in those of maximal length. From now on char($\mathbb{K}$)=0.

Lemma 5.1. *Let L be a split regular δ-Hom-Jordan-Lie algebra with $\mathrm{Z}(L) = 0$ and I an ideal of L. If $I \subset H$, then $I = \{0\}$.*

Proof. Suppose there exists a nonzero ideal I of L such that $I \subset H$. We get $[I, H] \subset [H, H] = 0$. We also get $[I, \oplus_{\alpha \in \Lambda} L_\alpha] \subset I \subset H$. Then taking into account $H = L_0$, we have $[I, \oplus_{\alpha \in \Lambda} L_\alpha] \subset H \cap (\oplus_{\alpha \in \Lambda} L_\alpha) = 0$. From here $I \subset \mathrm{Z}(L) = 0$, which is a contradiction. □

Lemma 5.2. *For any $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$ there exists $h_0 \in H$ such that $\alpha(h_0) \neq 0$ and $\alpha(h_0) \neq \beta(h_0)$.*

Proof. As $\alpha \neq \beta$, there exists $h \in H$ such that $\alpha(h) \neq \beta(h)$. If $\alpha(h) \neq 0$ we have finished. So let us suppose $\alpha(h) = 0$, which implies $\beta(h) \neq 0$. Since $\alpha \neq 0$, we can fix some $h^{'} \in H$ such that $\alpha(h^{'}) \neq 0$. We can distinguish two cases, in the first one $\alpha(h^{'}) \neq \beta(h^{'})$ and in the second one $\alpha(h^{'}) = \beta(h^{'})$. Then, by taking $h_0 := h^{'}$ in the first case and $h_0 := h + h^{'}$ in the second, we complete the proof. □

Lemma 5.3. *Let $L = H \oplus (\oplus_{\alpha\in\Lambda} L_\alpha)$ be a split regular δ-Hom-Jordan-Lie algebra. Suppose that I is an ideal of L and $x = h + \sum_{j=1}^{n} v_{\alpha_j} \in I$, with $h \in H$, $v_{\alpha_j} \in L_{\alpha_j}$ and $\alpha_j \neq \alpha_k$. Then any $v_{\alpha_j} \in I$.*

Proof. If $n = 1$ we have $x = h + v_{\alpha_1} \in I$. By taking $h^{'} \in H$ such that $\alpha_1(h^{'}) \neq 0$ we have $[h^{'}, x] = \alpha_1(h^{'})\phi(v_{\alpha_1}) \in I$ and so $\phi(v_{\alpha_1}) \in I$. Therefore $\phi^{-1}(\phi(v_{\alpha_1})) = v_{\alpha_1} \in I$.

Suppose now $n > 1$ and consider α_1 and α_2. By Lemma 5.2 there exists $h_0 \in H$ such that $\alpha_1(h_0) \neq 0$ and $\alpha_1(h_0) \neq \alpha_2(h_0)$. Then we have

$$I \ni [h_0, x] = \alpha_1(h_0)\phi(v_{\alpha_1}) + \alpha_2(h_0)\phi(v_{\alpha_2}) + \cdots + \alpha_n(h_0)\phi(v_{\alpha_n}) \tag{5.18}$$

and

$$I \ni \phi(x) = \phi(h) + \phi(v_{\alpha_1}) + \phi(v_{\alpha_2}) + \cdots + \phi(v_{\alpha_n}). \tag{5.19}$$

By multiplying (5.19) by $\alpha_2(h_0)$ and subtracting (5.18), one gets

$$\alpha_2(h_0)\phi(h) + (\alpha_2(h_0) - \alpha_1(h_0))\phi(v_{\alpha_1}) + (\alpha_2(h_0) - \alpha_3(h_0))\phi(v_{\alpha_3}) + \cdots + (\alpha_2(h_0) - \alpha_n(h_0))\phi(v_{\alpha_n}) \in I.$$

By denoting $\widetilde{h} := \alpha_2(h_0)\phi(h) \in H$ and $v_{\alpha_i\phi^{-1}} := (\alpha_2(h_0) - \alpha_i(h_0))\phi(v_{\alpha_i}) \in L_{\alpha_i\phi^{-1}}$, we can write

$$\widetilde{h} + v_{\alpha_1\phi^{-1}} + v_{\alpha_3\phi^{-1}} + \cdots + v_{\alpha_n\phi^{-1}} \in I. \tag{5.20}$$

Now we can argue as above, with (5.20), to get

$$\widetilde{\widetilde{h}} + v_{\alpha_1\phi^{-2}} + v_{\alpha_4\phi^{-2}} + \cdots + v_{\alpha_n\phi^{-2}} \in I$$

for $\widetilde{\widetilde{h}} \in H$ and $v_{\alpha_1\phi^{-2}} \in L_{\alpha_i\phi^{-2}}$. Following this process, we obtain

$$\overline{h} + v_{\alpha_1\phi^{-n+1}} \in I,$$

with $\overline{h} \in H$ and $v_{\alpha_1\phi^{-n+1}} \in L_{\alpha_i\phi^{-n+1}}$. As in the above case $n = 1$, we conclude that $v_{\alpha_1\phi^{-n+1}} \in I$ and consequently $v_{\alpha_1} = \phi^{-n+1}(v_{\alpha_1\phi^{-n+1}}) \in I$.

