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A GEOMETRIC INEQUALITY ON A COMPACT DOMAIN IN $\mathbb{R}^n$

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Abstract. In this paper, we study some topological structure of a compact domain in $\mathbb{R}^n$ in terms of the curvature conditions and develop a geometric inequality involving the volume and the integral of mean curvatures over the boundary of the compact domain.

1. Introduction

Geometric inequalities on the convex bodies have been studied by many mathematicians including ([1, 3, 4, 7, 11]).

Among them Minkowski's classical inequalities [7] are famous: for a surface S bounding a convex domain D with volume V

$$\int_S H dA \leq \frac{A^2}{3V}$$

and

$$[\int_S H dA]^3 \geq 48\pi^2 V, \tag{1}$$

where A is the area of S and H is mean curvature function .

A. Ros [8] showed that for a surface S bounding a compact domain D of volume V if the mean curvature H of S is positive everywhere, then

$$\int_S \frac{1}{H} dA \geq 3V.$$

Equality holds if and only if S is a standard sphere.

We deal with $C(o)$-compact domain in $\mathbb{R}^n$ which is a topological ball whose boundary has only odd number of negative principal normal curvatures almost everywhere if they exist.

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In this paper, we prove the following main results for the $C(o)$-compact domain in $\mathbb{R}^n$:

Main Theorem. *If W is a $C(o)$-compact domain in R^n with convex hull $K(W)$ and M_{n-2} is the integral of $(n-2)$nd mean curvature functional, then*

$$M_{n-2}(\partial W) \geq M_{n-2}(\partial K(W)) \tag{2}$$

and equality holds if W is convex.

This extends Minkowski's inequality (1) for a convex domain in $\mathbb{R}^3$ to the $C(o)$-compact domain in $\mathbb{R}^n$. In fact, from (1) and (2)

$$\int_{\partial W} HdA \geq \int_{\partial K(W)} HdA \geq 3 \cdot \left[\frac{4\pi}{3}\right]^{\frac{2}{3}} [V(K(W))]^{\frac{1}{3}} \geq 3 \cdot \left[\frac{4\pi}{3}\right]^{\frac{2}{3}} [V(W)]^{\frac{1}{3}}$$

or

$$\left[\int_{\partial W} HdA\right]^3 \geq 48\pi^2 V(W).$$

2. Preliminaries

For a compact domain W with C^2-boundary ∂W in $\mathbb{R}^n$, $(n-1)$ principal normal curvatures $\kappa_1, \kappa_2, \ldots, \kappa_{n-1}$ at a point of ∂W are the eigenvalues of the second fundamental form at the point. The integral of i-th mean curvature $M_i(\partial W)$ of W is defined by the integral,

$$M_i(\partial W) = \left(\begin{array}{c} n-1 \\ i \end{array} \right)^{-1} \int_{\partial W} \sigma_i(\kappa_1, \ldots, \kappa_{n-1}) dw,$$

where σ_i is the i-th elementary symmetric function.

Let $(x; e_i)$ be an orthonormal moving frame and consider the $(n-1)$-flat, L_{n-1} determined by $x, e_1, e_2, \ldots, e_{n-1}$. The density for L_{n-1} is differential n-form given by the following wedge product of 1-forms:

$$dL_{n-1} = \omega_n \wedge \omega_{n1} \wedge \omega_{n2} \wedge \cdots \wedge \omega_{nn-1}, \tag{3}$$

where $\omega_n = dx \cdot e_n$, $\omega_{nj} = de_j \cdot e_n$ for $j = 1, 2, \ldots, n-1$.

Then we introduce an integral formula proved in Santalo [9, p. 248]: for any compact domain W with C^2-boundary ∂W,

$$\int_{W \cap L_{n-1} \neq \emptyset} \chi(W \cap L_{n-1}) dL_{n-1} = M_{n-2}(\partial W), \tag{4}$$

where $\chi(W \cap L_{n-1})$ is the Euler Characteristic of the $(n-1)$-flat section $(W \cap L_{n-1})$.