In a similar way we can prove that $v_{\alpha_i} \in I$ for $i \in \{2, \ldots, n\}$, and the proof is complete. □

Definition 5.4. A split regular δ-Hom-Jordan-Lie algebra L is *root-multiplicative* if given $\alpha, \beta \in \Lambda$ such that $\delta\alpha\phi^{-1} + \delta\beta\phi^{-1} \in \Lambda$, then $[L_\alpha, L_\beta] \neq 0$.

Definition 5.5. A split regular δ-Hom-Jordan-Lie algebra L is of *maximal length* if for any $\alpha \in \Lambda$, we have $\dim L_\alpha = 1$ for any $\alpha \in \Lambda$.

Theorem 5.6. *Let L be a split regular δ-Hom-Jordan-Lie algebra of maximal length and root-multiplicative. Then L is simple if and only if $\mathrm{Z}(L) = 0$, $H = \sum_{\alpha\in\Lambda}[L_\alpha, L_{-\alpha}]$ and Λ has all its nonzero roots connected.*

Proof. Suppose L is simple. Since $\mathrm{Z}(L)$ is an ideal of L then $\mathrm{Z}(L) = 0$. From here, Theorem 4.4-2 completes the proof of the first implication. To prove the converse, consider I a nonzero ideal of L. By Lemma 5.3 we can write $I = (I \cap H) \oplus (\oplus_{\alpha\in\Lambda} I_\alpha)$, where $I_\alpha = I \cap L_\alpha$. By the maximal length of L, if we set $\Lambda_I := \{\alpha \in \Lambda : I_\alpha \neq 0\}$, we can write $I = (I \cap H) \oplus (\oplus_{\alpha\in\Lambda_I} L_\alpha)$, where $\Lambda_I \neq \emptyset$ as consequence of Lemma 5.1. Let us fix some $\alpha_0 \in \Lambda_I$ so that $0 \neq L_{\alpha_0} \subset I$. The fact $\phi(I) = I$ together with Lemma 2.4-1 allows us to assert that

$$\text{if } \alpha \in \Lambda_I, \text{ then } \{\alpha\phi^z : z \in \mathbb{Z}\} \subset \Lambda_I, \tag{5.21}$$

that is

$$\{L_{\alpha_0\phi^z} : z \in \mathbb{Z}\} \in I. \tag{5.22}$$

Now, let us take any $\beta \in \Lambda$ satisfying $\beta \notin \{\pm\alpha_0\phi^z : z \in \mathbb{Z}\}$. Since α_0 and β are connected, we have a connection $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$, $k \geq 2$, from α_0 to β satisfying:

$\alpha_1 = \alpha_0\phi^{-n}$ for some $n \in \mathbb{N}$, and
$\delta\alpha_1\phi^{-1} + \delta\alpha_2\phi^{-1} \in \Lambda$,
$\delta^2\alpha_1\phi^{-2} + \delta^2\alpha_2\phi^{-2} + \delta\alpha_3\phi^{-1} \in \Lambda$,
....
$\delta^i\alpha_1\phi^{-i} + \delta^i\alpha_2\phi^{-i} + \delta^{i-1}\alpha_3\phi^{-i+1} + \cdots + \delta\alpha_{i+1}\phi^{-1} \in \Lambda$,
....
$\delta^{k-2}\alpha_1\phi^{-k+2} + \delta^{k-2}\alpha_2\phi^{-k+2} + \delta^{k-3}\alpha_3\phi^{-k+3} + \cdots$
$+ \delta^{k-i}\alpha_i\phi^{-k+i} + \cdots + \delta\alpha_{k-1}\phi^{-1} \in \Lambda$,
$\delta^{k-1}\alpha_1\phi^{-k+1} + \delta^{k-1}\alpha_2\phi^{-k+1} + \cdots + \delta^{k-i+1}\alpha_i\phi^{-k+i-1} + \cdots + \delta\alpha_k\phi^{-1}$
$= \epsilon\beta\phi^{-m}$ for some $m \in \mathbb{N}$ and $\epsilon \in \{\pm 1\}$.