Let f be a real valued function on an n-manifold M. A point $p \in M$ is called a critical point of f if for any coordinate system $(x_1, \ldots, x_n)$ around p,

$$\frac{\partial f}{\partial x_1}(p) = \cdots = \frac{\partial f}{\partial x_n}(p) = 0.$$

A critical point p of f is called a non-degenerated critical point if the $n \times n$ matrix
$$(\frac{\partial^2 f}{\partial x_i \partial x_j})_{1 \le i,j \le n}$$
is non-singular. If f has only non-degenerate critical points, then f is called a Morse function.

If $p \in M$ is a non-degenerate critical point of $f : M \to R$, then *index* of p is defined to be the number of negative eigenvalues of the Hessian matrix, counting multiplicity.

As an example, for a fixed point a in $\mathbb{R}^n$ a distance function defined on a hyper-surface M of $\mathbb{R}^n$ by $d_a(x) = |x-a|^2$ is a Morse function if a is not focal point of M.

When M is a manifold without boundary, Morse theory [6] gives us valuable information about the topological behavior of the manifold near critical point $p \in M$: *if r is a critical value associated with only one non-degenerate critical point p of index λ, then for sufficiently small positive real number ϵ, $f^{-1}(r-\epsilon, r+\epsilon)$ has the same homotopy type as $f^{-1}(r-\epsilon, r)$ with λ-cell e^λ attached.*

Throughout this paper, we denote by $H_i(W)$ the i-dimensional reduced homology group of W with integer coefficients.

The following lemma is to make use of Morse theory on the manifold without boundary to deal with topology around critical point of a function on the manifold with boundary.

Lemma 1. *Let W be a compact n-manifold with C^2-boundary ∂W in $\mathbb{R}^n$ and $f : W \to R$ be a smooth function. If $s < t$, $f|_{\partial W}^{-1}[s,t]$ contains no critical points and $f|_{\partial W}$ is a Morse function, then for each $i \ge 0$*
$$H_i(f|_W^{-1}(-\infty, s]) \cong H_i(f|_W^{-1}(-\infty, t]).$$

Proof. See [2]. □

Remark. A simple modification of Lemma 1 allows to hold the statement with $f|_{\partial W}^{-1}(s,t]$ instead of $f|_{\partial W}^{-1}[s,t]$.

Let p be a critical point of a Morse function $f|_{\partial W}$ and $V(p)$ be an outward normal vector to ∂W at p. Then there is a neighborhood $U(p)$ of p so that the standard dot product $\nabla f(p') \cdot V(p)$ of $\nabla f(p')$ and $V(p)$ is either positive or negative for all $p' \in U(p) \setminus \{p\}$ where ∇f is nonzero. Such neighborhood is possible because critical point p is non-degenerate.

Definition 1. Let W be an n-manifold with C^2-boundary ∂W in $\mathbb{R}^n$ and let f be a C^2-function on $\mathbb{R}^n$ such that $f|_{\partial W}$ is a Morse function. Then a critical point p of $f|_{\partial W}$ is said to be negative if $\nabla f(p') \cdot V(p)$ is negative for all $p' \in U(p) \setminus \{p\}$. Otherwise p is said to be positive.

Consider the solid torus with boundary, the surface of revolution obtained by rotating the curve ${x_1}^2 + (x_2 - 2)^2 = 1$ about x_1-axis. Then for $v = (0,0,1)$, height function h_v defined by $h_v(x_1, x_2) = (x_1, x_2) \cdot v$ has four critical points corresponding critical values r_1, r_2, r_3, r_4 with $r_1 < r_2 < r_3 < r_4$: the first one is negative, the second one is positive, the third one is negative and the fourth one is positive. At the second critical point, we have

$$H_i(h|_W^{-1}(-\infty, r_2 - \epsilon]) \cong H_i(h|_W^{-1}(-\infty, r_2]) \cong H_i(h|_W^{-1}(-\infty, r_2 + \epsilon]),$$

since corresponding sets are of the same homotopy type. However, at the third negative critical point, we have

$$Z \oplus H_1(h|_W^{-1}(-\infty, r_3 - \epsilon]) \cong H_1(h|_W^{-1}(-\infty, r_3]),$$

where Z denotes ring of integers.