Taking into account that $\alpha_1, \alpha_2 \in \Lambda$ and $\delta\alpha_1\phi^{-1} + \delta\alpha_2\phi^{-1} \in \Lambda$, the root multiplicativity and maximal length of L allow us to assert $0 \neq [L_{\alpha_1}, L_{\alpha_2}] = L_{\delta\alpha_1\phi^{-1}+\delta\alpha_2\phi^{-1}}$. Since $0 \neq L_{\alpha_1} \subset I$ as consequence of equation (5.22) we get

$$0 \neq L_{\delta\alpha_1\phi^{-1}+\delta\alpha_2\phi^{-1}} \subset I.$$

A similar argument applied to $\delta\alpha_1\phi^{-1} + \delta\alpha_2\phi^{-1}$, α_3 and

$$\delta^2(\alpha_1\phi^{-1} + \alpha_2\phi^{-1})\phi^{-1} + \delta\alpha_3\phi^{-1} = \delta^2\alpha_1\phi^{-2} + \delta^2\alpha_2\phi^{-2} + \delta\alpha_3\phi^{-1}$$

gives us $0 \neq L_{\delta^2\alpha_1\phi^{-2}+\delta^2\alpha_2\phi^{-2}+\delta\alpha_3\phi^{-1}} \subset I$. We can follow this process with the connection $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ to get

$$0 \neq L_{\delta^{k-1}\alpha_1\phi^{-k+1}+\delta^{k-1}\alpha_2\phi^{-k+1}+\cdots+\delta^{k-i+1}\alpha_i\phi^{-k+i-1}+\cdots+\delta\alpha_k\phi^{-1}} \subset I$$

and then

$$\text{either } L_{\beta\phi^{-m}} \subset I \text{ or } L_{-\beta\phi^{-m}} \subset I.$$

From equations (5.21) and (5.22), we now get

(5.23) $\quad$ either $\{L_{\alpha\phi^{-z}} : z \in \mathbb{Z}\} \subset I$ or $\{L_{-\alpha\phi^{-z}} : z \in \mathbb{Z}\} \subset I$ for any $\alpha \in \Lambda$.

Equation (5.23) can be reformulated by asserting that given any $\alpha \in \Lambda$ either $\{\alpha\phi^{-z} : z \in \mathbb{Z}\}$ or $\{-\alpha\phi^{-z} : z \in \mathbb{Z}\}$ is contained in Λ_I. Taking now into account $H = \sum_{\alpha\in\Lambda}[L_\alpha, L_{-\alpha}]$, we obtain

$$H \subset I. \tag{5.24}$$

If we consider now any $\alpha \in \Lambda$, since $L_\alpha = [H, L_{\delta\alpha\phi}]$ by the maximal length of L, the inclusion (5.24) gives us $L_\alpha \subset I$ and so $I = L$. That is, L is simple. □

Theorem 5.7. *Let L be a split regular δ-Hom-Jordan-Lie algebra of maximal length, root multiplicative and satisfying* $\mathrm{Z}(L) = 0$, $H = \sum_{\alpha\in\Lambda}[L_\alpha, L_{-\alpha}]$. *Then*

$$L = \oplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]},$$

where any $I_{[\alpha]}$ is a simple (split) ideal having its roots system, $\Lambda_{I_{[\alpha]}}$, with all of its elements $\Lambda_{I_{[\alpha]}}$-connected.

Proof. By Corollary 4.7, we can write $L = \oplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$ as the direct sum of the family of ideals

$$I_{[\alpha]} = I_{0,[\alpha]} \oplus V_{[\alpha]} = \mathrm{span}_{\mathbb{K}}\{[L_\beta, L_{-\beta}] : \beta \in [\alpha]\} \oplus (\oplus_{\beta\in[\alpha]} L_\beta),$$

where each $I_{[\alpha]}$ is a split regular δ-Hom-Jordan-Lie algebra having as roots system $\Lambda_{I_{[\alpha]}} = [\alpha]$. To apply Theorem 5.6 to each $I_{[\alpha]}$, we have to observe that the root-multiplicativity of L and Proposition 4.2 show that $\Lambda_{I_{[\alpha]}}$ has all of its elements $\Lambda_{I_{[\alpha]}}$-connected, that is, connected through connections contained in $\Lambda_{I_{[\alpha]}}$. We also get that any of the $I_{[\alpha]}$ is root-multiplicative as consequence of the root-multiplicativity of L. Clearly $I_{[\alpha]}$ is of maximal length, and finally $Z_{I_{[\alpha]}}(I_{[\alpha]})$=0, (where $Z_{I_{[\alpha]}}(I_{[\alpha]})$ denotes the center of $I_{[\alpha]}$ in $I_{[\alpha]}$), as consequence of $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$, (Theorem 4.5), and $Z(L) = 0$. We can therefore apply Theorem 5.6 to any $I_{[\alpha]}$ so as to conclude $I_{[\alpha]}$ is simple. It is clear that the decomposition $L = \oplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$ satisfies the assertions of the theorem. □

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