The following lemma justifies the discussion above and shows that the only negative critical point plays the role to decide local behavior of the manifold near the critical point.

Lemma 2. *Let W be an n-manifold with C^2-boundary ∂W in $\mathbb{R}^n$ and let $f|_{\partial W}$ be a Morse function. If $r_1 > r$ and r_1 is a critical value of $f|_{\partial W}$ such that all the critical points at the level r_1 are of positive and $(f|_{\partial W}^{-1})[r, r_1)$ contains no critical points, then for every $i \geq 0$,*

$$H_i(f|_W^{-1}(-\infty, r_1]) \cong H_i(f|_W^{-1}(-\infty, r]).$$

Proof. See [2]. □

Remark. Lemma 2 tells that there is no topological change of sub-level set across positive critical points.

3. Main result

The following definition extends our ideas for convex bodies to the generalized convex bodies.

Definition 2. Let W be a compact connected topological ball with C^2-boundary ∂W in $\mathbb{R}^n$. Then W is called $C(o)$-compact if ∂W has odd number of negative principal normal curvatures almost everywhere if any.

It should be noted that

(1) every convex set with C^2-boundary is $C(o)$-compact vacuously.

(2) every topological ball with next-to-convex boundary defined in [5] is $C(o)$-compact.

(3) every topological ball with p-convex boundary defined in [10] is $C(o)$-compact.

Lemma 3. *Let W be a topological ball with C^2-boundary ∂W in $\mathbb{R}^n$ and $d : \mathbb{R}^n \to R$ be a distance function which is C^2-function on $\mathbb{R}^n$ such that $d|_{\partial W}$ is a Morse function. If $H_i(d|_W^{-1}[0, r_0]) \neq 0$ for some r_0 and for some odd i, then $r_1 = \inf\{r \geq r_0 \mid H_i(d|_W^{-1}[0, r]) = 0$ for all odd $i\}$ is a critical value of $d|_{\partial W}$ and $H_i(d|_W^{-1}[0, r_1]) = 0$ for all odd i.*

Proof. Suppose that r_1 is a regular value of $d|_{\partial W}$. By Lemma 1, for all sufficiently small positive real number ϵ and for all odd i,

$$H_i(d|_W^{-1}[0,r_1-\epsilon]) \cong H_i(d|_W^{-1}[0,r_1]) \cong H_i(d|_W^{-1}[0,r_1+\epsilon]).$$

This contradicts definition of r_1. Thus r_1 must be a critical value of $d|_{\partial W}$.

Now suppose that for some odd i, $H_i(d|_W^{-1}[0,r_1]) \neq 0$. Since r_1 is a critical value of a Morse function $d|_{\partial W}$, there exists a positive real number ϵ such that $d|_{\partial W}^{-1}(r_1, r_1+\epsilon]$ contains no critical points. Now by a simple modification of Lemma 1, for all small ϵ and for all odd i, $H_i(d|_W^{-1}[0,r_1]) \cong H_i(d|_W^{-1}[0,r_1+\epsilon])$. This contradicts definition of r_1. □

The following theorem shows how curvature conditions on a surface have influences on the local topology of a surface.

Theorem 1. *Let W be a $C(o)$-compact in $\mathbb{R}^n$. If $c \in \mathbb{R}^n$ is not a focal point of ∂W and d_c is the distance function defined on W from c, then $H_i(d_c^{-1}[0,r]) = 0$ for all odd i and for any positive real number r.*

Proof. Let ∂W be oriented by outward unit normal vector field N. Then the distance function $d_c|_{\partial W}$ restricted on ∂W is a Morse function. We may assume that $c = 0 \in \mathbb{R}^n$. Use a notation d for d_0. Suppose that $H_i(d|_W^{-1}[0,r_0]) \neq 0$ for some odd i, and for some $r_0 > 0$. Let

$$r_1 = \inf\{r \geq r_0 \mid H_i(d|_W^{-1}[0,r]) = 0 \text{ for all odd } i\}.$$

Then from Lemma 3 r_1 is a critical value of $d|_{\partial W}$ and $H_i(d^{-1}[0,r_1]) = 0$ for all odd i.

Since $d|_{\partial W}^{-1}(r_1)$ is compact and $d|_{\partial W}$ is a Morse function, $d|_{\partial W}^{-1}(r_1)$ contains only finitely many critical points, say $p_1, p_2, \ldots, p_k, p_{k+1}, \ldots, p_m$,where $p_1, p_2, \ldots, p_k$ are negative critical points and $p_{k+1}, \ldots, p_m$ are positive critical points. Existence of negative critical points is guaranteed from the definition of r_1 and Lemma 2.

Thicken W around critical points $p_{k+1}, \ldots, p_m$ of d to construct new compact domain W_* from W in the following way: Let $U_{k+1}, \ldots, U_m$ be the sufficiently small disjoint neighborhoods of $p_{k+1}, \ldots, p_m$ in ∂W, respectively and let $\eta_i : \partial W \to [0,1]$ be a function having support in U_i and equal to 1 on some neighborhood $V_i \subset U_i$ of $p_i, i = k+1, \ldots, m$. For a small positive real number δ_i we define W_* by

$$W_* = \{x + t\delta_i\eta_i(x)\cdot x/|x| | x \in \partial W, k+1 \leq i \leq m, 0 \leq t \leq 1\} \cup W.$$

As max δ_i tends to 0, $W_* \to W$. Then W_* is an n-manifold with C^2-boundary ∂W_*

Now it is easy to check that $d|_W^{-1}[0, r_1+\epsilon]$ is deformation retract of $d|_{W_*}^{-1}[0, r_1+\epsilon]$ and $d|_{W_*}^{-1}[0, r_1-\epsilon] = d|_W^{-1}[0, r_1-\epsilon]$. On the other hand, every critical point of $d|_{W_*^c}^{-1}$ at level r_1 is positive. So by Lemma 2,

$$H_i(d|_{W_*^c}^{-1}[0, r_1+\epsilon]) = H_i(d|_{W_*^c}^{-1}[0, r_1-\epsilon]).$$

Using the Mayer-Vietoris exact homology sequence, we have

$$H_i(d|_{\partial W_*}^{-1}[0, r_1 - \epsilon]) \cong H_i(d|_{W_*}^{-1}[0, r_1 - \epsilon]) \oplus H_i(d|_{W_*^c}^{-1}[0, r_1 - \epsilon])$$

and

$$H_i(d|_{\partial W_*}^{-1}[0, r_1 + \epsilon]) \cong H_i(d|_{W_*}^{-1}[0, r_1 + \epsilon] \oplus H_i(d|_{W_*^c}^{-1}[0, r_1 + \epsilon])$$

for all odd i.

Now consider the following exact homology sequence with a positive number ϵ such that $H_i(d|_{W_*}[0, r_1 - \epsilon]) \neq 0$ for some odd i:

$$\cdots \to H_{i+1}(d|_{\partial W_*}^{-1}[0, r_1 + \epsilon]), d|_{\partial W_*}^{-1}[0, r_1 - \epsilon]) \xrightarrow{f} H_i(d|_{\partial W_*}^{-1}[0, r_1 - \epsilon])$$
$$\xrightarrow{h} H_i(d|_{\partial W_*}^{-1}[0, r_1 + \epsilon]) \to \cdots$$

Since $H_i(d|_{W_*^c}^{-1}[0, r_1 - \epsilon]) \cong H_i(d|_{W_*^c}^{-1}[0, r_1 + \epsilon]), H_i(d|_{W_*^{-1}}[0, r_1 - \epsilon]) \neq 0$ and $H_i(d|_{W_*}^{-1}[0, r_1 + \epsilon]) = 0$, the kernel of h is nontrivial. Since the homology sequence is exact, image of f is nontrivial. Thus for cell $e^{\lambda(i)}$ of dimension which is the index $\lambda(i)$ of critical point p_i

$$H_i(d|_{\partial W_*}^{-1}[0, r_1 - \epsilon]) \cup e^{\lambda(1)} \cdots e^{\lambda(i)} \cdots \cup e^{\lambda(k)}, d|_{\partial W_*}^{-1}[0, r_1 - \epsilon]) \neq 0$$

for some odd i. This implies that there is a critical point $p \in \{p_i \,|\, 1 \leq i \leq k\}$ of index $i+1$. Thus we have $i+1$ number of negative principal normal curvature at p. This contradicts W is $C(o)$-compact. □

Remark. In the proof of Theorem 1, existence of $r_1 = \inf\{r \,|\, H_i(d|_W^{-1}[0, r]) = 0$ for all odd i, $r \geq r_0\}$ is guaranteed for a topological ball which is contained in the definition of $C(o)$-compact set. In fact, for a regular torus $\inf\{r \,|\, H_i(d|_W^{-1}[0, r]) = 0$ for all odd i, $r \geq r_0\}$ is not defined and the theorem fails to hold.

Corollary 1. *Let W be a compact connected n-manifold with C^2-boundary in $\mathbb{R}^n$. If W is $C(o)$-compact, then for almost all $(n-1)$-flat L_{n-1}, $H_i(W \cap L_{n-1}) = 0$ for all odd i.*

Proof. Let v be a unit vector perpendicular to L_{n-1} and p be a point in $W \cap L_{n-1}$. Then $l(t) = p + tv, -\infty < t < \infty$ is a normal line to L_{n-1}. Now we can take two balls A and B with centers $l(t_0)$ and $l(-t_0)$ respectively. Let us take sufficiently large number t_0 so that $W \cap L_{n-1}$ is deformation retraction of $A \cap B$. Then $A \cup B = W$ and $A \cap B \simeq W \cap L_{n-1}$. We have the Mayer-Vietories exact sequence: for odd i

$$\cdots \to H_{i+1}(A \cup B) \to H_i(A \cap B) \to H_i(A) \oplus H_i(B) \to \cdots .$$

By the definition of W for all i, $H_{i+1}(A \cup B) = 0$ and by Theorem 1, for all odd i, $H_i(A) = H_i(B) = 0$. Hence $H_i(W \cap L_{n-1}) = H_i(A \cap B) = 0$ for all odd i. □

Theorem 2. *If W is a $C(o)$-compact domain in $\mathbb{R}^n$ and $K(W)$ is the convex hull of W, then*

$$M_{n-2}(\partial W) \geq M_{n-2}(\partial K(W)).$$

Proof. Note that flat L_{n-1} of dimension $n-1$ intersecting $K(W)$ intersects W and so by the Corollary 1, $\chi(W \cap L_{n-1}) \geq 1$ for $W \cap L_{n-1} \neq \emptyset$. But $\chi(K(W) \cap L_{n-1}) = 1$ for such a flat L_{n-1}. So we have

$$\int_{W \cap L_{n-1} \neq \emptyset} \chi(W \cap L_{n-1}) dL_{n-1} \geq \int_{K(W) \cap L_{n-1} \neq \emptyset} \chi(K(W) \cap L_{n-1}) dL_{n-1}.$$

From (4), we have

$$\begin{aligned} M_{n-2}(\partial K(W)) &= \int_{K(W) \cap L_{n-1} \neq \emptyset} \chi(K(W) \cap L_{n-1}) dL_{n-1} \\ &\leq \int_{W \cap L_{n-1} \neq \emptyset} \chi(W \cap L_{n-1}) dL_{n-1} \\ &= M_{n-2}(\partial W). \end{aligned}$$

Comparing both sides of the inequality (9), we have

$$M_{n-2}(\partial W) \geq M_{n-2}(\partial K(W)). \tag{5}$$

□

Remark. If K_1 and K_2 are two convex bodies with C^2- boundaries with $K_1 \subset K_2$, then it is easy to show that $M_{n-2}(\partial K_1) \leq M_{n-2}(\partial K_2)$. Theorem 2 is unusual since the inequality is reversed in (5) for the $C(o)$-compact case even if $W \subset K(W)$:

Lemma 4. *Let W be a topological n-ball with C^2-boundary ∂W in $\mathbb{R}^n$. Then*

$$M_i'(\partial W) = (i+2) M_{i+1}(\partial W), \tag{6}$$

where $'$ denotes derivative for the functional M_i.

Proof. For a sufficiently small $r > 0$, Steiner's formula [7] states that

$$V(W_r)_h = V(W_r) + \sum_{i=0}^{n-1} M_i(\partial W_r) h^{i+1} \tag{7}$$

and

$$V(W_{r+h}) = V(W) + \sum_{i=0}^{n-1} M_i(\partial W)(r+h)^{i+1}. \tag{8}$$

Since $V(W_r)_h = V(W_{r+h})$ for a sufficiently small h, if we compare coefficients of right hand sides of (7)and (8) in h, we have

$$M_i(\partial W_r) = M_i(\partial W) + \sum_{k=2}^{n-i} \begin{pmatrix} i+k \\ k-1 \end{pmatrix} M_{i+k-1}(\partial W) r^{k-1}. \tag{9}$$

From (9),

$$M_i'(\partial W) = \lim_{r \to 0} \frac{M_i(\partial W_r) - M_i(\partial W)}{r} = (i+2) M_{i+1}(\partial W) \tag{10}$$

So our lemma is proved. □

Theorem 3. *If W is a $C(o)$-compact domain in $\mathbb{R}^n$ and $K(W)$ is the convex hull of W, then $M_{n-3}(\partial W)$ increases faster than or equal to $M_{n-3}(\partial K(W))$ around their boundaries.*

Proof. It is immediate from Theorem 2 and Lemma 4 since

$$\begin{aligned} M'_{n-3}(\partial W) &= (n-1)M_{n-2}(\partial W) \\ &\geq (n-1)M_{n-2}(\partial K(W)) \\ &= M'_{n-3}(\partial K(W)). \end{aligned}$$

□

References

[1] Y. D. Chai, *A new lower bound for the integral of the $(n-2)$nd mean curvature over the boundary of a compact domain in $\mathbb{R}^n$*, Differential Geom. Appl. **7** (1997), no. 1, 35–40.
[2] Y. D. Chai and G. Kim, *A Characterization of compact sets in $\mathbb{R}^n$ and its Application to a geometric inequality*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **17** (2010), no. 4, 543–553.
[3] S. S. Chern, *Global differential geometry*, MAA **27** (1989), 303–350.
[4] H. Flanders, *A proof of Minkowski's inequality for convex curves*, Amer. Math. Monthly **75** (1968), 581–593.
[5] M. Gromov, *Hyperbolic manifolds, group and actions*, Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), pp. 183–213, Ann. of Math. Stud., 97, Princeton Univ. Press, Princeton, N.J., 1981.
[6] J. Milnor, *Morse Theory*, Annals of Mathematics Studies, No. 51 Princeton University Press, Princeton, N. J., 1963.
[7] Polya and Szegö, *Isoperimetric inequalities in mathematical physics*, Annals of Mathematics Studies, no. 27, Princeton University Press, Princeton, N. J., 1951.
[8] A. Ros, *Compact hypersurfaces with constant higher order mean curvatures*, Rev. Mat. Iberoamericana **3** (1987), no. 3-4, 447–453.
[9] L. A. Santalo, *Integral Geometry and Geometric Probability*, Addison-Wesley Publishing Co., 1976.
[10] J. P. Sha, *p-convex Riemannian manifolds*, Invent. Math. **83** (1986), no. 3, 437–447.
[11] F. A. Valentine, *Convex Sets*, Mcgraw-Hill Book Co., 1964.

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